Advances and Applications in Mathematical Sciences
Volume 18, Issue 10, August 2019, Pages 1185-1202

NEW CONTRACTION MAPPING PRINCIPLE IN PARTIALLY ORDERED GENERALIZED INTUITIONISTIC FUZZY METRIC SPACES

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Abstract

This paper is to introduce a new type of coupled contraction mapping in generalized intuitionistic fuzzy metric spaces having partial ordering. The two mappings $F : X \times X \to X$ and $g : X \to X$ mentioned here are taken to be compatible. The t-norm $*$ and the t-conorm $\diamond$ are assumed to be of Hadžić type. The main result on the coupled coincidence point is obtained in generalized intuitionistic fuzzy metric spaces as an application of the coincidence point theorem on metric spaces. This work is intended to extend recent results on coupled coincidence points.

1. Introduction

In this paper we consider a coupled coincidence point problem in generalized intuitionistic fuzzy metric spaces. There are several independent

2010 Mathematics Subject Classification: 54H25, 47H10.
Keywords: fuzzy metric space, partial ordered set, intuitionistic fuzzy metric space.
Received July 21, 2019; Accepted September 12, 2019

definitions of fuzzy metric spaces in the literatures, for instance [7], [8], [12], [13] out of which we consider here the definition given by George and Veeramani [7]. Coupled fixed point was introduced by Guo et al. [9]. A coupled contraction mapping principle was established by Bhaskar et al. [3] in partially ordered metric spaces. The result was extended to coincidence point problems by Ciric et al. [6] under two different sets of conditions. In [4], the well known concept of compatible mappings was extended to the context of coupled and single mappings. In fuzzy metric spaces Zhu et al. [15] were the first to correctly work out a fuzzy fixed point theorem. Afterwards, a coupled coincidence point result was established by Choudhury et al. [5], Hu [11]. In 1986, Atanassov [1] introduced the notion of intuitionistic fuzzy metric space. Afterward, Park [14] gave the notion of intuitionistic fuzzy metric space and generalized the notion of a fuzzy metric space due to George and Veeramani. The purpose of this paper is to prove a coincidence point result for compatible mappings in a generalized intuitionistic fuzzy metric space which has Hadžić type t-norm and t-conorm under the assumption of a new inequality.

2. Preliminaries

Definition 2.1. A 5-tuple $(X, M, N, *, \Diamond)$ is said to be a generalized intuitionistic fuzzy metric space (shortly GIFM-Space), if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuoust-conorm and M, N are fuzzy sets on $X^3 \times (0, \infty)$ satisfying the following conditions:

For all $x, y, z, a \in X$ and $s, t > 0$,

(GIFM-1) $M(x, y, z, t) \leq 1, N(x, y, z, t) \leq 1$,

(GIFM-2) $M(x, y, z, t) > 0$,

(GIFM-3) $M(x, y, z, t) = 1$ if and only if $x = y = z$,

(GIFM-4) $M(x, y, z, t) = M(p\{x, y, z\}, t)$, where p is a permutation function,

(GIFM-5) $M(x, y, z, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$,

(GIFM-6) $M(x, y, z, \cdot)(0, \infty) \to [0, 1]$ is continuous, (GIFM-7) $N(x, y, z, t) < 1$,

(GIFM-8) $N(x, y, z, t) = 0$ if and only if $x = y = z$,

(GIFM-9) $N(x, y, z, t) = N(p\{x, y, z\}, t)$, where p is a permutation function,

(GIFM-10) $N(x, y, z, a, t) \lozenge N(a, z, z, s) \geq N(x, y, z, t + s)$,

(GIFM-11) $N(x, y, z, \cdot)$ is continuous.

In this case, the pair (M, N) is called a generalized intuitionistic fuzzy metric on X.

Definition 2.2. Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space. For $t > 0$, the open balls $B_M(x, r, t)$ and $B_N(x, r, t)$ with centre $x \in X$ and radius $0 < r < 1$ are defined by

$$B_M(x, r, t) = \{y \in X : M(x, y, y, t) > 1 - r\} \text{ and}$$

$$B_N(x, r, t) = \{y \in X : N(x, y, y, t) < r\}.$$

A subset A of X is called an open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_M(x, r, t)$ contained in A.

Example 2.3. Let X be a nonempty set with metric D^*. Denote $a * b = a \cdot b$ and $a \lozenge b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$. For any $t \in [0, \infty)$, define

$$M(x, y, z, t) = \frac{t}{t + D^*(x, y, z)} \text{ and}$$

$$N(x, y, z, t) = \frac{D^*(x, y, z)}{t + D^*(x, y, z)} \text{ for all } x, y, z \in X.$$

It is easy to see that $(X, M, N, *, \lozenge)$ is a generalized intuitionistic fuzzy metric space.

Definition 2.4. Let $(X, M, N, *, \lozenge)$ be a generalized intuitionistic fuzzy metric space and $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is said to be converging to a point $x \in X$ if

$$\lim_{n \to \infty} M(x, x, x_m, t) = 1 \text{ and } \lim_{n \to \infty} N(x, x, x_m, t) = 0 \text{ for all } t > 0.$$

(ii) $\{x_n\}$ is called Cauchy sequence if $\lim_{n \to \infty} M(x_{n+P}, x_{n+P}, x_n, t) = 1$ and $N(x_{n+P}, x_{n+P}, x_n, t) = 0$ for all $t > 0$ and $P > 0$.

(iii) An intuitionistic fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Lemma 2.5. *Let* $(X, M, N, *, \lozenge)$ *be a generalized intuitionistic fuzzy metric space. Then* $M(x, y, z, t)$ *and* $N(x, y, z, t)$ *are non-decreasing with respect to t for all* x, y, z *in X.*

Let $(X, \preceq)$ be a partially ordered set and F be a mapping from X to itself. The mapping F is said to be non-decreasing if for all $x_1, x_2 \in X, x_1 \preceq x_2$ implies $F(x_1) \preceq F(x_2)$ and non-increasing if for all $x_1, x_2 \in X, x_1 \preceq x_2$ implies $F(x_1) \succeq F(x_2)$.

Definition 2.6. Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X.$ Then the map F is said to have mixed monotone property if F is non-decreasing its first argument and is non-increasing in its second argument, that is, if for all $x_1, x_2 \in X, x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for fixed $y \in X$, and for all $y_1, y_2 \in X, y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2)$ for fixed $x \in X$.

Definition 2.7. Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. The mapping F is said to have the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, if, for all $x_1, x_2 \in X, g(x_1) \preceq g(x_2)$ implies $F(x_1, y) \preceq F(x_2, y)$, for any $y \in X$ and for all $y_1, y_2 \in X, g(y_1) \preceq g(y_2)$ implies $F(x, y_1) \succeq F(x, y_2)$ for any $x \in X$.

Definition 2.8. Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x, F(y, x) = y$.

Definition 2.9. Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x, y), g(y) = F(y, x)$.

Definition 2.10 Let $(X, M, N, *, \Diamond)$ be a generalized intuitionistic fuzzy metric space. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if for all

$$\lim_{n\to\infty} M(g(F(x_n, y_n)), g(F(x_n, y_n)), F(g(x_n), g(y_n), t) = 1 \text{ and}$$

$$\lim_{n\to\infty} M(g(F(y_n, x_n)), g(F(y_n, x_n)), F(g(y_n), g(x_n), t) = 1,$$

$$\lim_{n\to\infty} N(g(F(x_n, y_n)), g(F(x_n, y_n)), F(g(x_n), g(y_n), t) = 0 \text{ and}$$

$$\lim_{n\to\infty} N(g(F(y_n, x_n)), g(F(y_n, x_n)), F(g(y_n), g(x_n), t) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

Lemma 2.11. *Let $(X, M, N, *, \Diamond)$ be a generalized intuitionistic fuzzy metric space. If the pair (F, g) where $F : X \times X \to X$ and $g : X \to X$ are compatible then the pair (F, g) is also compatible.*

Lemma 2.12. *Let $(X, M, N, *, \Diamond)$ be a generalized intuitionistic fuzzy metric space with a Hadžić type t-norm $*$ and t-conorm $\Diamond$ such that $M(x, y, z, t) \to 1, N(x, y, z, t) \to 0$ as $t \to \infty$, for all $x, y, z \in X$. If the sequences $\{x_n\}$ and $\{y_n\}$ in X are such that for all $n \geq 1, t > 0$,*

$$M(x_n, x_n, x_{n+1}, t) * M(y_n, y_n, y_{n+1}, t) \geq M(x_{n-1}, x_{n-1}, x_n, t/k)$$
$$*M(y_{n-1}, y_{n-1}, y_n, t/k)$$

$$N(x_n, x_n, x_{n+1}, t) \Diamond N(y_n, y_n, y_{n+1}, t) \leq N(x_{n-1}, x_{n-1}, x_n, t/k)$$
$$\Diamond N(y_n, y_n, y_{n-1}, t/k)$$

where $0 < k < 1$, then the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

3. Main Results

Theorem 3.1. *Let* $(X, M, N, *, \Diamond)$ *be a complete generalized intuitionistic fuzzy metric space with a Hadžić type t-norm and t-conorm such that* $M(x, y, z, t) \to 1$, $N(x, y, z, t) \to 0$ *as* $t \to \infty$, *for all* $x, y, z \in X$. *Let* $\preceq$ *be a partial order defined on X. Let* $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings such that F has mixed g-monotone property and satisfies the following conditions*:

(i) $F(X \times X) \subseteq g(X)$,

(ii) *g is continuous and monotonic increasing*,

(iii) (g, F) *is a compatible pair*,

(iv) $M(F(x, y), F(x, y), F(u, v), kt) * M(F(y, x), F(y, x), F(v, u), kt)$

$$\geq M(g(x), g(x), g(u), t) * M(g(y), g(y), g(v), t) \quad (3.1.1)$$

$$N(F(x, y), F(x, y), F(u, v), kt) \Diamond N(F(y, x), F(y, x) F(v, u), kt)$$

$$\leq N(g(x), g(x), g(u), t) \Diamond N(g(y), g(y), g(v), t) \quad (3.1.2)$$

for all $x, y, u, v \in X, t > 0$ *with* $g(x) \preceq g(u)$ *and* $g(y) \succeq g(v)$, *where* $0 < k < 1$. *Also suppose either*

(a) *F is continuous or*

(b) *X has the following properties*:

(i) *If a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preceq x$ *for all* $n \geq 0$ (3.1.3)

(ii) *If a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \preceq y$ *for* all $n \geq 0$. (3.1.4)

If there are $x_0, y_0 \in X$ *such that* $g(x_0) \preceq F(x_0, y_0)$, $g(y_0) \succeq F(y_0, x_0)$ *then there exist* $x, y \in X$ *such that* $g(x) = F(x, y)$ *and* $g(y) = F(y, x)$, *that is, g and F have a coupled coincidence point in X.*

Proof. Starting with x_0, y_0 in X, we define the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$g(x_1) = F(x_0, y_0) \text{ and } g(y_1) = F(y_0, x_0)$$

$$g(x_2) = F(x_1, y_1) \text{ and } g(y_2) = F(y_1, x_1)$$

and, in general, for all $n \geq 0$,

$$g(x_{n+1}) = F(x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n). \tag{3.1.5}$$

This construction is possible by the condition (i) of the theorem.

Next, we prove that for all $n \geq 0$,

$$g(x_n) \preceq g(x_{n+1}) \tag{3.1.6}$$

$$\text{and } g(y_n) \succeq g(y_{n+1}). \tag{3.1.7}$$

From the conditions on x_0, y_0, we have

$$g(x_0) \preceq F(x_0, y_0) = g(x_1) \text{ and } g(y_0) \succeq F(y_0, x_0) = g(y_1).$$

Therefore (3.1.6) and (3.1.7) hold for $n = 0$.

Let (3.1.6) and (3.1.7) hold for some $n = m$. As F has the mixed g-monotone property and $g(x_m) \preceq g(x_{m+1})$, $g(y_m) \succeq g(y_{m+1})$, it follows that

$$g(x_{m+1}) = F(x_m, y_m) \preceq F(x_{m+1}, y_m) \text{ and}$$

$$F(y_{m+1}, x_m) \preceq F(y_m, x_m) = g(y_{m+1}). \tag{3.1.8}$$

Also, for the same reason, we have

$$F(x_{m+1}, y_m) \preceq F(x_{m+1}, y_{m+1}) = g(x_{m+2}) \text{ and}$$

$$g(y_{m+2}) = F(y_{m+1}, x_{m+1}) \preceq F(y_{m+1}, x_m). \tag{3.1.9}$$

Then, from (3.1.8) and (3.1.9),

$$g(x_{m+1}) \preceq g(x_{m+2}) \text{ and } g(y_{m+1}) \succeq g(y_{m+2}).$$

Then, by induction, (3.1.6) and (3.1.7) hold for all $n \geq 0$.

Due to (3.1.5), (3.1.6) and (3.1.7), from (3.1.1) and (3.1.2) for all $t > 0$, $n \geq 1$, we have

$$M(g(x_n), g(x_n), g(x_{n+1}), kt) * M(g(y_n), g(y_n), g(y_{n+1}), kt)$$

$$= M(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n), kt)$$

$$*M(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n), kt)$$

$\geq M(g(x_{n-1}), g(x_{n-1}), g(x_n), t) * M(g(y_{n-1}), g(y_{n-1}), g(y_n), t)$. (by (3.1.1)) (3.1.10)

$N(g(x_n), g(x_n), g(x_{n+1}), kt) \Diamond N(g(y_n), g(y_n), g(y_{n+1}), kt)$

$= N(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n), kt)$

$\Diamond N(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n), kt)$

$\leq N(g(x_{n-1}), g(x_{n-1}), g(x_n), t) \Diamond N(g(y_{n-1}), g(y_{n-1}), g(y_n), t)$. (by (3.1.2)) (3.1.11)

From (3.1.10) and (3.1.11) and by applying Lemma 2.12, we can conclude that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Since X is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x \text{ and } \lim_{n\to\infty} g(y_n) = y. \quad (3.1.12)$$

Therefore, $\lim_{n\to\infty} g(x_{n+1}) = \lim_{n\to\infty} F(x_n, y_n) = x$ and $\lim_{n\to\infty} g(y_{n+1}) = \lim_{n\to\infty} F(y_n, x_n) = y$.

Since (g, F) is a compatible pair, using continuity of g, we have

$$g(x) = \lim_{n\to\infty} g(g(x_{n+1})) = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(g(x_n), g(y_n)) \quad (3.1.13)$$

$$\text{and } g(y) = \lim_{n\to\infty} g(g(y_{n+1})) = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(g(y_n), g(x_n)). \quad (3.1.14)$$

Now assume that (a) holds. Then by continuity of F, from (3.1.13) and (3.1.14) and by using (3.1.12), we have

$$g(x) = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(g(x_n), g(y_n)) = F(\lim_{n\to\infty} g(x_n),$$

$$\lim_{n\to\infty} g(y_n)) = F(x, y)$$

$$g(x) = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(g(y_n), g(x_n)) = F(\lim_{n\to\infty} g(y_n),$$

$$\lim_{n\to\infty} g(x_n)) = F(y, x).$$

which implies that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Next we assume that (b) holds. By (3.1.6), (3.1.7) and (3.1.12), $\{g(x_n)\}$ is a non-decreasing sequence with $g(x_n) \to x$ and $\{g(y_n)\}$ is a non-increasing sequence with $g(y_n) \to x$ as $n \to \infty$.

By (3.1.3) and (3.1.4), it follows that, for all $n \geq 0$, $g(x_n) \preceq x$ and $g(y_n) \succeq y$.

Since g is monotonic increasing,

$$g(g(x_n)) \preceq g(x) \text{ and } g(g(y_n)) \succeq g(y). \tag{3.1.15}$$

Now, for all $t > 0$, $n \geq 0$, we have

$$M(F(x, y), F(x, y), g(F(x_n, y_n)), t) \geq M(F(x, y), F(x, y), g(g(x_{n+1})), kt)$$
$$*M(g(g(x_{n+1})), g(g(x_{n+1})), g(F(x_n, y_n)), (t - kt)) \tag{3.1.16}$$

$$M(F(y, x), F(y, x), g(F(x_n, y_n)), t) \geq M(F(y, x), F(y, x), g(g(x_{n+1})), kt)$$
$$*M(g(g(y_{n+1})), g(g(y_{n+1})), g(F(y_n, x_n)), (t - kt)) \tag{3.1.17}$$

$$N(F(x, y), F(x, y), g(F(x_n, y_n)), t) \leq N(F(x, y), F(x, y), g(g(x_{n+1})), kt)$$
$$\Diamond N(g(g(x_{n+1})), g(g(x_{n+1})), g(F(x_n, y_n)), (t - kt)) \tag{3.1.18}$$

$$N(F(y, x), F(y, x), g(F(y_n, x_n)), t) \geq N(F(y, x), F(y, x), g(g(y_{n+1})), kt)$$
$$\Diamond N(g(g(y_{n+1})), (g(g(y_{n+1})), (g(F(y_n, x_n)), (t - kt)) \tag{3.1.19}$$

From (3.1.16), (3.1.17), (3.1.18) and (3.1.19), for all $t > 0$ we have

$$M(F(x, y), F(x, y), g(F(x_n, y_n)), t) * M(F(y, x), F(y, x), g(F(y_n, x_n)), t)$$

$$\geq \{M(F(x, y), F(x, y), g(g(x_{n+1})), kt) * M(g(g(x_{n+1})), g(g(x_{n+1})),$$

$$g(F(x_n, y_n)), (t - kt))\}$$

$$* \{M(F(y, x), F(y, x), g(g(y_{n+1})), kt) * M(g(g(y_{n+1})), g(g(y_{n+1})),$$

$$g(F(y_n, x_n)), (t - kt))\}$$

$$N(F(x, y), F(x, y), g(F(x_n, y_n)), t) \Diamond N(F(y, x), F(y, x), g(F(y_n, x_n)), t)$$

$$\leq \{N(F(x, y), F(x, y), g(g(x_{n+1})), kt) \lozenge N g(g(x_{n+1})), g(g(x_{n+1})),$$

$$g(F(x_n, y_n)), (t - kt))\}$$

$$\lozenge \{N(F(y, x), F(y, x), g(g(y_{n+1})), kt)$$

$$\lozenge N(g(g(y_{n+1})), g(g(y_{n+1})), g(F(y_n, x_n)), (t - kt))\}.$$

Taking $n \to \infty$ on the both sides of the above inequality, for all $t > 0$,

$$\lim_{n\to\infty} \{M(F(x, y), F(x, y), g(F(x_n, y_n)), t) * M(F(x, y), F(x, y), g(F(y_n, x_n)), t)\}$$

$$\geq \lim_{n\to\infty} \{M(F(x, y), F(x, y), g(g(x_{n+1})), kt) * M(g(g(x_{n+1})), (g(g(x_{n+1})),$$

$$(g(F(x_n, y_n)), (t - kt))\} * \lim_{n\to\infty} \{M(F(x, y), F(x, y), g(g(y_{n+1})), kt)$$

$$* M(g(g(y_{n+1})), (g(g(y_{n+1})), (g(F(y_n, x_n)), (t - kt))\}$$

$$\lim_{n\to\infty} \{N(F(x, y), F(x, y), g(F(x_n, y_n)), t) \lozenge N(F(y, x), F(y, x), g(F(y_n, x_n)), t)\}$$

$$\leq \lim_{n\to\infty} \{N(F(x, y), F(x, y), g(g(x_{n+1})), kt) \lozenge N(g(g(x_{n+1})), g(g(x_{n+1})),$$

$$g(F(x_n, y_n)), (t - kt))\}$$

$$\lozenge \lim_{n\to\infty} \{N(F(y, x), F(y, x), g(g(y_{n+1})), kt) \lozenge N(g(g(y_{n+1})),$$

$$(g(g(y_{n+1})), (g(F(y_n, x_n)), (t - kt))\}.$$

Hence

$$M(F(x,y), F(x,y), \lim_{n\to\infty} g(F(x_n, y_n)), t) * M(F(y,x), F(y,x), \lim_{n\to\infty} g(F(y_n, x_n)), t)$$

$$\geq \{M(F(x,y), F(x,y), \lim_{n\to\infty} g(g(x_{n+1})), kt)$$

$$* M(\lim_{n\to\infty} g(g(x_{n+1})), \lim_{n\to\infty} g(g(x_{n+1})),$$

$$\lim_{n\to\infty} g(F(x_n, y_n)), (t - kt))\}$$

$$* \{M(F(y,x), F(y,x), \lim_{n\to\infty} g(g(y_{n+1})), kt)$$

$$*M(\lim_{n\to\infty} g(g(y_{n+1})), \lim_{n\to\infty} g(g(y_{n+1})),$$

$$\lim_{n\to\infty} g(F(y_n, x_n)), (t-kt))\}$$

$$N(F(x, y), F(x, y), \lim_{n\to\infty} g(F(x_n, y_n)), t) \Diamond N(F(y, x), F(y, x), \lim_{n\to\infty} g(F(y_n, x_n)), t)$$

$$\leq \{N(F(x, y), F(x, y), \lim_{n\to\infty} g(g(x_{n+1})), kt)$$

$$\Diamond N(\lim_{n\to\infty} g(g(x_{n+1})), \lim_{n\to\infty} g(g(x_{n+1})),$$

$$\lim_{n\to\infty} g(F(x_n, y_n)), (t-kt))\}$$

$$\Diamond \{N(F(y, x), F(y, x), \lim_{n\to\infty} g(g(y_{n+1})), kt)$$

$$\Diamond N(\lim_{n\to\infty} g(g(y_{n+1})), \lim_{n\to\infty} g(g(y_{n+1})),$$

$$\lim_{n\to\infty} g(F(y_n, x_n)), (t-kt))\}.$$

Therefore

$$M(F(x, y), F(x, y), g(x), t) * M(F(y, x), F(y, x), g(y), t)$$

$$\geq \{M(F(x, y), F(x, y), \lim_{n\to\infty} g(F(x_n, y_n)), kt) * M(g(x), g(x), g(y), (t-kt))\}$$

$$* \{M(F(y, x), F(y, x), \lim_{n\to\infty} g(F(y_n, x_n)), kt) * M(g(y), g(y), g(x), (t-kt))\},$$

$$N(F(x, y), F(x, y), g(x), t) \Diamond N(F(y, x), F(y, x), g(y), t)$$

$$\leq \{N(F(x, y), F(x, y), \lim_{n\to\infty} g(F(x_n, y_n)), kt) \Diamond N(g(x), g(x), g(y), (t-kt))\}$$

$$\Diamond \{N(F(y, x), F(y, x), \lim_{n\to\infty} g(F(y_n, x_n)), kt) \Diamond N(g(y), g(y), g(x), (t-kt))\},$$

that is,

$$M(F(x, y), F(x, y), g(x), t) * M(F(y, x), F(y, x), g(y), t)$$

$$\geq \lim_{n\to\infty} \{M(F(x, y), F(x, y), g(F(x_n, y_n)), kt) * 1)\}$$

$$* \lim_{n\to\infty}\{M(F(y,\, x),\, F(y,\, x),\, g(g(F(y_n,\, x_n))),\, kt) * 1\}$$

$$\geq \lim_{n\to\infty}\{M(F(g(x_n),\, g(y_n)),\, F(g(x_n),\, g(y_n)),\, F(y,\, x),\, kt)$$

$$* M(F(g(y_n),\, g(x_n)),\, g(y_n),\, g(x_n)),\, F(y,\, x),\, kt)\}$$

$$\geq \lim_{n\to\infty}\{M(g(g(x_n),\, g(g(x_n)),\, g(x),\, t) * M(g(g(y_n),\, g(g(y_n)),\, g(y),\, t)\}$$

$$= M(\lim_{n\to\infty} g(g(x_n)),\, \lim_{n\to\infty} g(g(x_n)),\, g(x),\, t) * M(\lim_{n\to\infty} g(g(y_n))$$

$$\lim_{n\to\infty} g(g(y_n)),\, g(y),\, t)$$

$$= M(g(x),\, g(x),\, g(x),\, t) * M(g(y),\, g(y),\, g(y),\, t)$$

$$= 1 * 1 = 1,$$

$$N(F(x,\, y),\, F(x,\, y),\, g(x),\, t) \lozenge N(F(y,\, x),\, F(y,\, x),\, g(y),\, t)$$

$$\leq \lim_{n\to\infty}\{N(F(x,\, y),\, F(x,\, y),\, g(F(x_n,\, y_n),\, kt) \lozenge 0)$$

$$\lozenge \lim_{n\to\infty}\{N(F(y,\, x),\, F(y,\, x),\, g(g(F(y_n,\, x_n))),\, kt) \lozenge 0\}$$

$$\leq \lim_{n\to\infty}\{N(F(g(x_n)),\, g(y_n)),\, F(g(x_n),\, g(y_n)),\, F(y,\, x),\, kt)$$

$$\lozenge N(F(g(y_n),\, g(x_n)),\, F(g(y_n),\, g(x_n)),\, F(y,\, x),\, kt)\}$$

$$\leq \lim_{n\to\infty}\{N(g(g(x_n)),\, g(g(x_n)),\, g(x),\, t) \lozenge N(g(g(y_n),\, g(g(y_n)),\, g(y),\, t)\}$$

$$= N(\lim_{n\to\infty} g(g(x_n)),\, \lim_{n\to\infty} g(g(x_n)),\, g(x),\, t) \lozenge N(\lim_{n\to\infty} g(g(y_n)),\, \lim_{n\to\infty} g(g(y_n)),\, g(y),\, t)$$

$$= N(g(x),\, g(x),\, g(x),\, t) \lozenge N(g(y),\, g(y),\, g(y),\, t)$$

$$= 0 \lozenge 0 = 0.$$

That is, $M(F(x,\, y),\, F(x,\, y),\, g(x),\, t) * M(F(y,\, x),\, F(y,\, x),\, g(y),\, t) \geq 1$ and $N(F(x,\, y),\, F(x,\, y),\, g(x),\, t) \lozenge N(F(y,\, x),\, F(y,\, x),\, g(y),\, t) \leq 0$. Therefore $M(F(x,\, y),\, F(x,\, y),\, g(x),\, t) = 1$ and $M(F(y,\, x),\, F(y,\, x),\, g(y),\, t) = 1$, $N(F(x,\, y),\, F(x,\, y),\, g(x),\, t) = 0$ and $N(F(y,\, x),\, F(y,\, x),\, g(y),\, t) = 0$, which implies that $g(x) = F(x, y)$ and $g(y) = F(y, x)$. This completes the proof of the theorem.

Corollary 3.2. *Let* $(X, M, N, *, \Diamond)$ *be a complete generalized intuitionistic fuzzy metric space with a Hadžić type t-norm and t-conorm such that* $M(x, y, z, t) \to 1$, $N(x, y, z, t) \to 0$ *as* $t \to \infty$, *for all* $x, y \in X$. *Let* $\preceq$ *be a partial order defined on X. Let* $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings such that F has mixed g-monotone property and satisfies the following conditions*:

(i) $F(X \times X) \subseteq g(X)$,

(ii) *g is continuous and monotonic increasing*,

(iii) (g, F) *is a commuting pair*,

(iv) $M(F(x, y), F(x, y), F(u, v), kt) * M(F(y, x), F(y, x), F(v, u), kt)$

$$\geq M(g(x), g(x), g(u), t) * M(g(y), g(y), g(v), t),$$

$N(F(x, y), F(x, y), F(u, v), kt) \Diamond N(F(y, x), F(y, x), F(v, u), kt)$

$$\leq N(g(x), g(x), g(u), t) \Diamond N(g(y), g(y), g(v), t),$$

for all $x, y, u, v \in X, t > 0$ *with* $g(x) \preceq g(u)$ *and* $g(y) \succeq g(v)$ *where* $0 < k < 1$.

Also suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) If a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$,

(ii) If a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \geq 0$.

If there are $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, g and F have coupled coincidence point in X.

Proof. Since a commuting pair is also a compatible pair, the result of the Corollary 3.2 follows from Theorem 3.1. Later, by an example, we will show that the Corollary 3.2 is properly contained in Theorem 3.1. The following corollary is a result on fixed point.

Corollary 3.3. *Let* $(X, \preceq)$ *be a partially ordered set and let* $(X, M, N, *, \lozenge)$ *be a complete generalized intuitionistic fuzzy metric space with a Hadžić type t-norm and t-conorm such that* $M(x, y, z, t) \to 1$, $N(x, y, z, t) \to 0$ *as* $t \to \infty$, *for all* $x, y, z \in X$. *Let* $\preceq$ *be a partial order defined on X. Let* $F : X \times X \to X$ *be a mapping such that F has mixed monotone property and satisfies the following conditions*:

$$M(F(x, y), F(x, y), F(u, v), kt) * M(F(y, x), F(y, x), F(v, u), kt)$$

$$\geq M(x, x, u, t) * M(y, y, v, t),$$

$$N(F(x, y), F(x, y), F(u, v), kt) \lozenge N(F(y, x), F(y, x), F(v, u), kt)$$

$$\leq N(x, x, u, t) \lozenge N(y, y, v, t),$$

for all $x, y, u, v \in X, t > 0$ *with* $x \preceq u$ *and* $y \succeq v$, *where* $0 < k < 1$.

Also suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) If a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$,

(ii) If a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$, for all $n \geq 0$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point in X.

Proof. The proof follows by putting $g = I$, the identity function, in Theorem 3.1.

Example 3.4. Let $(X, \preceq)$ be a partially ordered set with $X = [0, 1]$ and the natural ordering $\leq$ of the real numbers as the partial ordering $\preceq$. Let for all $t > 0$ and $x, y, z \in X$,

$$M(x, y, z, t) = e^{-\frac{|x-y|+|y-z|+|z-x|}{t}}, \quad N(x, y, z, t) = \frac{\left(e^{-\frac{|x-y|+|y-z|+|z-x|}{t}}\right) - 1}{\left(e^{-\frac{|x-y|+|y-z|+|z-x|}{t}}\right)}.$$

Let $a * b = \min\{a, b\}$ and $a \Diamond b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$.

Then $(X, M, N, *, \Diamond)$ is a complete generalized intuitionistic fuzzy metric space such that $M(x, y, z, t) \to 1$ and $N(x, y, z, t) \to 0$ as $t \to \infty$, for all $x, y, z \in X$.

Let the mapping $g : X \to X$ be defined as $g(x) = \frac{5}{6}x^2$ for all $x \in X$ and the mapping $F : X \times X \to X$ be defined as $F(x, y) = \frac{x^2 - y^2}{4}$. Then $F(X \times X) \subseteq g(X)$ and F satisfies the mixed g-monotone property.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = a$, $\lim_{n\to\infty} g(x_n) = a$, $\lim_{n\to\infty} F(x_n, y_n) = b$ and $\lim_{n\to\infty} g(y_n) = b$.

Now, for all $n \geq 0$,

$$g(x_n) = \frac{5}{6}x_n^2, \; g(x_n) = \frac{5}{6}y_n^2, \; F(x_n, y_n) = \frac{x_n^2 - y_n^2}{4} \text{ and}$$

$$F(y_n, x_n) = \frac{y_n^2 - x_n^2}{4}.$$

Then necessarily $a = 0$ and $b = 0$. It then follows that, for all $t > 0$,

$$\lim_{n\to\infty} M(g(F(x_n, y_n)), (g(F(x_n, y_n)), F(g(x_n), g(y_n), t) = 1,$$

$$\lim_{n\to\infty} N(g(F(x_n, y_n)), (g(F(x_n, y_n)), F(g(x_n), g(y_n), t) = 0, \text{ and}$$

$$\lim_{n\to\infty} M(g(F(y_n, x_n)), (g(F(y_n, x_n)), F(g(y_n), g(x_n), t) = 1,$$

$$\lim_{n\to\infty} N(g(F(y_n, x_n)), (g(F(y_n, x_n)), F(g(y_n), g(y_n), t) = 0.$$

Therefore the mappings F and g are compatible in X.

We show that the conditions (3.1) and (3.2) hold.

$$| F(x, y) - F(u, v) | \leq \frac{1}{2} | g(x) - g(u) | + \frac{1}{2} | g(y) - g(v) |, x \geq u, y \leq v, \text{ and} \tag{3.4.1}$$

$$| F(y, x) - F(v, u) | \leq \frac{1}{2} | g(y) - g(v) | + \frac{1}{2} | g(x) - g(u) |, x \geq u, y \leq v. \tag{3.4.2}$$

From (3.1.18), for all $t > 0$ and $0 < k < 1$, we have

$$e^{-\frac{| F(x, y) - F(u, v) |}{ks}} \geq e^{-\frac{1/2 | g(x) - g(u) | + 1/2 | g(y) - g(v) |}{t}} \geq e^{-\frac{| g(x) - g(v) |}{2t}} \cdot e^{-\frac{| g(y) - g(u) |}{2t}}$$

$$\geq \sqrt{e^{-\frac{| g(x) - g(u) |}{t}} \cdot e^{-\frac{| g(y) - g(v) |}{t}}} \geq \min \left\{ e^{-\frac{| g(x) - g(u) |}{t}}, e^{-\frac{| g(y) - g(v) |}{t}} \right\}$$

$$e^{-\frac{| F(x, y) - F(u, v) |}{kt}} \geq \min \{ M(g(x), g(x), g(u), t), M(g(y), g(y), g(v), t) \} \tag{3.4.3}$$

and

$$\frac{\left(e^{-\frac{| F(x, y) - F(u, v) |}{ks}} \right) - 1}{\left(e^{-\frac{| F(x, y) - F(u, v) |}{ks}} \right)} \leq \max \{ N(g(x), g(x), g(u), t), N(g(y), g(y), g(v), t) \}. \tag{3.4.4}$$

Similarly from (3.4.2), we get

$$e^{-\frac{| F(y, x) - F(v, u) |}{kt}} \geq \min \{ M(g(x), g(x), g(u), t), M(g(y), g(y), g(v), t) \} \tag{3.4.5}$$

$$\frac{\left(e^{-\frac{|F(y,x)-F(v,u)|}{ks}}\right)-1}{\left(e^{-\frac{|F(y,x)-F(v,u)|}{ks}}\right)} \leq \max\{N(g(x), g(x), g(u), t), N(g(y), g(y), g(v), t)\}. \quad (3.4.6)$$

From (3.4.3), (3.4.4),(3.4.5) and (3.4.6) we have

$$\min\{M(F(x, y), F(x, y), F(u, v), kt), M(F(y, x), F(y, x), F(v, u), kt)\}$$

$$\geq \min\{M(g(x), g(x), g(u), t), M(g(y), g(y), g(v), t)\}$$

$$\max\{N(F(x, y), F(x, y), F(u, v), kt), N(F(y, x), F(y, x), F(v, u), kt)\}$$

$$\leq \max\{N(g(x), g(x), g(u), t), N(g(y), g(y), g(v), t)\}$$

that is,

$$M(F(x, y), F(x, y), F(u, v), kt) * M(F(y, x), F(y, x), F(v, u), kt)$$

$$\geq M(g(x), g(x), g(u), t) * M(g(y), g(y), g(v), t)$$

$$N(F(x, y), F(x, y), F(u, v), kt) \Diamond N(F(y, x), F(y, x), F(v, u), kt)$$

$$\leq N(g(x), g(x), g(u), t) \Diamond N(g(y), g(y), g(v), t).$$

Hence (3.1.1) and (3.1.2) hold.

Thus all the conditions of Theorem 3.1 are satisfied. Hence we can conclude that g and F have a coupled coincidence point. Here $(0, 0)$ is a coupled coincidence point of g and F in X.

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