Advances and Applications in Mathematical Sciences
Volume 21, Issue 8, June 2022, Pages 4289-4297

GENERALIZED METRIC SPACES WITH INTERVAL POINTS AND SOME EXTENDED FIXED POINT RESULTS OF PEROV

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Abstract

The immense applications of fixed point results encourages the researchers for a collaborative study to various areas of mathematics and other disciplines of science where ever its applications works. Interval analysis and fixed point theory are now connecting for obtaining results which are mostly useful in studies including signal processing and space research. The interval valued notion of generalized metric space is already defined. In this article we have portrayed the Perov's fixed point results found in [8], [9] for the generalized interval metric space. We also redefined Czerwik generalized metric space with interval points with an illustration.

1. Introduction

Lack of accuracy in mathematical calculations is an important constraint in many areas of research. Interval arithmetic approach responds to these questions which appears in scientific computing problems and in the areas where ever the problems of accuracy and efficiency occured. So interval approach needs to develop one step forward under these circumstances. But all the studies are mainly focused on the background of real arithmetic only. Moore (1966) and Sunaga (1958) were marked as the pioneers in the development of interval arithmetic. The topic of interval metric was first introduced in 2010 and we can't find much works in the ares of interval metric and fixed point theory. Here we are trying to find fixed point results

2020 Mathematics Subject Classification: 47H10, 46N99.
Keywords: Generalized metric spaces, fixed point results.
Received January 10, 2022; Accepted March 2, 2022

which are suited for metric spaces with interval points by satisfying all the required conditions of the results.

In the first section you will get basic notes about generalized interval metric space and its properties which is required to follow this paper. The main results are included in section 3 and 4, there we established fixed point results and defined Czerwik generalized metric space in $I(R_m)$ space. Final section is nothing but the concluding remarks.

2. Preliminary Notes

Genaralized interval metric space: In [11], the interval distance $M_{ei}(X, Y)$ for the intervals $X = [\underline{x}, \bar{x}], Y = [\underline{y}, \bar{y}]$ is defined as follows:

$$M_{ei}(X, Y) = \begin{cases} [\underline{y} - \bar{x}, \bar{y} - \underline{x}], & \text{if } X \leq Y \text{ and } X \cap Y = \phi \\ [0, \bar{y} - \underline{x}], & \text{if } X \leq Y \text{ and } X \cap Y \neq \phi \\ [0, \max\{\bar{x}, \underline{y}, \bar{y} - \underline{x}\}], & \text{if } X \subseteq Y \end{cases}$$

Let $I(R_m)$ be the set of all $m \times 1$ real interval matrices. $X, Y \in I(R_m)$ such that $X = (X_1, X_2, \ldots, X_m)^T$ and $X = (Y_1, Y_2, \ldots, Y_m)^T$ where each X_i, Y_i are of the form $X_i = [\underline{x_i}, \overline{x_i}]$ and $Y_i = [\underline{y_i}, \overline{y_i}]$. Then the interval distance of the m tuples X and Y, we can denote it by d_{id} can be explicate as:

$$d_{id}(X, Y) = (M_{ei}(X_i, Y_i)) = (M_{ei}(X_1, Y_1), M_{ei}(X_2, Y_2), \ldots, M_{ei}(X_m, Y_m))$$

Now we can define genaralized interval metric of m tuples.

Let X_I, a non empty interval set. A function $d_{id} : X_I \times X_I \to I(R_m)$ is called an interval metric of m tuples if the given properties are appeased:

1. $O \in d_{id}(X, X)$ where $O = ([0, 0], \ldots, [0, 0])$ and $[0, 0] \subseteq M_{ei}(X_i, X_i)$ for every i

2. $d_{id}(X, Y) = d_{id}(Y, X)$

3. $d_{id}(X, Y) \leq d_{id}(X, Z) + d_{id}(Z, Y)$

4. If $O \in d_{id}(X, X) = d_{id}(X, Y) = d_{id}(Y, Y)$, then $X = Y$

Interval Matrix convergence [7], [10]: An interval matrix $A \in M_{m\times m}I(R_+)$ is of the form

$$\begin{bmatrix} [\underline{a_{11}}, \overline{a_{11}}] & \dots & [\underline{a_{1m}}, \overline{a_{1m}}] \\ & \dots & [\underline{a_{2m}}, \overline{a_{2m}}] \\ \dots & \dots & \dots \\ & \dots & [\underline{a_{mm}}, \overline{a_{mm}}] \end{bmatrix}$$

Then the multiplication of two interval matrices A and B is defined as:

$$C_{ij} = \sum_k A_{ik} * B_{kj}$$

and each element in this matrix C_{ij} is of the form $[\underline{c_{ij}}, \overline{c_{ij}}]$ where

$$[\underline{c_{ij}}, \overline{c_{ij}}] = [\underline{a_{ik}}, \overline{a_{ik}}] * [\underline{b_{kj}}, \overline{b_{kj}}]$$

such that

$$\underline{c_{ij}} = \min\{\underline{a_{ik}}\underline{b_{kj}}, \underline{a_{ik}}\overline{b_{kj}}, \overline{a_{ik}}\underline{b_{kj}}, \overline{a_{ik}}\overline{b_{kj}}\}$$

and

$$\overline{c_{ij}} = \max\{\underline{a_{ik}}\underline{b_{kj}}, \underline{a_{ik}}\overline{b_{kj}}, \overline{a_{ik}}\underline{b_{kj}}, \overline{a_{ik}}\overline{b_{kj}}\}$$

The convergence of an interval matrix can be defined as follows. A is convergent iff the sequence A^k converges to the zero matrix O. That is

$$\lim_{k\to\infty} \underline{a_{ij}}^{(k)} = 0, \lim_{k\to\infty} \overline{a_{ij}}^{(k)} = 0, i, j = 1, 2, \dots, m$$

The if and only if condition for the convergence of an interval matrix is that the spectral radius $\rho(|A_{ij}|) < 1$ where $|A_{ij}|$ is the absolute real matrix of A by taking absolute values of each elements $[\underline{a_{ij}}, \overline{a_{ij}}]$ [5].

Completeness of Interval sequence: Here we are defining the ideas of convergence and completeness for the case of generalized interval metric space. As in the case of real sequences, convergence and completeness are defined in the same way for interval sequences. Studies related to the interval convergence can be seen in [6]. All the definitions depicted in [6] for interval sequences is also applicable for generalized interval space.

Let X_k be a sequence of intervals. We say that X_k is convergent if there exists an interval X^* such that for every $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that $d_{id}(X_k, X^*) < \varepsilon$ whenever $k > N$. As in the case of real sequences, we write

$$X^* = \lim_{k\to\infty} X_k \text{ or } X_k \to X^*$$

and refer to X^* as the limit of X_k

A sequence (X_k) in (X_I, d_{id}) is said to be Cauchy if for every $\varepsilon \geq 0$ there is a positive integer N such that for all positive integer $m, n > N$, the distance $d_{id}(X_m, X_n) < \varepsilon$. The generalized interval metric space (X_I, d_{id}) is said to be complete if every Cauchy sequence of intervals (X_k) in (X_I) is convergent.

Definition: The Czerwik vector valued metric on X is the mapping $e : X \times X \to R_m$ such that

1. $e(u, v) \geq 0$; if $e(u, v) = 0$ then $u = v$

2. $e(u, v) = e(u, v)$

3. $e(u, v) \leq S[e(u, w) + e(w, v)]$

where $S = (s_{ij})$ is a square matrix of order m such that $s_{ij} = \begin{cases} s & i = j \\ 0 & i \neq j \end{cases}$

3. Fixed Point Results in Generalized Interval Metric Space

In this section we are going to establish the Perov's extensions of Banach contraction principle to the generalized interval metric for finding fixed point solutions. The important thing here to keep in mind is that the considered points are not single values but intervals and the intervals are considered as the points of the metric space.

So here we are finding fixed intervals as solution. From now onwards we write the complete generalized metric space as 'cgms' and complete generalized interval metric space as 'cgims' in short. Here we are stating only the results and it is only an implementations of the theorem to the interval valued metric space. The proofs, if needed, nothing but the proofs of the base theorems.

Theorem 1. *As in* [2], *Let* (Y, e) *be a cgms and* $g : Y \to Y$ *be a mapping for which there exists a square matrix B of order m such that*

$e(g_x, g_y) \leq B\, e(x, y)$ *forever* $x, y \in Y$

And the convergence of the matrix B implies;

1. $fix(g) = x^*$, $fix(g) = \{x \in Y : gx = x\}$

2. The successive sequence (x_n) such that $x_n = g^n(x_0)$ is convergent and attains the limit x^* for all $x_0 \in Y$.

Theorem 2. *Let* (X_I, d_{id}) *be a cgims and* $f_I : X_I \to X_I$ *be a mapping for which there exist an interval matrix* $A_I \in M_{m \times m} I(R_+)$ *such that*

$d_{id}(f_I X, f_I Y) \leq A_I d_{id}(X, Y)$ *for all* $x, y \in X$

If $\rho(A_I) < 1$ *for the interval matrix* A_I *then*;

1. $fix(f_I) = X^*$ *where* $fix(f_I) = \{X \in X_I : f_I X = X\}$

2. *The successive sequence of intervals* (X_n) *such that* $X_n = f_I^n(X_0)$ *is convergent and attains the limit* X^* *for all* $X_0 \in X_I$.

Theorem 3. *As in* [3], *let* (Y, e) *be a cgms and* $g : Y \to Y$ *be a mapping for which there exists a matrix* $B \in M_{m \times m}(R_+)$ *such that*

$e(g_x, g_y) \leq Be(x, y)$ for all $x, y \in Y$

If B is a convergent matrix to zero, then;

1. $fix(g) = x^*$ where $fix(g) = \{x \in Y : g_x = x\}$

2. The sequence of successive approximations x_n such that $x_n = g^n(x_0)$ is convergent and admits the limit x^* for all $x_0 \in X_I$.

3. $e(x_n, x^*) \leq B^n(I_m - B)^{-1}(e(x_0, x_n)), n \in N$

4. if $h : Y \to Y$ satisfies the situation $e(g(x), h(x)) \leq c$ for all $x \in Y$ and some $c \in R_m$, and when the sequence $y_n = h^n(x_0), n \in N$, is considered, one has

$$e(y_n, x^*) \leq (I_m - B)^{-1}(c) + B^n(I_m - B)^{-1}(e(x_0, x_1)), n \in N$$

Now we are going to extend this result on a setting of generalized interval metric space.

Theorem 4. *Let* $(X_I - d_{id})$ *be a cgims and* $f_I : X_I \to X_I$ *be a mapping for which there exists an interval matrix* $A_I \in M_{m \times m} I(R_+)$ *such that*

$$d_{id}(f_I X, f_I Y) \leq A_I d_{id}(X, Y) \text{ for all } X, Y \in X_I$$

If $\rho(| A_I |) < 1$ *then*;

1. $fix(f_I) = X^*$ where $fix(f_I) = \{X \in X_I : f_I X = X\}$

2. The approximate sequence in succession X_n such that $X_n = f_I^n(X_0)$ is convergent and agrees the limit X^* for all $X_0 \in X_I$

3. $d_{id}(X_n, X^*) \leq A_I^n(I_m - A_I)^{-1}(d_{id}(X_0, X_n)), n \in N$

4. If $g_I : X_I \to X_I$ satisfies the situation $d_{id}(f_I(X), g_I(X)) \leq C_I$ for all $X \in X_I$ and some $C_I \in IR_m$, and when the sequence $Y_n = g_I^n(X_0), n \in N$, is considered, one has

$$d_{id}(Y_n, X^*) \leq (I_m - A_I)^{-1}(C_I) + A_I^n(I_m - A_I)^{-1}(d_{id}(X_0, X_1)), n \in N$$

4. Czerwik Generalized Metric Space with Interval Points

We starts this section by reaching out the definition of Czerwik gms [2] for the case of interval points.

Definition. A mapping $d_{id}; X_I \times X_I \to I(R)_m$ is called Czerwik vector valued metric on X_I, when there exists an interval matrix $S_I \in M_{m,m} I(R_+)$ and for each $U, V, W \in X_I$, the succeeding conditions holds:

1. $O \in d_{id}(U, U)$ where $O = ([0, 0], \ldots, [0, 0])$ and $[0, 0] \subseteq M_{ei}(U_i, U_i)$ for every i

2. $d_{id}(U, V) = d_{id}(V, U)$

3. $d_{id}(U, V) = S_I[d_{id}(U, W) + d_{id}(W, V)]$

4. If $O \in d_{id}(U, U) = d_{id}(U, V) = d_{id}(V, V)$, then $U = V$

where $S_I = ([\underline{s_{ij}}, \overline{s_{ij}}]$ is an $m \times m$ interval matrix such that

$$[\underline{s_{ij}}, \overline{s_{ij}}] = \begin{cases} [s, s], & i = j \\ [0, 0], & i \neq j \end{cases}$$

where $s \geq 1$

Then, a non empty set X_I equipped with Czerwik vector valued metric d_{id} with interval points is called Czerwik gms with interval points, denoted by (X_I, d_{id}, S_I).

Example. Let $X_I = I(R^2)$. Then the mapping $d_{id} : X_I \times X_I \to I(R_2)$ defined by

$$d_{id}(X, Y) = d_{id}(([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}]), ([\underline{y_1}, \overline{y_1}], [\underline{y_2}, \overline{y_2}]))$$

$$= \begin{bmatrix} M_{ei}([\underline{x_1}, \overline{x_1}], [\underline{y_1}, \overline{y_1}]) \\ M_{ei}([\underline{x_2}, \overline{x_2}], [\underline{y_2}, \overline{y_2}]) \end{bmatrix}$$

for every $X, Y \in X_I$ is a Czerwik generalized metric with interval points with interval matrix $S_I = \begin{bmatrix} [2, 2] & [0, 0] \\ [0, 0] & [2, 2] \end{bmatrix}$

Illustration: Let $X = ([1, 2], [3, 9])$ and $Y = ([4, 8], [0, 1])$. Then

$$d_{id}(X, Y) = \begin{bmatrix} M_{ei}([1, 2], [4, 8]) \\ M_{ei}([3, 9], [0, 1]) \end{bmatrix} = \begin{bmatrix} [2, 7] \\ [2, 9] \end{bmatrix}$$

Let $Z = ([1, 3], [4, 5])$. Then

$$d_{id}(X, Z) = \begin{bmatrix} [0, 2] \\ [0, 5] \end{bmatrix}, \ d_{id}(Z, Y) = \begin{bmatrix} [1, 7] \\ [3, 5] \end{bmatrix}$$

Then $S_I(d_{id}(X, Z) + d_{id}(Z, Y)) = \begin{bmatrix} [2, 18] \\ [6, 20] \end{bmatrix}$ and

$d_{id}(X, Y) \leq S_I(d_{id}(X, Z) + d_{id}(Z, Y))$

5. Concluding Remarks

In this article we established some Perov extensions of Banach fixed point theorem for the generalized interval metric space. Also we defined Czerwik generalized metric space in interval platform with an example. The applications of these results are entended to the areas wherever real mathematics is transformed into interval mathematics and for all disciplines mathematics where fixed point results works.

References

[1] S. Sergio Aguiar, et al., Using the Interval Metric for Modeling Entities Geometrics in R^2 -Case Study Interval Circumference, TEMA (São Carlos) 21 (2020), 65-82.

[2] Ali, Muhammad Usman and Jong Kyu Kim, An Extension of Vector-Valued Metric Spaces and Perov's Fixed Point Theorem (Nonlinear Analysis and Convex Analysis), RIMS Kokyuroku 2114 (2019), 12-20.

[3] Cvetković Marija, On the equivalence between Perov fixed point theorem and Banach contraction principle, Filomat 31(11) (2017), 3137-3146.

[4] K. Ganesan, On some properties of interval matrices, International Journal of Computational and Mathematical Sciences 1(2) 2007.

[5] Mayer Günter, On the convergence of powers of interval matrices, Linear Algebra and its applications 58 (1984), 201-216.

[6] Moore, E. Ramon, R. Baker Kearfott and Michael J. Cloud, Introduction to interval analysis, Society for Industrial and Applied Mathematics, 2009.

[7] T. Nirmala, et al., Inverse interval matrix: A new approach, Applied Mathematical Sciences 5(13) (2011), 607-624.

[8] A. I. Perov, On the Cauchy problem for a system of ordinary differential equations, (1964), 115-134.

[9] Perov Anatoliy Ivanovich and A. V. Kibenko, On a certain general method for investigation of boundary value problems, Izvestiya Rossiiskoi Akademii Nauk Seriya Matematicheskaya 30(2) (1966), 249-264.

[10] Rohn Jirí, Inverse interval matrix, SIAM Journal on Numerical Analysis 30(3) (1993), 864-870.

[11] Trindade Roque Mendes Prado, et al., An interval metric, New Advanced Technologies, Intech Open, 2010.