Advances and Applications in Mathematical Sciences
Volume 21, Issue 4, February 2022, Pages 2125-2135

FEKETE-SZEGÖ INEQUALITIES FOR CERTAIN ANALYTIC FUNCTIONS ASSOCIATED WITH q DERIVATIVE OPERATOR

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Abstract

Using the concept of q derivative operator and subordination principle we introduce and study new subclasses of analytic functions. We derive Fekete-Szegö inequalities for the functions belonging to the new subclasses. Some special cases of the established results are discussed.

1. Introduction

Let A represent the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

in the open unit disc $U = \{z : z \in C \text{ and } |z| < 1\}$.

The q calculus or quantum calculus is a generalization of the ordinary calculus without using the limit notation. The study of q calculus was initiated at the beginning of 19th century, it has many applications in the fields of special functions and many other areas. The q derivative operator is one of the tool used to explore many number of subclasses of analytic

2020 Mathematics Subject Classification: 30C45.
Keywords: Analytic functions, Subordination, q derivative operator, Fekete-Szegö inequalities.
Received November 2, 2021; Accepted December 29, 2021

functions, it plays significant role in the development of Geometric function theory. The application and usage of the q calculus was introduced by Jackson [10], [11]. Recently many researchers paid more attention to the area of q derivative operator and many new operators have been introduced and studied (refer [6], [7], [8], [9], [13], [16], [17], [20]).

Some basic notations and definitions of q calculus which are used in this paper are provided below. The q derivative of the function $f(z)$ is defined as

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \ (z \neq 0, \ 0 < q < 1) \tag{2}$$

In view of equation (2) it is clear that if $f(z)$ and $g(z)$ are two functions, then

$$D_q(f(z) + g(z)) = D_q f(z) + D_q g(z). \tag{3}$$

Observe that as $q \to 1^-, D_q f(z) \to f'(z)$, where $f'(z)$ is the ordinary derivative of the function $f(z)$. Further by (2) the q derivative of the function $h(z) = z^k$, is as follows

$$D_q h(z) = [k]_q z^{k-1} \tag{4}$$

where $[k]_q$ is given as:

$$[k]_q = \frac{1-q^k}{1-q}, \ (0 < q < 1). \tag{5}$$

Note that as $q \to 1^-, [k]_q \to k$, therefore as $q \to 1^-, D_q h(z) = h'(z)$ where $h'(z)$ denotes the ordinary derivative of the function $h(z)$ with respect to z.

The q derivative of the function $f(z)$, given by equation (1) is defined as follows

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} (0 < q < 1) \tag{6}$$

where $[k]_q$ is given by (5).

For $f(z) \in A$ and $0 < q < 1$, we define a new q derivative operator as follows.

$$D^n_{\delta, \lambda, l, q} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{(l + \delta - \lambda) + (1 - \delta + \lambda)[k]_q}{l + 1} \right)^n a_k z^k \tag{7}$$

where $\delta, \lambda, l \geq 0, n \in N_0 = NU\{0\}$. Let

$$D^n_{\delta, \lambda, l, q} f(z) = z + \sum_{k=2}^{\infty} L(\delta, \lambda, l, n)([k]_q) a_k z^k$$

where $L(\delta, \lambda, l, n)([k]_q) = \left(\frac{(l + \delta - \lambda) + (1 - \delta + \lambda)[k]_q}{l + 1} \right)^n$

It can be seen that as $q \to 1^-$, by specializing the parameters the new q differential operator $D^n_{\delta, \lambda, l, q}$ reduces to various operators studied by Al-oboudi [2], Catas [3], Cho and Srivastava [4], Latha and Shilpa [12], Maslina Darus and Rabha W Ibrahim [13], Salagean [14], Uralegaddi and Somanatha [18]. For example letting $q \to 1^-, \delta = \lambda, l = 0$ we get Salagean operator and letting $q \to 1^-, \delta = 1, l = 0$ we get Al-oboudi operator.

For the analytic functions $f(z)$ and $g(z)$ in U, we say that the function $g(z)$ is subordinate to $f(z)$ in U [15], and write $g(z) \prec f(z)$ if there exists a Schwarz function $\omega(z)$, which is analytic in with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$g(z) = f(\omega(z)), (z \in U). \tag{8}$$

Let P denote the class of all functions $\varphi(z)$ which are analytic U and univalent in and for which $\varphi(z)$ is convex with $\varphi(0) = 1$ and $\Re\{\varphi(z)\} > 0$ for all $z \in U$.

Now using the q derivative operator $D^n_{\delta, \lambda, l, q}$ and the concept of the subordination we introduce the new subclasses of analytic functions as follows.

Definition 1.1. A function $f(z)$ belongs to the class $R^n_{\delta, \lambda, l, q}(\varphi)$ if it satisfies the following subordination condition where and

$$D^n_{\delta, \lambda, l, q}(f(z)) \prec \varphi(z) \tag{9}$$

Where $\varphi(z) \in P$ and $0 < q < 1$.

Definition 1.2. A function $f(z) \in A$ belongs to the class $N^n_{\delta, \lambda, l, q}(\varphi)$ if it satisfies the following subordination condition

$$(1 - \alpha)\frac{f(z)}{z} + \alpha D^n_{\delta, \lambda, l, q}(f(z)) \prec \varphi(z) \tag{10}$$

where $\varphi(z) \in P$ and $0 \le \alpha \le 1$, $0 < q < 1$.

Note that, for suitable choices of parameters the new classes $R^n_{\delta, \lambda, l, q}(\varphi)$ and $N^n_{\delta, \lambda, l, q}(\varphi)$ reduces to the classes $R_q(\varphi)$ and $N_q(\varphi)$ studied in [1] respectively.

2. Main Results

The Fekete-Szegö problem [5] is to obtain the coefficient estimates for the second and third coefficients of functions belonging to class of analytic functions with a specific geometric properties. Now we find the Fekete-Szegö inequalities for functions belonging to the classes $R^n_{\delta, \lambda, l, q}(\varphi)$ and $N^n_{\delta, \lambda, l, q}(\varphi)$.

The following lemma is necessary to prove our main results.

Lemma 2.1 [19]. *Let* $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$, $(z \in U)$ *be a function with positive real part in and* μ *is a complex number, then*

$$| c_2 - \mu c_1^2 | \le 2 \max\{1; | 2\mu - 1 |\}. \tag{11}$$

The result is sharp for the functions given by $p(z) = \dfrac{1 + z}{1 - z}$ *and*

$p(z) = \frac{1 + z^2}{1 - z^2}$.

Theorem 2.1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots \in P$. *If* $f(z)$ *given by* (1) *belongs to the class* $R^n_{\delta, \lambda, l, q}(\varphi)$ *then*

$$| a_3 - \mu a_2^2 | \leq \frac{B_1}{L(\delta, \lambda, l, n)([3]_q)[3]_q} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{L(\delta, \lambda, l, n)([3]_q)[3]_q \mu B_1}{[L(\delta, \lambda, l, n)([2]_q)[2]_q]^2}\right|\right\} \tag{12}$$

where μ *is a complex number*, *and* $0 < q < 1$. *The result is sharp*.

Proof. If $f(z) \in R^n_{\delta, \lambda, l, q}(\varphi)$, then in view of Definition (1.1) there is a Schwarz function $\omega(z)$ in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ in U such that

$$D^n_{\delta, \lambda, l, q}(f(z)) = \varphi(\omega(z)). \tag{13}$$

We define the function

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1 z + p_2 z^2 + \ldots \tag{14}$$

Since $\omega(z)$ is a Schwarz function, we have $\Re\{p(z)\} > 0$ and $p(0) = 1$. Let

$$g(z) = D^n_{\delta, \lambda, l, q}(z) = 1 + d_1 z + d_2 z^2 + \ldots \tag{15}$$

Using equations (13), (14) and (15) we obtain

$$g(z) = \varphi\left(\frac{p(z) - 1}{p(z) + 1}\right) \tag{16}$$

Since

$$\frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 + \frac{p_1^3}{4} - p_1 p_2\right)z^3 + \ldots\right) \tag{17}$$

which yields

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right) = 1 + \frac{1}{2}B_1p_1z + \left(\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2\right)z^2 + \dots \tag{18}$$

Using equations (15) and (18) we obtain

$$d_1 = \frac{1}{2}B_1p_1 \tag{19}$$

$$d_2 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2. \tag{20}$$

A simple computation gives

$$D^n_{\delta, \lambda, l, q}(f(z)) = 1 + L(\delta, \lambda, l, n)([2]_q)[2]_q a_2 z + L(\delta, \lambda, l, n)([3]_q)[3]_q a_3 z^2 + \dots \tag{21}$$

Inequality (15), yields

$$d_1 = L(\delta, \lambda, l, n)([2]_q)[2]_q a_2 \tag{22}$$

$$d_2 = L(\delta, \lambda, l, n)([3]_q)[3]_q a_3 \tag{23}$$

now comparing the coefficients of z and z^2 and simplifying we get

$$a_2 = \frac{B_1p_1}{2L(\delta, \lambda, l, n)([2]_q)[2]_q} \tag{24}$$

and

$$a_3 = \frac{B_1}{2L(\delta, \lambda, l, n)([3]_q)[3]_q}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{B_2p_1^2}{4L(\delta, \lambda, l, n)([3]_q)[3]_q} \tag{25}$$

hence

$$a_3 - \mu a_2^2 = \frac{B_1}{2L(\delta, \lambda, l, n)([3]_q)[3]_q}(p_2 - \Upsilon p_1^2) \tag{26}$$

where

$$\Upsilon = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{L(\delta, \lambda, l, n)([3]_q)[3]_q \mu B_1}{[L(\delta, \lambda, l, n)([2]_q)[2]_q \alpha]^2}\right) \tag{27}$$

Hence, by Lemma 2.1, the result follows.

Note that, for suitable choices of parameters in Theorem 2.1 we get the following Corollary derived in [1].

Corollary 2.1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots \in P$, *with* $B_1 \neq 0$. *If* $f(z)$ *given by* (1) *belongs to the class* $R_{(q)}(\varphi)$ *and* μ *is a complex number*, *then*

$$| a_3 - \mu a_2^2 | \leq \frac{B_1}{[3]_q} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{[3]_q \mu B_1}{[2]_q^2}\right|\right\} \tag{28}$$

The result is sharp.

Similarly, we can obtain upper bound for the Fekete-Szegö inequalities for functions belonging to the class $N^n_{\delta,\, gl,\, l,\, q}(\varphi)$ as follows.

Theorem 2.2. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots \in P$. *If* $f(z)$ *given by* (1) *belongs to the class* $N^n_{\delta,\, gl,\, l,\, q}(\varphi)$ *then*

$$| a_3 - \mu a_2^2 | \leq \frac{B_1}{[(1-\alpha) + L(\delta, \lambda, l, n)([3]_q)[3]_q \alpha]} \max\left\{1, \left|\frac{B_2}{B_1}\right.\right.$$

$$\left.\left. - \frac{\mu B_1 [(1-\alpha) + L(\delta, \lambda, l, n)([3]_q)[3]_q \alpha]}{[(1-\alpha) + L(\delta, \lambda, l, n)([2]_q)[2]_q \alpha]^2}\right|\right\} \tag{29}$$

where μ *is a complex number*, *and* $0 < q < 1$. *The result is sharp*.

Proof. If $f(z) \in N^n_{\delta,\, \lambda,\, l,\, q}(\varphi)$, then in view of Definition (1.1) there is a Schwarz function $\omega(z)$ in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ in U such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha D^n_{\delta,\, \lambda,\, l,\, q}(f(z)) = \varphi(\omega(z)). \tag{30}$$

We define the function

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1 z + p_2 z^2 + \ldots \tag{31}$$

Since ω is a Schwarz function, we have $\Re\{p(z)\} > 0$ and $p(0) = 1$. Let

$$g(z)=(1-\alpha)\frac{f(z)}{z}+\alpha D^{n}_{\delta,\lambda,l,q}(f(z))=1+d_1z+d_2z^2+\ldots \tag{32}$$

using equations (30), (31) and (32) we obtain

$$g(z)=\varphi\left(\frac{p(z)-1}{p(z)+1}\right) \tag{33}$$

Since

$$\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left(p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3-\frac{p_1^3}{4}-p_1p_2\right)z^3+\ldots\right) \tag{34}$$

which gives

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right)=1+\frac{1}{2}B_1p_1z+\left(\frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2\right)z^2+\ldots \tag{35}$$

using equations (32) and (35) we obtain

$$d_1=\frac{1}{2}B_1p_1 \tag{36}$$

$$d_2=\frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2 \tag{37}$$

A computation gives

$$\begin{aligned}(1-\alpha)\frac{f(z)}{z}+\alpha D^{n}_{\delta,\lambda,l,q}(f(z))=&\\ &1+[(1-\alpha)+L(\delta,\lambda,l,n)([2]_q)[2]_q\alpha]a_2z\\ &+[(1-\alpha)+L(\delta,\lambda,l,n)([3]_q)[3]_q\alpha]a_3z^2+\ldots\end{aligned} \tag{38}$$

Inequality (32), yields

$$d_1=[(1-\alpha)+L(\delta,\lambda,l,n)([2]_q)[2]_q\alpha]a_2 \tag{39}$$

$$d_2=[(1-\alpha)+L(\delta,\lambda,l,n)([3]_q)[3]_q\alpha]a_3 \tag{40}$$

or equivalently we get

$$a_2 = \frac{B_1 p_1}{2[(1-\alpha) + L(\delta, \lambda, l, n)([2]_q)[2]_q \alpha]} \tag{41}$$

and

$$a_3 = \frac{B_1}{2[(1-\alpha) + L(\delta, \lambda, l, n)([3]_q)[3]_q \alpha]}\left(p_2 - \frac{p_1^2}{2}\right)$$

$$+ \frac{B_2 p_1^2}{4[(1-\alpha) + L(\delta, \lambda, l, n)([3]_q)[3]_q \alpha]} \tag{42}$$

hence

$$a_3 - \mu a_2^2 = \frac{B_1}{2[(1-\alpha) + L(\delta, \lambda, l, n)([3]_q)[3]_q \alpha]}(p_2 - \Upsilon p_1^2) \tag{43}$$

where

$$\gamma = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{\mu B_1[(1-\alpha) + L(\delta, \lambda, l, n)([3]_q)[3]_q \alpha]}{[(1-\alpha) + L(\delta, \lambda, l, n)([2]_q)[2]_q \alpha]}\right) \tag{44}$$

Hence, by applying Lemma 2.1, the result follows.

Note that, for suitable choices of parameters in Theorem 2.2 we get the following Corollary derived in [1].

Corollary 2.2. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots \in P$, *with* $B_1 \neq 0$. *If* $f(z)$ *is given by* (1) *belongs to the class* $N_q(\varphi)$ *and* μ *is a complex number*, *then*

$$|\, a_3 - \mu a_2^2 \,| \leq \frac{B_1}{[(1-\alpha) + [3]_q \alpha]} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{\mu B_1[(1-\alpha) + [3]_q \alpha]}{[(1-\alpha) + [2]_q \alpha]^2}\right|\right\} \tag{45}$$

The result is sharp.

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