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S-PRIME IDEAL GRAPH OF A FINITE COMMUTATIVE RING

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Abstract

Let R be a finite commutative ring with unity and $S \subseteq R$ be a multiplicative subset. Let S_d be an S-prime ideal of R and it is generated by the divisor d of the order of R. In this paper, we introduce a new graph called S-prime ideal graph which is an undirected graph whose vertex set is the set of elements of R and two vertices a and b are joined by an edge if $sa \in S_d$ or $sb \in S_d$ for some $s \in S$. Further, the graph theoretic properties like diameter, girth and chromatic number and the algebraic properties like rank and nullity of the graph are interpreted.

1. Introduction

The investigation of graphs from groups is a vast research area. In 1964, Bosak [3] studied about certain graph over semigroups. Many author's developed this area such as intersection graphs of finite abelian groups, undirected power graphs of semigroups, the power graph of a finite group, the centre graph of a group, power graphs of finite groups of even order, power graph of finite abelian groups, normal subgroup based power graphs of a finite group, etc.,

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Recently, Kiruthika and Kalamani [11] introduced and studied about some aspects of the vertex order graph and its complements. The vertex order graph is a simple graph and there is an edge between any two distinct vertices if and only if its orders are equal. The author [12] partitioned the set of vertices and edges of the power graph of a finite abelian group $\mathbb{Z}_{pq}$ where p and q are primes. In 2021, Ramya and Kalamani [13] examined for some cordial labeling for commuting graph of a subset of the dihedral group.

Also the notion of connecting and developing a graph with a commutative ring with unity was initially proposed by Istvan Beck [2] in 1988. Consequently, Anderson and Livingston [4] introduced the zero-divisor graph whose vertices are the set of non-zero zero divisors of R with $Z(R)^* = Z(R) \setminus \{0\}$, and x and y are adjacent iff $xy = 0$ and is denoted as $\Gamma(R)$. Frequently, the graphs were introduced with algebraic properties such as commuting graph, prime graph, comaximal ideal graph, non-comaximal graph $NC(R)$, total graph $T(\Gamma(R))$, unit graph, annihilating ideal graph $AG(R)$, nilpotent graph, etc., Likewise, the S-prime ideal graph is defined.

In 2021, Kalamani and Ramya [9, 10] introduced and investigated a new graph called the product maximal graph of a finite commutative ring R whose vertices are all the elements of R and if the two distinct vertices x and y are adjacent if and only if the product xy is in the maximal ideal of R.

In 2019, Ahmed Hamed and Achraf Malek [1] developed the idea of S-prime ideals of a commutative ring. The ideal I disjoint with the multiplicative subset S is called S-prime ideal if there exists an $s \in S$ such that for all $a, b \in R$, either $sa \in I$ or $sb \in I$ whenever $ab \in I$. It is the generalization of the definition of prime ideal. Every prime ideal I of R is an S-prime ideal but the converse need not be true. In 2021, Fuad Ali Ahmahdi, El Mehdi Bouba and Mohammed Tamekkante [6] introduced the concept of weakly S-prime ideals of commutative rings. The ideal I is weakly S-prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $0 \neq ab \in I$ then $sa \in I$ or $sb \in I$ which is the generalization of weakly prime ideals.

Our aim is to introduce a new graph called the S-prime ideal graph which is denoted by $G_{S_d}(R)$ where d is the divisor of the cardinality of R.

We study some basic definitions and notations for algebra from Dummit and Foote [5] abstract algebra and Gallian [7] contemporary abstract algebra and for graph theory Dougles B. West [14] introduction to graph theory. The algebraic and graph theoretical properties of the S-prime ideal graph will be discussed in the proposed paper.

2. Preliminaries

In this section, the basic definitions are given.

Definition 2.1. A subring I of a ring R is called a (two-sided) *ideal* of R if for every $r \in R$ and every $a \in I$ both ra and ar are in I.

Definition 2.2. An ideal P is called a *prime ideal* if $P \neq R$ and whenever the product ab of two elements $a, b \in R$ is an element of P, then at least one of a and b is an element of P.

Definition 2.3. Let R be a ring. A non-zero element a of R is called a *zero divisor* if there is a non-zero element b in R such that either $ab = 0$ or $ba = 0$.

Definition 2.4. An element u of R is called a *unit* in R if there is some v in R such that $uv = vu = 1$.

Definition 2.5. If G has a u, v path, then the distance from u to v, written as $d(u, v)$, is the least length of a u, v-path. If G has no path, then $d(u, v) = \infty$. The *diameter* is the maximum distance of $d(u, v)$. It is denoted by $diam(G)$. The *radius* is the minimum distance of $d(u, v)$. It is denoted by $rad(G)$.

Definition 2.6. The *girth* of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth.

Definition 2.7. A *complete graph* is a simple graph whose vertices are pairwise adjacent. The complete graph with n vertices is denoted by K_n.

Definition 2.8. The *chromatic number* of a graph G written as $\chi(G)$ is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors.

Definition 2.9. In a graph, a set of pairwise adjacent vertices is called a *clique*. The size of a maximum clique in G is called the *clique number* of G

and is denoted as $\omega(G)$. A clique of a graph is a maximal complete subgraph.

Definition 2.10. The *rank* of a graph G, denoted by $\rho(G)$, is the number of non-zero eigenvalues of the adjacency matrix A of the graph G. The *nullity* of a graph is defined as the multiplicity of the eigenvalue zero of the adjacency matrix A of the graph. It is denoted by $\eta(G)$. The dimension of a matrix A is $\dim A = \rho(A) + \eta(A)$.

Definition 2.11. Let R be any ring. A *multiplicative subset* S of R is a subset such that $1 \in S$ and S is closed under multiplication.

Definition 2.12. Let R be a commutative ring of order n. The set $S \subseteq R$ is a multiplicative subset S of R. The ideal I of R disjoint with S is called *S-prime ideal* if there exists an $s \in S$ such that for all $a, b \in R$ either $sa \in I$ or $sb \in I$ whenever $ab \in I$.

The S-prime ideal is denoted by S_d.

3. Main Results

Let R be a commutative ring of order n with unity. Let S_d be the S-prime ideal of R where d is the divisor of n. Consider the multiplicative subset S of R disjoint from S_d. The S-prime ideal graph of R is introduced and defined as follows.

Definition 3.1. Let R be a finite commutative ring with unity. Let $S \subseteq R$ be a multiplicative subset of R and S_d be an S-prime ideal of R generated by d where d is the divisor of order of R. The S-prime ideal graph of R, denoted by $G_{S_d}(R)$ is an undirected graph whose vertex set is the set of elements of R and two vertices a and b are joined by an edge if $sa \in S_d$ or $sb \in S_d$ for some $s \in S$.

Example 1. Let $R = \mathbb{Z}_{12}$.

By the definition, the ideals $\langle 2\rangle, \langle 3\rangle, \langle 6\rangle$ and $\langle 12\rangle$ are S-prime ideals and are denoted by S_2, S_3, S_6 and S_{12} respectively and the corresponding S-prime ideal graphs are shown in Figure 1.

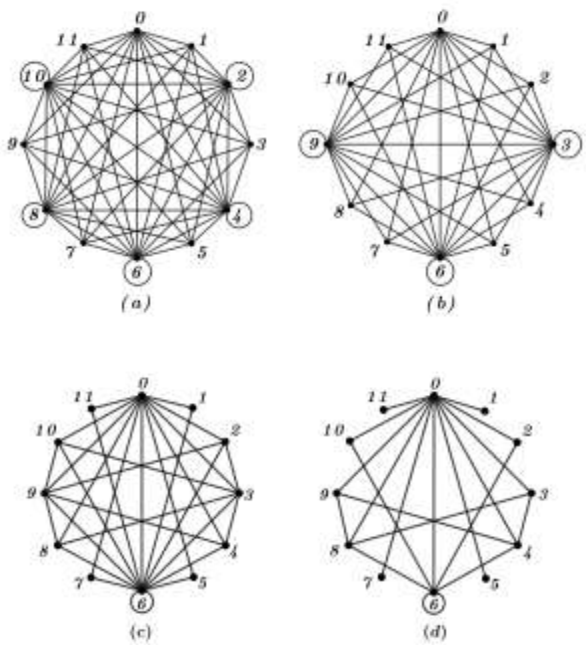

Figure 1. S-prime ideal graph G_{S_d} of $\mathbb{Z}_{12}$ for (a) S_2 ideal (b) S_3 ideal (c) S_6 ideal (d) S_{12} ideal.

If S_d is the S-prime ideal of a commutative ring of order n then d is the divisor of n. But the converse is not true. For example, the ideal $\langle 4 \rangle$ is not the S-prime ideal of $\mathbb{Z}_{12}$.

Let $G_{S_d}(R)$ be the S-prime ideal graph of a finite commutative ring R with unity of order n. It is not a simple graph. The vertices which are in S-prime ideals is adjacent to all the vertices in S-prime ideal graph $G_{S_d}(R)$ and also the elements in S-prime ideal S_d form the self-loop except $0 \in R$.

The degree of the non-zero elements in the S-prime ideal is $n+1$, the degree of the additive identity is $n-1$ and the degree of the units of R is the cardinality of the S-prime ideal S_d.

3.1 Graph theoretical properties of S-prime ideal graph $G_{S_d}(R)$

Some of the well-known graphs namely the graphs of semigroups, the zero-divisor graph of a commutative ring, the product maximal graph of a finite commutative ring, etc., are introduced with algebraic concepts and their properties [3], [4], [9, 10] are studied.

In this section the properties of the S-prime ideal graph are generalized with theorems and examples.

Theorem 3.1. *Let* $G_{S_d}(R)$ *be the S-prime ideal graph of a finite commutative ring R with unity and* S_d *be an S-prime ideal of R. Then the total number of edges* $= |N|$ *where* $N = \{(a, b) \mid ab \equiv \bmod d$ *for all* $a, b \in R$ *and* $\langle d \rangle = S_d\}$.

Proof. Let S be a multiplicative subset of R and $S \cap S_d \neq 0$. It is clear that every S-prime ideal S_d of R has an identity element 0 of R. The identity element 0 is adjacent to all the other elements of R in $G_{S_d}(R)$. If $a = 0$ then $ab \in S_d$ for every $b \in R$ this implies that $sa \in S_d$ for some $s \in S$. It gives $n - 1$ edges for n-number of vertices of $G_{S_d}(R)$. Hence the edge set $E \neq 0$.

Let $N = \{(a, b) \mid ab \equiv 0 \bmod d$ for all $a, b \in R\}$ where d is the divisor of the order of R.

Claim. $(a, b) \in E$ if and only if $(a, b) \in N$.

Assume that $(a, b) \in E$. This implies that either $sa \in S_d$ or $sb \in S_d$ for some $s \in S$ whenever $ab \in S_d$.

If $ab \in S_d$ then d divides ab since d is the generator of S_d.

Thus $ab \equiv 0 \bmod d$.

$\therefore (a, b) \in N$.

Next, assume that $(a, b) \in N$.

This implies that $ab \equiv 0 \bmod d$ and ab is an element of S_d since d divides ab.

If either $a \in S_d$ or $b \in S_d$ then take $s = 1$. In this case, $sa \in S_d$ or $sb \in S_d$. (1)

If both $a, b \notin S_d$, then choose $s =$ divisors of d such that $sa \in S_d$ or $sb \in S_d$. (2)

From (1) and (2) we can conclude that $(a, b) \in E$.

Hence the total number of edges $= |N|$. □

The graph $G_{S_d}(R)$ is a connected graph and hence the diameter of the graph is finite. It is proved in the following theorem.

Theorem 3.2. *Let R be a finite commutative ring with unity and $G_{S_d}(R)$ be the S-prime ideal graph of R. Then $diam[G_{S_d}(R)] = 2$.*

Proof. Let R be a finite commutative ring of order n with unity. Let S_d be the S-prime ideal of R where d is the divisor of n. We denote the distance between any two vertices a and b of the graph $G_{S_d}(R)$ by $d(a, b)$.

Case (i). Either $a \in S_b$ or $b \in S_d$.

Every element of an S-prime ideal S_d is adjacent to all the other elements of the S-prime ideal graph $G_{S_d}(R)$.

Hence the distance $d(a, b) = 1$.

Case (ii). Both a and b not in S_d

By using the previous Theorem 3.1., we have

$$d(a, b) = \begin{cases} 1 \text{ if } (a, b) \in N \\ 2 \text{ if } (a, b) \notin N. \end{cases}$$

Therefore, the maximum distance of vertices a and b is 2.

Thus $diam[G_{S_d}(R)] = 2$.

Corollary 3.3. *Let R be a finite commutative ring with unity and $G_{S_d}(R)$ be the S-prime ideal graph of R. Then $rad[G_{S_d}(R)] = 1$.*

In the following theorem, the girth of the S-prime ideal graph $G_{S_d}(R)$ is detemined by using the number of edges of $G_{S_d}(R)$ [Theorem 3.1].

Theorem 3.4. *Let R be a finite commutative ring with unity of order n and S_d be the S-prime ideal of R. Then the girth of S-prime ideal graph is*

$$gr[G_{S_d}(R)] = \begin{cases} 3 \text{ if } n \neq p \\ \infty \text{ otherwise} \end{cases}$$

where p is a prime.

Proof. Let R be a finite commutative ring and S_d be the S-prime ideal of R.

Case (i). n is not a prime.

Let a and b be any two vertices of S-prime ideal graph $G_{S_d}(R)$.

By Theorem 3.1, if $(a, b) \in N$ then it forms an adjacency and they are adjacent to all the elements of S-prime ideal S_d. So the S-prime ideal graph $G_{S_d}(R)$ form a cycle $a - b - c - a$ of length 3 where $c \in S_d$.

$$\therefore gr[G_{S_d}(R)] = 3$$

Case (ii). n is prime.

In this case, R has no non-zero zero divisor and will have only zero ideal. It is adjacent with all other vertices in $G_{S_d}(R)$ and it is a star graph.

Thus $gr[G_{S_d}(R)] = \infty$.

The chromatic number of the S-prime ideal graph $G_{S_d}(R)$ is studied in the following theorem.

Theorem 3.5. *Let $G_{S_d}(R)$ be the S-prime ideal graph of a finite commutative ring R with unity and S_d be an S-prime ideal of R. Then $\chi[G_{S_d}(R)] = m + r$, where $m = |S_d|$ and r is the clique number of $G_{S_d}(R) - S_d$.*

Proof. In a S-prime ideal graph, the elements of S-prime ideal S_d are adjacent with each other and it forms a complete graph K_m with self-loops, $m \geq 1$. Hence there are m distinct colors for the complete graph K_m with self-loops. Therefore, the chromatic number of a graph of ideal elements is m, where m is the number of elements of S_d.

Since every element of the S-prime ideal is adjacent to all the other elements of the graph $G_{S_d}(R)$, the S-prime ideal and non S-prime ideal elements will have different colors.

The graph $G_{S_d}(R) - S_d$ of non S-prime ideal elements will contain a complete graph K_r with self-loops, where $1 \leq r \leq n - m$ and n is the number of elements of R. Hence there are r distinct colors for a complete graph K_r with self-loops. Therefore, the chromatic number of $G_{S_d}(R) - S_d$ is r, where r is the clique number of $G_{S_d}(R) - S_d$.

$\therefore$ Chromatic number of the S-prime ideal graph $G_{S_d}(R)$ is $\chi[G_{S_d}(R)] = m + r$.

Example 2. Let $R = \mathbb{Z}_6$.

The S-prime ideals of $\mathbb{Z}_6$ are S_2, S_3 and S_6 and the corresponding S-prime ideal graphs are shown in Figure 2.

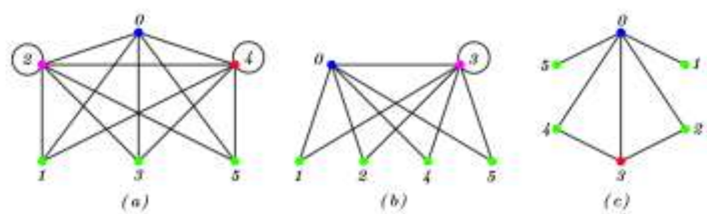

Figure 2. S-prime ideal graph G_{S_d} of $\mathbb{Z}_6$ for (a) S_2 ideal (b) S_3 ideal (c) S_6 ideal.

By Theorem 3.5. the chromatic number of S-prime ideal graph G_{S_d} of $\mathbb{Z}_6$ is

$$\chi[G_{S_d}(R)] = \begin{cases} 4 \text{ if } S_d = \langle 2 \rangle \\ 3 \text{ if } S_d = \langle 3 \rangle \\ 3 \text{ if } S_d = \langle 6 \rangle. \end{cases}$$

3.2 Algebraic properties of *S*-prime ideal graph $G_{S_d}(R)$

In this section, the rank and nullity of the S-prime ideal graph $G_{S_d}(R)$ are investigated.

Theorem 3.6. *Let S_d be the S-prime ideal of a commutative ring R of order n with unity and $G_{S_d}(R)$ be the S-prime ideal graph of R. Then the rank of $G_{S_d}(R)$ is*

$$\rho[G_{S_d}(R)] = \begin{cases} \tau(d)+1 & \text{if } S_d \neq 0 \\ \tau(n) & \text{otherwise} \end{cases}$$

where $\tau(x)$ is the number of divisors of x, $x \in \mathbb{Z}_+$.

Proof. Let $G_{S_d}(R)$ be the S-prime ideal graph of a finite commutative ring R and E be the edge set of the S-prime ideal $G_{S_d}(R)$ and $A = [a_{ij}]$ be an adjacency matrix of $G_{S_d}(R)$ whose entries are

$$a_{ij} = \begin{cases} 1 \text{ if } (u_i, u_j) \in E \\ 0 \text{ otherwise} \end{cases}$$

for all $u_i, u_j \in R, i, j = 1, 2, \ldots, n$.

Now the characteristic equation of the adjacency matrix A is

$$\lambda^n - c_1\lambda^{n-1} + \ldots + (-1)^n c_n = 0.$$

where c_k is the sum of the minors of $(n-k)$ diagonal elements of A.

If u is the unit of R then u is adjacent to all the elements of S_d and is not adjacent to other elements of R.

Since the degree of units is $|S_d|$, there is no pairwise adjacency between the units of R.

In the adjacency matrix of $G_{S_d}(R)$, the rows corresponding to the units of R are identical. This implies that, $|A| = 0$. That is, $c_n = 0$.

So, $\rho[G_{S_d}(R)] < n$.

Case (i). Let S_d be the non-zero ideal of R. It has additive identity element and some non-zero zero divisors of R.

The set $R - S_d$ contains all units and the remaining non-zero zero divisors of R.

Let d_i be the divisors of d where $i = 1, 2, 3, \ldots, \tau(d)$ and choose h such that $d_i h \neq d_j k$ for $i < j$ and $h = 1, 2, 3, \ldots, n$. The rows corresponding to $d_i h$ are identical for $i = 1, 2, \ldots, \tau(d)$.

The different non-zero rows of the adjacency matrix A are as follows:

(i) The row corresponding to the additive identity.

(ii) The rows corresponding to $d_i h$ for $i = 1, 2, 3, \ldots, \tau(d)$.

So the number of different non-zero rows is $\tau(d) + 1$. It gives $\tau(d) + 1$ non-zero eigenvalues.

Case (ii). Let S_d be the zero ideal of R. In this case, $R - S_d$ contains all units and non-zero zero divisors of R.

Let $D_i, i = 1, 2, \ldots, \tau(n)$ be the divisors of n and choose h such that $d_i h \neq d_j k$ for $i < j$ and $h = 1, 2, 3, \ldots, n$. Now the rows corresponding to $D_i h$ are identical for $i = 1, 2, \ldots, \tau(n)$.

The number of different non-zero rows is $\tau(n)$. Hence it gives $\tau(n)$ non-zero eigenvalues.

Therefore, $\rho[G_{S_d}(R)] = \begin{cases} \tau(d) + 1 & \text{if } S_d \neq 0 \\ \tau(n) & \text{otherwise.} \end{cases}$

Corollary 3.7. *The rank of the S-prime ideal graph of a ring of prime order is* 2.

Corollary 3.8. *Let* $G_{S_d}(R)$ *be the S-prime ideal graph of a commutative ring with unity of order* n. *Then the nullity of the* $G_{S_d}(R)$ *is*

$$\eta[G_{S_d}(R)] = \begin{cases} n - \tau(d) - 1 & \textit{if } S_d \neq 0 \\ n - \tau(n) & \textit{otherwise.} \end{cases}$$

4. Conclusion

In this paper, the S-prime ideal graph $G_{S_d}(R)$ is determined and the graph theoretic properties namely diameter, radius, girth, number of edges, chromatic number and the algebraic properties namely rank and nullity are studied and explained with suitable examples.

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