Advances and Applications in Mathematical Sciences
Volume 20, Issue 9, August 2021, Pages 2393-2418

INVESTIGATION OF TORSIONAL WAVE PROPAGATION IN SANDWICHED MAGNETO-POROELASTIC DISSIPATIVE TRANSVERSELY ISOTROPIC MEDIUM

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Abstract

This paper is to investigate the torsional waves in transversely isotropic poroelastic solid. The solid consists of magneto poroelastic dissipative transversely isotropic medium sandwiched between two transversely isotropic poroelastic half-spaces, all are under initial stress. Employing the boundary conditions at the interfaces, frequency equation is obtained. Wave characteristics namely, phase velocity and attenuation coefficient are computed against wavenumber at fixed heterogeneity and magneto-poroelastic coupling. Graphical representations are made to exhibit the results.

2010 Mathematics Subject Classification: 76S05.

Keywords: Magneto-poroelasticity, Torsional waves, Frequency equation, Phase velocity, Attenuation coefficient.

Received June 1, 2020; Accepted April 11, 2021

1. Introduction

In general, the study of torsional waves in magneto-poroelastic isotropic medium has many theoretical and practical applications in several fields such as Seismology, Acoustics, Geophysics, Soil-mechanics, Bio-mechanics, Civil engineering, and Mechanical engineering. Particularly, the Earth contains different layers, heterogeneous in nature, and has significant effect on the propagation of elastic waves. The Earth crust is made of diversity of igneous, metamorphic and sedimentary rocks. These rocks are capable to generate magnetic field due to presence of iron, nickel, and cobalt, etc., in them. Hence, these rocks are the magneto dissipative poroelastic in nature. The theory of linear isotropic poroelasticity was developed by Biot [3]. Studies of torsional vibrations in isotropic poroelastic cylinder are reported in the papers [14-16]. In the paper [1], the analytical solutions for pore pressure and stress fields for inclined borehole and the cylinder, induced Investigation of Torsional Wave Propagation in Sandwiched Magneto-Poroelastic Dissipative Transversely Isotropic Medium by boundary stress perturbation in an anisotropic poroelastic medium are presented. The Shale rock and Berea sandstone exhibit transversely isotropic behavior at low effective stress [8-9]. Employing theory of Poroelasticity, torsional vibrations in composite transversely isotropic poroelastic solid cylinder are investigated by Rani et al., [13]. Because of various processes in artificial structures and natural phenomena in the Earth, initial stresses do present in them. In the paper [4], the authors discussed the propagation of shear waves in an isotropic, visco-elastic, heterogeneous layer lying over a homogeneous half-space under initial stress. Kundu et al., [11] investigated propagation of a torsional surface wave in a non-homogeneous anisotropic layer over a heterogeneous half-space. In the paper [11], it is assumed that homogeneity varies exponentially with depth in layer, and in half-space. Three types of heterogeneities, namely, quadratic, hyperbolic and exponential are assumed. Magneto elastic shear waves in irregular monoclinic layer are studied by Chattopadhyay et al., [6]. In the paper [6], the propagation of horizontally polarized shear waves in an internal magneto elastic monoclinic stratum with irregularity in lower interface is investigated. Shear waves in magneto-elastic transversely isotropic layer bonded between two heterogeneous elastic media is studied by Kundu et al., [12]. Torsional surface wave propagation in anisotropic layer sandwiched between heterogeneous half-space is studied by Vaishnav et al., [17]. In the paper [17], the dispersion relation of torsional surface waves has been obtained in presence of heterogeneity, initial stress, and anisotropic. Shear wave propagation in magneto-elastic medium sandwiched between two elastic half-spaces is reported in several papers [7, 5, 10]. In the present paper, torsional wave propagation in magneto-poroelastic dissipative transversely isotropic medium sandwiched between two transversely isotropic poroelastic half-spaces is investigated.

The rest of the paper is organized as follows: In section 2, formulation and solution of the problems are given. Boundary conditions and frequency equation are presented in section 3. In section 4, numerical results are discussed. Finally, conclusion is given in section 5.

2. Formulation and Solution of the Problem

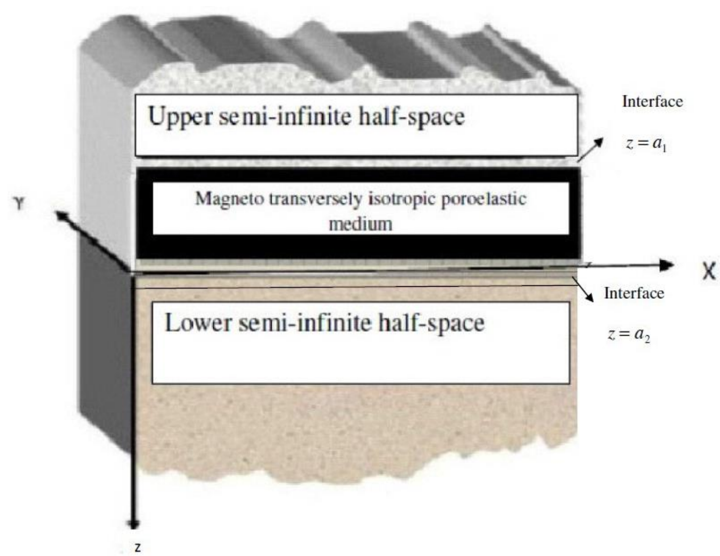

Figure 1. Geometry of the problem.

Consider the magneto transversely isotropic poroelastic medium sandwiched between two heterogeneous, and semi-infinite transversely isotropic poroelastic half-spaces, all assumed to be under initial stress. The constitutive stress-strain relations for transversely isotropic solid [1] are

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{r\theta} \\ \theta_{\theta z} \\ \sigma_{rz} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & 0 & 0 & 0 \\ M_{12} & M_{11} & M_{13} & 0 & 0 & 0 \\ M_{13} & M_{13} & M_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{55} \end{bmatrix} \begin{bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \\ e_{r\theta} \\ e_{\theta z} \\ e_{rz} \end{bmatrix} - \begin{bmatrix} \alpha \\ \alpha \\ \alpha' \\ 0 \\ 0 \\ 0 \end{bmatrix} p, \tag{1}$$

where $p = M[\varepsilon' - \alpha(e_{rr} + e_{\theta\theta}) - \alpha' e_{zz}]$.

In the Equation (1), e_{jj} and e_{jk} are normal and shear strain components, p is the pore pressure, α and α' are Biot's effective stress coefficients in the isotropic plane $(r - \theta$ plane) and in the z-diction, respectively. M is Biot's modulus, ε' is the variation of the fluid content per unit reference volume, and M_{jk} are components of the drained elastic modulus which depend on $E, E', \upsilon, \upsilon', G$ and G'. E and υ are drained Young's modulus and Poisson ratio in the isotropic plane, E' and υ' are similar quantities as that of E and υ pertaining to the direction of the axis of symmetry, G and G' are the shear modulus related to the direction of the isotropic plane and axis of the symmetry, respectively. For given anisotropic ratios of $N_E = \frac{E'}{E}$ and $N_\upsilon = \frac{\upsilon'}{\upsilon}$, E' and υ' can be determined. Different N_E and N_υ ratios define different degrees of anisotropy. For torsional waves, it is convenient to consider the cylindrical polar coordinate system (r, θ, z). The strain-displacement relation in cylindrical system are

$$e_{rr} = \frac{\partial u_i}{\partial r}, \; e_{\theta\theta} = \frac{u_i}{r} + \frac{1}{r}\frac{\partial v_i}{\partial \theta}, \; e_{zz} = \frac{\partial w_i}{\partial z}, \; e_{r\theta} = \frac{1}{2}\left[\frac{\partial u_i}{\partial \theta} + \frac{\partial v_i}{\partial r} - \frac{v_i}{r}\right],$$

$$e_{rz} = \frac{1}{2}\left[\frac{\partial u_i}{\partial z} + \frac{\partial w_i}{\partial r}\right], \; e_{z\theta} = \frac{1}{2}\left[\frac{\partial v_i}{\partial z} + \frac{1}{r}\frac{\partial w_i}{\partial \theta}\right]. \tag{2}$$

The x-axis is taken along the propagation of torsional wave, and z-axis is in downward direction as shown in figure 1. The equations of motion in this case are as follows:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} - p'\frac{\partial \omega_z}{\partial \theta} + p'\frac{\partial \omega_z}{\partial z} = \frac{\partial^2}{\partial t^2}(\rho_{11} u_i + \rho_{12} U_i) + b\frac{\partial}{\partial t}(u_i - U_i)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} - p'\frac{\partial \omega_z}{\partial r} = \frac{\partial^2}{\partial t^2}(\rho_{11} v_i + \rho_{12} V_i) + b\frac{\partial}{\partial t}(v_i - V_i)$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} - p'\frac{\partial \omega_\theta}{\partial r} = \frac{\partial^2}{\partial t^2}(\rho_{11} w_i + \rho_{12} W_i) + b\frac{\partial}{\partial t}(w_i - W_i)$$

$$\frac{\partial s}{\partial r} = \frac{\partial^2}{\partial t^2}(\rho_{12} u_i + \rho_{22} U_i) - b\frac{\partial}{\partial t}(u_i - U_i)$$

$$\frac{\partial s}{\partial \theta} = \frac{\partial^2}{\partial t^2}(\rho_{12} v_i + \rho_{22} V_i) - b\frac{\partial}{\partial t}(v_i - V_i)$$

$$\frac{\partial s}{\partial z} = \frac{\partial^2}{\partial t^2}(\rho_{12} w_i + \rho_{22} W_i) - b\frac{\partial}{\partial t}(w_i - W_i), \tag{3}$$

where, (u_i, v_i, w_i) and (U_i, V_i, W_i) are the displacement components of solid and fluid, respectively. $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}, \sigma_{r\theta}$ and $\sigma_{\theta z}$ are the stress components, ρ_{ij} are mass coefficients, b is the dissipative coefficient, t is time, s is fluid pressure, p' is initial stress, and ω_θ, ω_z are rotational components given by

$$\omega_\theta = \frac{1}{2}\left(\frac{\partial u_i}{\partial z} - \frac{\partial w_i}{\partial r}\right), \; \omega_z = \frac{1}{2}\left(\frac{\partial v_i}{\partial r} - \frac{\partial u_i}{\partial \theta}\right). \tag{4}$$

The solutions for three parts, namely, upper semi-infinite poroelastic half-space, intermediate medium of magneto-poroelastic, and lower semi-infinite poroelastic half-space as shown in Figure 1 are presented separately in the following sub sections.

2.1. Upper semi-infinite poroelastic half-space

Let (u_1, v_1, w_1) and (U_1, V_1, W_1) be the displacement components of solid and fluid parts, respectively, in the upper semi-infinite porous heterogeneous half-space M_1 (say). For the torsional vibrations, $u_1 = w_1 = 0, v_1 = v_1(r, z, t)$, and $U_1 = W_1 = 0, V_1 = V_1(r, z, t)$. In this case, the equations of motion (3) are reduced to the following:

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} - p'\frac{\partial \omega_z}{\partial r} = \frac{\partial^2}{\partial t^2}(\rho_{11}v_1 + \rho_{12}V_1) + \frac{\partial}{\partial t}(v_1 - V_1),$$

$$0 = \frac{\partial^2}{\partial t^2}(\rho_{12}v_1 + \rho_{22}V_1) - b\frac{\partial}{\partial t}(v_1 - V_1). \tag{5}$$

As the half-space M_1 under consideration is heterogeneous, the following variations in densities and initial stress with z coordinate are considered.

$$\rho_{11} = \rho_{110}\left(1 + \frac{z}{\xi_1}\right)^2, \ \rho_{12} = \rho_{120}\left(1 + \frac{z}{\xi_1}\right)^2, \ \rho_{22} = \rho_{220}\left(1 + \frac{z}{\xi_1}\right)^2,$$

$$p' = p'_{100}\left(1 + \frac{z}{\xi_1}\right)^2, \tag{6}$$

where $\rho_{110}, \rho_{120}, \rho_{220}$ are initial values of mass coefficients, and p'_{100} is initial value of initial stress. Substitution of Equation (3) and Equation (6) in Equation (5) gives the following equations:

$$2\left(M_{44} - p'_{100}\left(1 + \frac{z}{\xi_1}\right)^2\right)\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}(M_{44} - 1)\frac{\partial v_1}{\partial r} - \frac{M_{44}}{r^2}v_1 + \frac{M_{55}}{2}\frac{\partial^2 v_1}{\partial z^2}$$

$$= \rho_{110}\left(1 + \frac{z}{\xi_1}\right)^2\frac{\partial^2 v_1}{\partial t^2} + \rho_{120}\left(1 + \frac{z}{\xi_1}\right)^2\frac{\partial^2 V_1}{\partial t^2} + b\left(\frac{\partial v_1}{\partial t} - \frac{\partial V_1}{\partial t}\right),$$

$$0 = \rho_{120}\left(1 + \frac{z}{\xi_1}\right)^2\frac{\partial^2 v_1}{\partial t^2} + \rho_{220}\left(1 + \frac{z}{\xi_1}\right)^2\frac{\partial^2 V_1}{\partial t^2} - b\left(\frac{\partial v_1}{\partial t} - \frac{\partial V_1}{\partial t}\right). \tag{7}$$

For harmonic torsionsl waves, the displacement components can be taken as follows:

$$v_1(r, z, t) = f_1(z)e^{ik(z-ct)}, \ V_1(r, z, t) = F_1(r)e^{ik(z-ct)}. \tag{8}$$

In Equation (8), k is wavenumber, t is time and i is complex unity. Substitution of Equation (8) in Equation (7) gives

$$2\left(M_{44} - p'_{100}\left(1+\frac{z}{\xi_1}\right)^2\right)f''_1(r) + \frac{1}{r}(M_{44}-1)f'_1(r)$$

$$-\left(\frac{M_{44}}{r} + \frac{M_{55}}{2}k^2 - \rho_{110}k^2c^2\left(1+\frac{z}{\xi_1}\right)^2 - bikc\right)$$

$$f_1(r) - \left(bikc - \rho_{120}k^2c^2\left(1+\frac{z}{\xi_1}\right)^2\right)F_1(r) = 0,$$

$$\left(\rho_{120}(kc)^2\left(1+\frac{z}{\xi_1}\right)^2 - ibkc\right)f_1(r) + \left(\rho_{220}(kc)^2\left(1+\frac{z}{\xi_1}\right)^2 + ibkc\right)F_1(z) = 0. \quad (9)$$

The Equation (9) can be re-written as

$$\frac{d^2f_1}{dr^2} + L\frac{df_1}{dr} + q^1f_1 = 0. \quad (10)$$

Solution of Equation (10) is

$$f_1(r) = e^{\frac{-1x}{2}}(C_1\cos(q_1r) + C_2\sin(q_1r)). \quad (11)$$

In Equation (11), C_1 and C_2 are arbitrary constants, and

$$q_1 = \frac{\sqrt{L^2 - 4q^1}}{2},\ L = (M_{44}-1)\left(2r\left(M_{44} - p'_{100}\left(1+\frac{z}{\xi_1}\right)^2\right)\right)^{-1}.$$

$$q^1 = \frac{1}{2}\left(M_{44} - p'_{100}\left(1+\frac{z}{\xi_1}\right)^2\right)^{-1}\rho_{110}\,\omega^2\left(1+\frac{z}{\xi_1}\right)^2 + bi\,\omega - \frac{M_{44}}{r} - \frac{M_{55}k^3}{2}$$

$$-\left(\rho_{120}\,\omega^2\left(1+\frac{z}{\xi_1}\right)^2 - bi\,\omega\right)\left(\left(\rho_{120}\,\omega\left(1+\frac{z}{\xi_1}\right)^2 - ib\right)\right)\left(\rho_{220}\,\omega\left(1+\frac{z}{\xi_1}\right)^2 + ib\right)^{-1}.$$

Substitution of Equation (11) in Equation (8) gives

$$F_1(r) = -G_1G_2^{-1}f_1(r), \quad (12)$$

where, $G_1 = -\rho_{120}\,\omega^2\left(1+\frac{z}{\xi_1}\right)^2 + ib\omega$, and $G_2 = \rho_{220}\,\omega^2\left(1+\frac{z}{\xi_1}\right)^2 + ib\omega$.

Substituting Equation (11) in Equation (8), and then using stress-displacement relations, the following stress components in this half-space are obtained:

$$(\sigma_{r\theta})_{M_1} = -\frac{M_{44}}{2} e^{\frac{-Lr}{2}}\left[C_1\left(\frac{1}{r}\cos(q_1 r) - \frac{L_1 q_1}{2}\right)\sin(q_1 r)\right.$$

$$\left.+ C_2\left(\frac{L_1 q_1}{2}\cos(q_1 r) + \frac{1}{r}\sin(q_1 r)\right)\right] e^{ik(z-ct)},$$

$$(\sigma_{\theta z})_{M_1} = \frac{M_{55}}{2} ik\, e^{\frac{-Lr}{2}}[C_1\cos(q_1 r) + C_2\sin(q_1 r)]e^{ik(z-ct)}. \tag{13}$$

2.2. Sandwiched magneto poroelastic medium

Let (u_2, v_2, w_2) and (U_2, V_2, W_2) be the solid and fluid displacement components, respectively, in the sandwiched magneto transversely isotropic poroelastic medium M_2 (say). For torsional waves, Equation (3) can be written as

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} - p'\frac{\partial \omega_t}{\partial r} + (\vec{J}\times\vec{B})_\theta = \frac{\partial^2}{\partial t^2}(\rho_{11}v_2 + \rho_{12}V_2) + b\frac{\partial}{\partial t}(v_2 - V_2),$$

$$0 = \frac{\partial^2}{\partial t^2}(\rho_{12}v_2 + \rho_{22}V_2) - b\frac{\partial}{\partial t}(v_2 - V_2). \tag{14}$$

In Equation (14), $(\vec{J}\times\vec{B})_\theta$ is the θ-component of electromagnetic force, $\vec{J}$ is the electric current density and $\vec{B}$ the magnetic induction vector. The Maxwell's equations of the electromagnetic field are

$$\vec{\nabla}\cdot\vec{B} = 0,\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t},\ \vec{\nabla}\times\vec{H} = \vec{J},$$

$$\vec{B} = \mu_e\vec{H},\ \vec{J} = \sigma\left(\vec{E} + \frac{\partial v_2}{\partial t}\times\vec{B}\right), \tag{15}$$

here, $\vec{E}$ is the induced electric field, $\vec{H}$ is the magnetic field consisting both primary and induced magnetic field, μ_e and σ are the induced permeability, and conduction coefficient, respectively. The Maxwell stress tensor $(\tau_{ij}^0)^{Mx} = \mu_e(H_i p_i - H_j p_i - H_k p_k \delta_{ij})$. Let $\vec{H} = (H_r, H_\theta, H_z)$ and change in magnetic field be (p_1, p_2, p_3). If the displacement current is absent. With the help of Equation (15), the following equation is obtained.

$$\nabla^2 H = \mu_e \sigma \left(\frac{\partial \vec{H}}{\partial t} \bar{\nabla} \times \left(\frac{\partial \vec{v}_2}{\partial t} \times \vec{H} \right) \right). \tag{16}$$

The component form of Equation (16) can be written as follows:

$$\frac{\partial H_r}{\partial t} = \frac{1}{\mu_e \sigma} \nabla^2 H_r, \ \frac{\partial H_z}{\partial t} = \frac{1}{\mu_e \sigma} \nabla^2 H_z, \tag{17}$$

and

$$\frac{\partial H_\theta}{\partial t} = \frac{1}{\mu_e \sigma} \nabla^2 H_\theta + \frac{\partial}{\partial r}\left(H_r \frac{\partial v_2}{\partial t} \right) + \frac{\partial}{\partial z}\left(H_z \frac{\partial v_2}{\partial t} \right). \tag{18}$$

If the medium is perfectly conductor (i.e., $\sigma \to \infty$), the equations (17) and (18) reduce to

$$\frac{\partial H_r}{\partial t} = \frac{\partial H_z}{\partial t} = 0, \ \frac{\partial H_\theta}{\partial t} = \frac{\partial}{\partial r}\left(H_r \frac{\partial v_2}{\partial t} \right) + \frac{\partial}{\partial z}\left(H_z \frac{\partial v_2}{\partial t} \right). \tag{19}$$

Here, it is assumed that the primary magnetic field is uniform throughout the space. It is clear from Equation (19) that there is no perturbation in H_r and H_z, but there is perturbation in H_θ. If p_2 is perturbation in, H_θ, then $H_r = H_{01}$, $H_\theta = H_{02} + p_2$ and $H_z = H_{03}$, where H_{01}, H_{02}, H_{03} are components of the initial magnetic field $\vec{H}_0$. Since $(H_0 \cos \phi, p_2, H_0 \sin \phi)$, where $H_0 = |\vec{H}_0|$ and, ϕ is the angle at which wave crosses the magnetic field. Thus, $\vec{H}$ can be expressed as

$$\vec{H} = (H_0 \cos \phi, p_2, H_0 \sin \phi). \tag{20}$$

Here, it is assumed that p_2 is zero initially. Using Equation (20) in Equation (19), one obtains

$$\frac{\partial p_2}{\partial t} = \frac{\partial}{\partial r}\left(H_0 \cos\phi \frac{\partial v_2}{\partial t}\right) + \frac{\partial}{\partial z}\left(H_0 \sin\phi \frac{\partial v_2}{\partial t}\right). \tag{21}$$

The above equation gives

$$p_2 = H_0 \cos\phi \frac{\partial v_2}{\partial r} + H_0 \sin\phi \frac{\partial_2}{\partial z}. \tag{22}$$

The relation, $\nabla\left(\frac{H^2}{2}\right) = -(\vec{\nabla} \times \vec{H}) \times \vec{H} + (\vec{H} \cdot \vec{\nabla}) \cdot \vec{H}$ and Equation (15) give the electromagnetic force as given below:

$$(\vec{J} \times \vec{B}) = \mu_e\left(-\vec{\nabla}\left(\frac{H^2}{2}\right) + (\vec{H} \cdot \vec{\nabla})\vec{H}\right). \tag{23}$$

In this case, the components of $\vec{J} \times \vec{B}$ are $(\vec{J} \times \vec{B})_r = 0$, $(\vec{J} \times \vec{B})_z = 0$, and

$$(\vec{J} \times \vec{B})_\theta = \mu_e H_0^2\left(\sin^2\phi \frac{\partial^2 v_2}{\partial z^2} + \sin 2\phi \frac{\partial^2 v_2}{\partial r \partial z} + \cos^2\phi \frac{\partial^2 v_2}{\partial r^2}\right). \tag{24}$$

Substitution of Equation (24) in Equation (14) gives

$$\left(2\left(M_{44} - p'_{100}\left(1 + \frac{z}{\xi_1}\right)^2\right) + \mu_e H_0^2 \cos^2\phi\right)\frac{\partial^2 v_2}{\partial r^2} + \frac{1}{r}(M_{44} - 1)\frac{\partial v_2}{\partial r}$$

$$+ \mu_e H_0^2 \sin 2\phi \frac{\partial^2 v_2}{\partial r \partial z} - \frac{M_{44}}{r^2} v_2 + \left(\frac{M_{55}}{2} + \mu_e H_0^2 \sin^2\phi\right)\frac{\partial^2 v_2}{\partial z^2}$$

$$= \rho_{110}\left(1 + \frac{z}{\xi_1}\right)^2 \frac{\partial^2 v_2}{\partial t^2} + \rho_{120}\left(1 + \frac{z}{\xi_1}\right)^2 \frac{\partial^2 V_2}{\partial t^2} + b_1\left(\frac{\partial v_2}{\partial t} - \frac{\partial V_2}{\partial t}\right),$$

$$0 = \rho_{120}\left(1 + \frac{z}{\xi_1}\right)^2 \frac{\partial^2 v_2}{\partial t^2} + \rho_{220}\left(1 + \frac{z}{\xi_1}\right)^2 \frac{\partial^2 V_2}{\partial t^2} - b_1\left(\frac{\partial v_2}{\partial t} - \frac{\partial V_2}{\partial t}\right). \tag{25}$$

For torsional harmonic waves, the displacement components can be expressed as

$$v_2(r, z, t) = f_2(r)e^{ik(z-ct)}, \; V_2(r, z, t) = F_2(r)e^{ik(z-ct)}. \tag{26}$$

Substitution of Equation (26) in Equation (25) gives

$$\frac{d^2 f_2}{dr^2} + \frac{x_2}{x_1}\frac{df_2}{dr} - \frac{q^2}{x_1} f_2 = 0. \tag{27}$$

The solution of Equation (27) is

$$f_2(r) = e^{-\frac{x_2 r}{2x_1}}(C_3 \cos(q_2 r) + C_4 \sin(q_2 r)). \tag{28}$$

In Equation (28), C_3 and C_4 are arbitrary constants,

where,

$$q_2 = \frac{\sqrt{x_2^2 x_1^{-2} + 4q^2 x_1^{-1}}}{2}, \; x_1 = 2\left(M_{44} - p'_{100}\left(1 + \frac{z}{\xi_1}\right)^2 + \mu_e H_0^2 \cos^2\phi\right),$$

$$x_2 = \frac{1}{r}(M_{44} - 1) + ik\mu_e H_0^2 \sin 2\phi,$$

$$q^2 = \frac{M_{44}}{r^2} + \frac{M_{55}k^2}{2} + \mu_e H_0^2 k^2 \sin^2\phi - \rho_{110}\,\omega^2\left(1 + \frac{z}{\xi_1}\right)^2 - bi\,\omega$$

$$- \left(\rho_{120}\,\omega^2\left(1 + \frac{z}{\xi_1}\right)^2 + bi\,\omega\right)\left(\rho_{120}\,\omega\left(1 + \frac{z}{\xi_1}\right)^2 - ib\right)\left(\rho_{220}\,\omega\left(1 + \frac{z}{\xi_1}\right)^2 + ib\right)^{-1}.$$

Using Equations (25), (26), and (28) the following equation is obtained.

$$F_2(r) = -G_{12}G_{22}^{-1} f_2(r), \tag{29}$$

where, $G_{12} = \rho_{120} + ib\,\omega^{-1}$ and $G_{22} = \rho_{220} - ib\,\omega^{-1}$. Substitution of Equation (28) in Equation (26), gives

$$(\sigma_{r\theta})_{M_2} = -\frac{M_{44}}{2} e^{\frac{-x_2 r}{2x_1}}\left[C_3\left(\frac{1}{2}\cos(q_2 r) - \frac{x_2 q_2}{2x_1}\right)\sin(q_2 r)\right.$$

$$\left. + C_4\left(\frac{x_2 q_2}{2x_1}\cos(q_2 r) + \frac{1}{r}\sin(q_2 r)\right)\right] e^{ik(z-ct)},$$

$$(\sigma_{\theta z})_{M_2} = \frac{M_{55}}{2} ike^{-\frac{x_2 r}{2x_1}}[C_3 \cos(q_2 r) + C_4 \sin(q_2 r)]e^{ik(z-ct)}. \tag{30}$$

2.3. Lower inhomogeneous poroelastic half-space

In the lower heterogeneous poroelastic half-space M_3 (say), let (u_3, v_3, w_3) and (U_3, V_3, W_3) be the displacement components of solid and fluid, respectively. For torsional wave, the equation of motion is reduced to

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} - p'\frac{\partial \omega_z}{\partial r} = \frac{\partial^2}{\partial t^2}(\rho_{11}, v_3 + \rho_{12}V_3) + b\frac{\partial}{\partial t}(v_3 - V_3),$$

$$0 = \frac{\partial^2}{\partial t^2}(\rho_{12}v_3 + \rho_{22}V_3) - b\frac{\partial}{\partial t}(v_3 - V_3). \tag{31}$$

As the half-space is heterogeneous, the mass coefficients, and initial stress are assumed as follows:

$$\rho_{11} = \rho_{110}(1 + nz), \rho_{12} = \rho_{120}(1 + nz), \rho_{22} = \rho_{220}(1 + nz)$$

and

$$p' = p'_{100}(1 + p^* z), \tag{32}$$

where, p'_{100} is initial value of the initial stress, and n, p^* are constants. Substitution of Equations (3) and (32) in Equation (31) gives,

$$2(M_{44} - p'_{100}(1 + p^* z))\frac{\partial^2 v_3}{\partial r^2} + \frac{1}{r}(M_{44} - 1)\frac{\partial v_3}{\partial r} - \frac{M_{44}}{r^2}u_3 + \frac{M_{55}}{2}\frac{\partial^2 v_3}{\partial z^2}$$

$$= \rho_{110}(1 + nz)\frac{\partial^2 v_3}{\partial t^2} + \rho_{120}(1 + nz)\frac{\partial^2 V_3}{\partial t^2} + b\left(\frac{\partial v_3}{\partial t} - \frac{\partial V_3}{\partial t}\right),$$

$$0 = \rho_{120}(1 + nz)\frac{\partial^2 v_3}{\partial t^2} + \rho_{220}(1 + nz)\frac{\partial^2 V_3}{\partial t^2} - b\left(\frac{\partial v_3}{\partial t} - \frac{\partial V_3}{\partial t}\right). \tag{33}$$

For torsional harmonic waves, the displacement components in lower semi-infinite porous half-space are taken as follows:

$$v_3(r, z, t) = f_3(r)e^{ik(z-ct)}, V_3(r, z, t) = F_3(r)e^{ik(z-ct)}. \tag{34}$$

Substitution of Equation (34) in Equation (33) gives

$$\frac{d^2 f_3}{dr^2} + \frac{x_4}{x_3}\frac{df_3}{dr} - \frac{q^3}{x_3} f_3 = 0. \tag{35}$$

The solution of Equation (35) is

$$f_3(r) = e^{\frac{-x_4 r}{2x_3}} (C_5 \cos(q_3 r) + C_6 \sin(q_3 r)). \tag{36}$$

In Equation (36), C_5 and C_6 are arbitrary constants, and

$$q_3 = \frac{\sqrt{(x_4 x_3^{-1}) + 4q^3 x_3^{-1}}}{2}, \; x_3 = 2(M_{44} - p'_{100}(1 + p^* z)), \; x_4 = r^{-1}(M_{44} - 1).$$

$$q^3 = \frac{M_{44}}{r^2} + \frac{M_{55} k^2}{2} - \rho_{110}\omega^2(1 + nz) - bi\omega - (\rho_{120}\omega^2(1 + nz)$$

$$- bi\omega)\left(\frac{\rho_{120}\omega(1 + nz) - ib}{\rho_{220}\omega(1 + nz) + ib}\right).$$

After long calculation, the following equation is obtained:

$$F_3(r) = -D_{12} D_{22}^{-1} f_3(r), \tag{37}$$

where, $D_{12} = \rho_{120}(1 + nz) - ib\omega^{-1}$ and $D_{22} = \rho_{220}(1 + nz) + ib\omega^{-1}$. Substitution of Equation (36) in Equation (33), and using stress-displacement relations, the following stress components are obtained.

$$(\sigma_{r\theta})_{M_3} = -\frac{M_{44}}{2} e^{\frac{-x_4 r}{2x_3}}\left[C_5\left(\frac{1}{2}\cos(q_3 r) - \frac{x_4 q_3}{2x_3}\right)\sin(q_3 r)\right.$$

$$\left. + C_6\left(\frac{x_4 q_3}{2x_3}\cos(q_3 r) + \frac{1}{r}\sin(q_3 r)\right)\right] e^{ik(z-ct)},$$

$$(\sigma_{\theta z})_{M_3} = \frac{M_{55}}{2} ike^{-\frac{x_4 r}{2x_3}}[C_5 \cos(q_3 r) + C_6 \sin(q_3 r)] e^{ik(z-ct)}. \tag{38}$$

3. Boundary Conditions and Frequency Equation

The displacement components and stresses are assumed to be continuous at $z = a_1$, the interface between upper half-space (M_1) and intermediate medium (M_2). That is,

$$(v_1)_{M_1} = (v_2)_{M_2}$$

$$(\sigma_{r\theta})_{M_1} = (\sigma_{r\theta})_{M_2},$$

$$(\sigma_{\theta z})_{M_1} = (\sigma_{\theta z})_{M_2}. \quad (39)$$

The displacement components and stresses are assumed to be continuous at the lower interface $z = a_2$. That is,

$$(v_2)_{M_2} = (v_3)_{M_3}$$

$$(\sigma_{r\theta})_{M_2} = (\sigma_{r\theta})_{M_3},$$

$$(\sigma_{\theta z})_{M_2} = (\sigma_{\theta z})_{M_3}. \quad (40)$$

The above boundary conditions lead to the following system of homogeneous equations:

$$\lfloor A_{ij} \rfloor \lfloor C_j \rfloor = [0], \; (i, \; j = 1, 2, 3, 4, 5, 6), \quad (41)$$

where,

$$A_{11} = \frac{M_{44}}{2} e^{-\frac{La_1}{2}} \cos (q_1 a_1), \; A_{13} = \frac{M_{44}}{2} e^{-\frac{x_2 a_1}{2 x_1}} \cos (q_2 a_1),$$

$$A_{21} = \frac{M_{44}}{2} e^{-\frac{-La_1}{2}} \left(\frac{1}{a_1} \cos (q_1 a_1) - \frac{La_1}{2} \sin (q_1 a_1) \right),$$

$$A_{23} = \frac{M_{44}}{2} e^{-\frac{-x_2 a_1}{2 x_1}} \left(\frac{1}{a_1} \cos (q_1 a_1) - \frac{x_2 q_1}{2 x_1} \sin (q_1 a_1) \right),$$

$$A_{31} = \frac{M_{55}}{2} e^{-\frac{-La_1}{2}} \cos (q_1 a_1),$$

$$A_{15} = A_{16} = A_{25} = A_{26} = A_{35} = A_{36} = A_{41} = A_{42} = A_{51} = A_{52} = A_{61} = A_{62} = 0,$$

$A_{12}, A_{14}, A_{22}, A_{32}, A_{34}$ = Similar expression as $A_{11}, A_{13}, A_{21}, A_{33}$ with cos integer replaced by sin integer, respectively,

A_{24} = Similar expression as A_{23} with cos integer replaced by sin integer, respectively,

A_{33}, A_{43} = Similar expression as A_{11} with $e^{-\frac{-La_1}{2}}$ and q_1 replaced by $\frac{M_{55}}{2}$; $e^{-\frac{x_2 a_1}{2x_1}}$ and q_2, respectively,

A_{44} = Similar expression as A_{24} with cos integer and q_1 replaced bysin integer and q_2, respectively,

$A_{45}, A_{46}, A_{55}, A_{56}$ = Similar expression as $A_{43}, A_{44}, A_{53}, A_{54}$ with x_1, x_2, q_2 replaced by x_3, x_4, q_5, respectively,

A_{53}, A_{54} = Similar expression as A_{23}, A_{24} with a_1 replaced by a_2, respectively,

A_{63}, A_{64} = Similar expression as A_{11}, A_{12} with q_1 and $e^{-\frac{-La_1}{2}}$ replaced by q_2 and $\frac{M_{55}}{2}$, respectively,

A_{65}, A_{66} = Similar expression as A_{63}, A_{64} with q_2 replaced by q_3 respectively,

The equation (41) results in a system of six homogeneous equations in six arbitrarily constants C_1, C_2, C_3, C_4, C_5 and C_6. For a non-trivial solution, determinant of coefficients is zero. Accordingly, the following complex valued frequency equation is obtained:

$$|a_{ij}| + i|a'_{ij}| = 0, \ (i, j = 1, 2, 3, 4, 5, 6). \tag{42}$$

The elements a_{ij} and a'_{ij} in Equation (42) are given in Appendix-A.

4. Numerical Results

For numerical process, the following materials are used. Both upper and lower semi-infinite porous half-spaces are Berea sandstone saturated with water [9] (say) Mat-I. The parameter values of the said material are as follows:

$$\rho_{11} = 2407.64 \text{ kg}/\text{m}^3, \ \rho_{12} = -266 \text{ kg}/\text{m}^3, \ \rho_{22} = 456 \text{ kg}/\text{m}^3.$$

Intermediate layer is magneto poroelastic medium is Shale Rock [8] Mat-II. The parameter values of the said material are as follows:

$$\rho_{11} = 1398.72 \text{ kg}/\text{m}^3, \ \rho_{12} = -257.28 \text{ kg}/\text{m}^3, \ \rho_{22} = 771.84 \text{ kg}/\text{m}^3.$$

Employing these values in frequency equation, the phase velocity, attenuation coefficient, against wavenumber are computed when the anisotropic ratio $N_E = 1$ at various N_V values $N_V = 1$, at various N_E values, and at an arbitrary chosen initial stress. The material constants M_{44}, M_{55} involves E, E', v and v'. The values of Young's modulus (E), Poisson's ratio (v) for Berea sandstone saturated with water and Shale rock are taken to be 14.4Gpa,0.20 and 1.854Gpa,0.22 as suggested in the paper [9, 8]. The heterogeneous parameters $\left(\frac{n}{k}, \frac{p}{k}, \frac{1}{k\xi_1} = 0.01\right)$ and magneto-poroelastic coupling factor are taken to be $\left(\frac{\mu_e, H_0^2}{N} = 0.01\right)$ [2]. The attenuation coefficient (Q^{-1}) is $Q^{-1} = \frac{2 \operatorname{Im}(\omega)}{\operatorname{Re}(\omega)}$. the formula, $\operatorname{Im}(\omega)$ is frequency of imaginary part in frequency equation and $\operatorname{Re}(\omega)$ is frequency of real part in frequency equation. The values are computed using the bisection method implemented in MATLAB, and the results are depicted in Figures 2-5. Figures. 2-5 depict the variation of phase velocity and attenuation coefficient against wavenumber at fixed heterogeneous and magneto poroelastic medium in Mat-I and Mat-II, respectively. From figure 2, it is clear that attenuation coefficient values are greater than that of phase velocity. In Figures 3-5, it is seen that the phase velocity, in general, lower than that of attenuation coefficient.

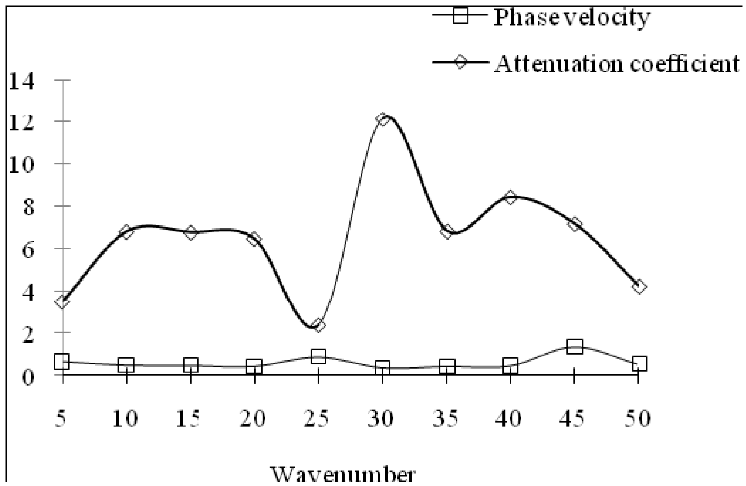

Figure 2. Variation of phase velocity and attenuation coefficient against wavenumber of interface between upper half-space and intermediate medium at Mat-I.

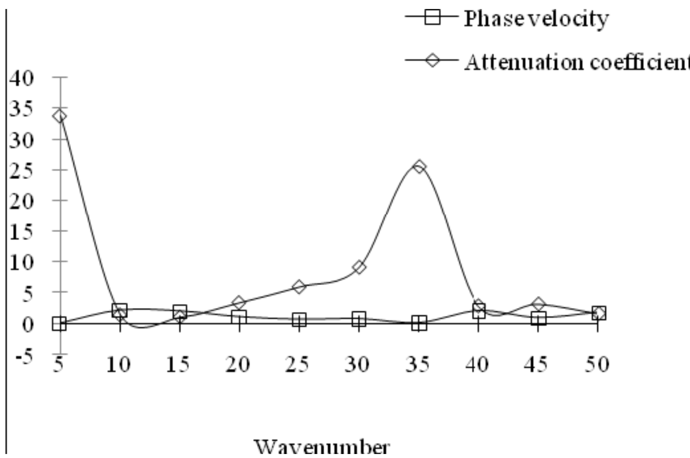

Figure 3. Variation of phase velocity and attenuation coefficient against wavenumber of interface between upper half-space and intermediate medium at Mat-II.

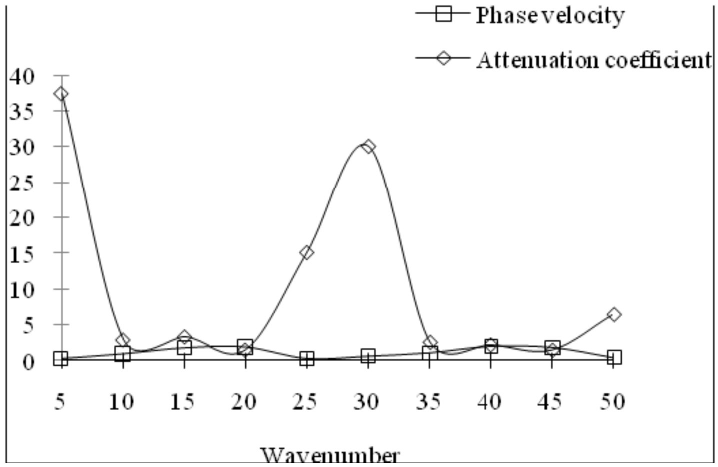

Figure 4. Variation of phase velocity and attenuation coefficient against wavenumber of interface between intermediate medium and lower half-space at Mat-I.

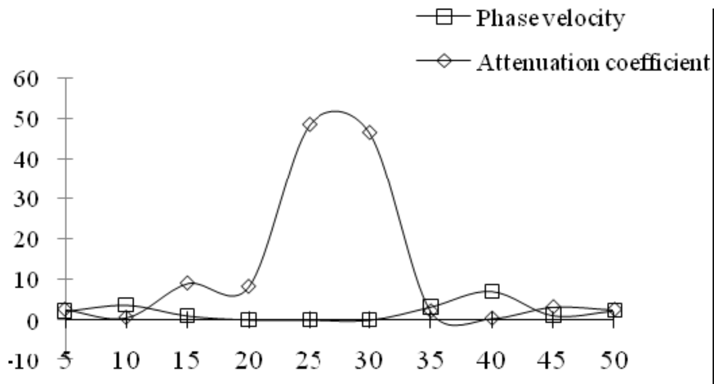

Figure 5. Variation of phase velocity and attenuation coefficient against wavenumber of interface between intermediate medium and lower half-space at Mat-II.

5. Conclusion

Torsional wave propagation in magneto-poroelastic dissipative transversely isotropic medium sandwiched between two transversely isotropic poroelastic half-spaces all under initial stress is investigated in the framework of Biot's theory. Employing the boundary conditions at the interfaces, frequency equation is obtained. Frequency equation and attenuation coefficient are computed against wavenumber at the fixed heterogeneous and magneto-poroelastic coupling factor. From numerical results, it is seen that the phase velocity, in general, less than that of attenuation coefficient.

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Appendix-A

$$a_{11} = \frac{M_{44}}{2} e^{-\frac{L}{2}} (R_1 a_1 \cos T_1 \sin (R_1 a_1 \cos T_1) \cosh (R_1 a_1 \sin T_1)$$

$$- R_1 a_1 \sin T_1 \cos (R_1 a_1 \cos T_1) \sinh (R_1 a_1 \sin T_1))$$

$$a_{12} = \frac{M_{44}}{2} e^{-\frac{L}{2}} (R_1 a_1 \cos T_1 \cos (R_1 a_1 \cos T_1) \cosh (R_1 a_1 \sin T_1)$$

$$+ R_1 a_1 \sin T_1 \sin (R_1 a_1 \cos T_1) \sinh (R_1 a_1 \sin T_1))$$

$$a_{21} = \frac{M_{55}}{2} e^{-\frac{L}{2}} \cos (R_1 a_1 \cos T_1) \cosh (R_1 a_1 \sin T_1);$$

$$a_{22} = \frac{M_{55}}{2} e^{-\frac{L}{2}} \sin (R_1 a_1 \cos T_1) \cosh (R_1 a_1 \sin T_1);$$

$$a_{23} = -\frac{M_{44}}{2} (x_{11} \cos (R_2 a_1 \cos T_2) \cosh (R_2 a_2 \sin T_2)$$

$$+ x_{12} \sin (R_2 a_1 \cos T_2) \sinh (R_2 a_1 \sin T_2));$$

$$a_{24} = \frac{M_{55}}{2} (x_{11} \sin (R_2 a_1 \cos T_2) \cosh (R_2 a_1 \sin T_2)$$

$$-x_{12} \cos(R_2 a_1 \cos T_2) \sinh(R_2 a_1 \cos T_2));$$

$$a_{31} = e^{-\frac{L}{2}} \cos(R_1 a_1 \cos T_1) \cosh(R_1 a_1 \sin T_1);$$

$$a_{32} = e^{-\frac{L}{2}} \sin(R_1 a_1 \cos T_1) \cosh(R_1 a_1 \sin T_1);$$

$$a_{33} = -e^{\frac{(1-M_{44})}{x_1}} (x_{11} \cos(R_2 a_1 \cos T_2) \cosh(R_2 a_1 \sin T_2)$$

$$+x_{12} \sin(R_2 a_1 \cos T_2) \sinh(R_2 a_1 \cos T_2))$$

$$a_{34} = -e^{\frac{(1-M_{44})}{x_1}} (x_{12} \cos(R_2 a_1 \cos T_2) \sinh(R_2 a_1 \sin T_2)$$

$$-x_{11} \sin(R_2 a_1 \cos T_2) \cosh(R_2 a_1 \sin T_2))$$

$$a_{45} = -\frac{M_{44}}{2} \frac{x_4 a_2}{2x_3} e^{-\frac{x_4 a_2}{2x_3}} (R_3 a_2 \cos T_3 \cos(R_3 a_2 \cos T_3) \cosh(R_3 a_2 \sin T_3)$$

$$+R_3 a_2 \sin T_3 \sin(R_3 a_2 \cos T_3) \sinh(R_3 a_2 \sin T_3))$$

$$a_{46} = -\frac{M_{44}}{2} \frac{x_4 a_2}{2x_3} e^{-\frac{x_4 a_2}{2x_3}} (R_3 a_2 \cos T_3 \sin(R_3 a_2 \cos T_3) \cosh(R_3 a_2 \sin T_3)$$

$$-R_3 a_2 \sin T_3 \cos(R_3 a_2 \cos T_3) \sinh(R_3 a_2 \sin T_3))$$

$$a_{55} = \frac{M_{55}}{2} e^{-\frac{x_4 a_2}{2x_3}} \cos(R_3 a_2 \cos T_3) \cosh(R_3 a_2 \sin T_3);$$

$$a_{56} = -\frac{M_{55}}{2} e^{-\frac{x_4 a_2}{2x_3}} \sin(R_3 a_2 \cos T_3) \cosh(R_3 a_2 \cos T_3);$$

$$a_{63} = e^{\frac{(1-M_{44})}{x_1}} (x_{13} \cos(R_2 a_2 \cos T_2) \cosh(R_2 a_2 \sin T_2)$$

$$+x_{14} \sin(R_2 a_2 \cos T_2) \sinh(R_2 a_2 \sin T_2))$$

$$a_{64} = e^{\frac{(1-M_{44})}{x_1}} (x_{13} \sin(R_2 a_2 \cos T_2) \cosh(R_2 a_2 \sin T_2)$$

$$-x_{14}\cos(R_2a_2\cos T_2)\sinh(R_2a_2\sin T_2))$$

$$a_{65} = -e^{-\frac{x_4a_2}{2x_3}}\cos(R_3a_2\cos T_3)\cosh(R_3a_2\cos T_3);$$

$$a_{66} = -e^{-\frac{x_4a_2}{2x_3}}\sin(R_3a_2\cos T_3)\cosh(R_3a_2\cos T_3);$$

$$a'_{11} = \frac{M_{44}}{2}e^{-\frac{L}{2}}(R_1a_1\cos T_1\cos(R_1a_1\cos T_1)\sinh(R_1a_1\sin T_1)$$

$$+R_1a_1\sin T_1\sin(R_1a_1\cos T_1)\cosh(R_1a_1\sin T_1))$$

$$a'_{12} = \frac{M_{44}}{2}e^{-\frac{L}{2}}(R_1a_1\sin T_1\cos(R_1a_1\cos T_1)\cosh(R_1a_1\sin T_1)$$

$$-R_1a_1\cos T_1\sin(R_1a_1\cos T_1)\sinh(R_1a_1\sin T_1))$$

$$a'_{31} = e^{-\frac{L}{2}}\sin(R_1a_1\cos T_1)\sinh(R_1a_1\sin T_1);$$

$$a'_{32} = e^{-\frac{L}{2}}\cos(R_1a_1\cos T_1)\cosh(R_1a_1\sin T_1);$$

$$a'_{45} = -\frac{M_{44}}{2}\frac{x_4a_2}{2x_3}e^{-\frac{x_4a_2}{2x_3}}(R_3a_2\sin T_3\cos(R_3a_2\cos T_3)\cosh(R_3a_2\sin T_3)$$

$$-R_3a_2\sin T_3\sin(R_3a_2\cos T_3)\sinh(R_3a_2\sin T_3))$$

$$a'_{46} = -\frac{M_{44}}{2}\frac{x_4a_2}{2x_3}e^{-\frac{x_4a_2}{2x_3}}(R_3a_2\cos T_3\cos(R_3a_2\cos T_3)\sinh(R_3a_2\sin T_3)$$

$$+R_3a_2\sin T_3\sin(R_3a_2\cos T_3)\cosh(R_3a_2\cos T_3))$$

$$a_{15} = a_{16} = a_{25} = a_{26} = a_{35} = a_{36} = a_{41} = a_{42} = a_{51} = a_{52} = a_{61} = a_{62}$$

$$= a'_{15} = a'_{16} = a'_{25} = a'_{26} = a'_{35} = a'_{36} = a'_{41} = a'_{42} = a'_{51}$$

$$= a'_{52} = a'_{61} = a'_{62} = 0,$$

$$L = (M_{44} - 1)\left\{2\left(M_{44} - p'_{100}\left(1 + \frac{z}{\xi_1}\right)^2\right)^{-1}\right\};\ T_1 = \frac{1}{2}Tan^{-1}\left(\frac{a_4}{a_3}\right),\ R_1 = (a_3^2 + a_4^2)^{\frac{1}{4}},$$

$$x_1 = \left(M_{44}^2 - p'_{100}\left(1 + \frac{z}{\xi_1}\right)^2\right) + \mu_e H_0^2 \cos^2\phi;$$

$$x_2 = M_{44} - 1 + \mu_e H_0^2 \sin 2\phi;\ x_3 = 2((M_{44} - p'_{100}(1 - pz)),$$

$$x_5 = (M_{44}^2 + 1 - 2M_{44}) - (ka)^2(\mu_e H_0^2 \sin 2\phi)^2;$$

$$x_6 = 2(M_{44} - 1)ka\,\mu_e H_0^2 \sin 2\phi;\ x_4 = M_{44} - 1,$$

$$x_7 = \left(2M_{44} - 2p'_{100}\left(1 + \frac{z}{\xi_1}\right)^2 + \mu_e H_0^2 \cos^2\phi\right)^2;$$

$$x_8 = 2\left(M_{44} - p'_{100}\left(1 + \frac{z}{\xi_1}\right)^2\right) + \mu_e H_0^2 \cos^2\phi;$$

$$x_{11} = R_2 a_1 \cos T_2 \cos(-ka\,\mu_e H_0^2 \sin 2\phi) - R_2 a_1 \sin T_2 \sin(-ka\,\mu_e H_0^2 \sin 2\phi),$$

$$x_{12} = R_2 a_1 \cos T_2 \cos(-ka\,\mu_e H_0^2 \sin 2\phi) + R_2 a_1 \sin T_2 \sin(-ka\,\mu_e H_0^2 \sin 2\phi),$$

$$a_1 = \frac{1}{2\left(M_{44}\left(1 + \frac{z}{\xi_1}\right)^{-2} - p'_{100}\right)}\left[\rho_{110}\rho_{220}^2\omega^6\left(1 + \frac{z}{\xi_1}\right)^6 - b^2(\rho_{110} + \rho_{120}^2 + 2\rho_{220})\right.$$

$$\left.\omega^4 - \left(1 + \frac{z}{\xi_1}\right)^2\rho_{220}^2\omega^4\left(\frac{M_{44}}{r} - \frac{M_{55}k^2}{2}\right) - \omega^2\left(\left(1 + \frac{z}{\xi_1}\right)^{-2}\right)\left(\frac{M_{44}b^2}{r} - \frac{M_{55}k^2}{2} + b^2\right)\right]$$

$$a_2 = b\left[\rho_{220}\omega^5(\rho_{220} + 2\rho_{110})\left(1 + \frac{z}{\xi_1}\right)^2 - \omega^3\left\{\begin{array}{l} b^2\left(1 + \frac{z}{\xi_1}\right)^2 \\ -\frac{2}{r^2}M_{44}\rho_{220} - 2M_{55}\rho_{220}k^2 + 2\rho_{120}\end{array}\right\}\right].$$

$$a_3 = \frac{L^2 - 4a_1}{4};\ a_4 = -a_2;\ a_5 = M_{44} + \left(\frac{M_{55}}{2} + \mu_e H_0^2 \sin^2\phi\right)$$

$$(ka)^2 \rho_{220}\, \omega^2 \left(1 + \frac{z}{\xi_1}\right)^2 - (\rho_{110}\, \rho_{120} + \rho_{120}^2)$$

$$\omega^4 \left(1 + \frac{z}{\xi_1}\right)^4; \; a_6 = M_{44} + \frac{M_{55}}{2} (ka)^2 \omega + \mu_e H_0^2 \sin^2 \phi$$

$$- \rho_{110} \left(1 + \frac{z}{\xi_1}\right)^2 - \rho_{220} \left(1 + \frac{z}{\xi_1}\right)^2 \omega^3,$$

$$a_9 = \frac{1}{(\rho_{220}^2 (1 + nz)^2 + b\omega^2)} \begin{bmatrix} M_{44} + \frac{M_{55}}{2} (kb)^2 (\rho_{220}^2 (1 + nz)^2 + b\omega^2) \\ - \rho_{120}\, \rho_{220}^2\, \omega^2 (1 + nz)^3 - b^2 \rho_{110} (1 + nz)\omega^4 \\ - \rho_{220}\, \rho_{120}^2\, \omega^4 (1 + nz)^3 + b^2 \rho_{220} (1 + nz)\omega^2 \\ - b^3 \omega^3 + 2b^2 \rho_{120} (1 + nz)\omega^4 \end{bmatrix}$$

$$a_{10} = \frac{1}{(\rho_{220}^2 (1 + nz)^2 + b\omega^2)} \begin{bmatrix} b^3 \omega^3 + 2b\omega^3 \rho_{120}\, \rho_{220} (1 + nz)^2 \\ + b^2 \rho_{120}^2 (1 + nz)^2 \omega^3 - b\rho_{220}^2 (1 + nz)^2 \omega \\ + b\rho_{120}^2 (1 + nz)^2 \omega^5 \end{bmatrix}$$

$$a_{111} = \frac{(x_4 x_3^{-1})^2 + 4x_3^{-1} a_9}{4}; \; a_{112} = a_{10} x_3^{-1},$$

$$a_7 = \frac{x_5 x_8 + 4a_5 x_7}{4x_7 x_8}; \; a_8 = \frac{x_6 x_8 + 4a_6 x_7}{4x_7 x_8};$$

a_{43}, a_{44} = Similar expression as a_{13}, a_{14} with x_{11}, x_{12} replaced by x_{13}, x_{14} respectively,

a_{53}, a_{54} = Similar expression as a_{13}, a_{14} with M_{44}, x_{11}, x_{12} replaced by M_{55}, x_{13}, x_{14}, respectively,

a_{13}, a_{14} = Similar expression as a_{63}, a_{64} with a_2, x_{13}, x_{14} replaced by a_1, x_{11}, x_{12}, and multiplied by $\frac{M_{44}}{2}$, M respectively,

x_{13}, x_{14} = Similar expression as x_{11}, x_{12} with a_1 replaced by a_2 respectively,

a'_{13}, a'_{14} = Similar expression as a'_{63}, a'_{64} with a_2, x_{13}, x_{14} replaced by a_1, x_{11}, x_{12} and multiplied by $\frac{M_{44}}{2}$, respectively,

a'_{23} = Similar expression as a_{23} with x_{11}, x_{12} and positive sign replaced by x_{12}, x_{11} and negative sign, respectively,

a'_{24} = Similar expression as a_{24} with cos, sin and negative sign replaced by sin, cos and positive sign, respectively,

a'_{21}, a'_{22} = Similar expression as a'_{31}, a'_{32} with multiplied by $\frac{M_{55}}{2}$, respectively,

a'_{33}, a'_{34} = Similar expression as a_{33}, a_{34} with x_{11}, x_{12} replaced by x_{12}, x_{11}, respectively,

a'_{43} = Similar expression as a_{43} with x_{13}, x_{14} replaced by x_{14}, x_{13} respectively,

a'_{44} = Similar expression as a_{44} with cos, sin and negative sign replaced by sin, cos and positive sign, respectively,

a'_{53} = Similar expression as a_{53} with x_{13}, x_{14} and positive sign replaced by x_{14}, x_{13} and negative sign, respectively,

a'_{54} = Similar expression as a'_{53} with cos, sin and negative sign replaced by sin, cos and positive sign, respectively,

a'_{55}, a'_{56} = Similar expression as a'_{31}, a'_{32} with multiplied by $\frac{M_{55}}{2} e^{\frac{-x_4 a_2}{2x_3}}$, respectively,

a'_{63} = Similar expression as a_{63} with x_{13}, x_{14} and positive sign replaced by x_{14}, x_{13} and negative sign, respectively,

a'_{64} = Similar expression as a'_{63} with x_{13}, x_{14}, cosh and negative sign replaced by x_{14}, x_{13}, sinh and positive sign, respectively,

a'_{65}, a'_{66} = Similar expression as a'_{31}, a'_{32} with multiplied by $e^{\frac{-x_4 a_2}{2x_3}}$, respectively,

T_2; T_3 = Similar expression as T_1 with a_3, a_4 replaced by a_7, a_8; a_{111}, a_{112} respectively,

R_2; R_3 = Similar expression as R_1 with a_3, a_4 replaced by a_7, a_8; a_{111}, a_{112} respectively.