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A RESULT ON BANACH SPACE USING COMMON LIMIT RANGE PROPERTY

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Abstract

The focus of this paper is to generate a fixed point theorem on Banach space using the concepts of CLR property and weakly compatible mappings. Further our result is substantiated by the provision of a suitable example.

1. Introduction

Fixed point theory is one of the most interesting topics of modern mathematics and might be taken as the main subject of analysis. For the past many years, fixed point theory has been evolved as the area of interest for many researchers. Husain and Latif proved some results for multi-valued contractive-type and non expansive type maps on complete metric spaces and on certain closed bounded convex subsets of Banach spaces. Further Husain and Sehgal [2] proved common fixed point theorems for a family of mappings. For the study of discontinuous and noncompatible mappings in fixed point theory a significant work has been contributed by Sharma and Deshpande [7] and Sharma, Deshpande and Tiwari [11]. Pathak, khan, M.S and Tiwari [1] established a fixed point theorem using the continuity and weakly compatible

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mappings on complete metric space. Thereafter Sushil Sharma, Bhavana Deshpande, and Alok Pandey [3] proved the some more results on Banach space. Further several theorems [4], [5], [6], [8], [9] and [10] are being generated on Banach space using various conditions. Now the aim of this paper is to prove a common fixed point theorem on Banach space using common limit range property and weakly compatible mappings. Further a suitable example is discussed to validate our theorem.

2. Preliminaries

Definition 2.1. Mappings G and H defined on Banach space X then the pair (G, H) is said to be *weakly commuting* on X if, $\| GH\alpha - HG\alpha \| \leq \| G\alpha - H\alpha \| \forall \alpha \in X$.

Definition 2.2. In a Banach space X, we define mappings G and H are *compatible* if $\| GH\alpha_k - HG\alpha_k \| = 1$ as $k \to \infty$, whenever $\{\alpha_k\}$ is a sequence in X such that $G\alpha_k = H\alpha_k = \mu$ for some $\alpha \in X$.

Definition 2.3. We define mappings G and H of a Banach space $(X, \| \cdot \|)$ in which if $G\mu = H\mu$ for some $\mu \in X$ such that $GH\mu = HG\mu$ holds then G and H are known as *weakly compatible mappings*.

Definition 2.4. We define mappings G and J of a Banach space $(X, \| \cdot \|)$ in which if for some $\mu \in X$ there exists a sequence $\{\alpha_k\}$ in X such that $\lim_{n \to \infty} G\alpha_k = \lim_{n \to \infty} J\alpha_k = J\mu$ for some, then G and J are said to satisfy the common limit in the range of J property and it is denoted by CLR_J property.

3. Main Result

The following Theorem was proved in [1].

Theorem 3.1. *Suppose X is a complete metric space, G, H, I and J are mappings defined on X holding the conditions*

(C1) $G(X) \subseteq H(X)$ *and* $I(X) \subseteq J(X)$

(C2) $d(G\alpha, I\beta)^{2p} \leq [a\phi_0(d(J\alpha, H\beta)^{2p}) + (1 - \alpha)\max\{\phi_1(d(J\alpha, H\beta)^{2p}),$

$$\phi_2(d(J\alpha, G\alpha)^q d(H\beta, I\beta)^{q'}), \phi_3(d(J\alpha, I\beta)^r d(H\beta, G\alpha)^{r'}),$$

$$\phi_4(d(j\alpha, G\alpha)^s d(H\beta, G\alpha)^{s'}), \phi_5(d(J\alpha, I\beta)^I d(H\beta, I\beta)^{l'})\}]$$

for all $\alpha, \beta \in X$, *where* $\phi_k \in \phi\, k = 0, 1, 2, 3, 4, 5, 0 \leq a \leq 1, 0 < p, q, q', r,$ $r's, s', l, l' \leq 1$ *such that* $2p = q + q' = r + r' = s + s' = l + l'$.

(C3) *either of the mappings G or I is continuous*

(C4) *the pair of mappings* (G, J) *and* (I, H) *are weakly compatible.*

Then the above mappings will be having unique common fixed point.

We prove the existence of above Theorem on Banach space under some modified conditions.

For this we need to recall the following lemmas.

Lemma 3.2 [6]**.** *If* $\phi_k \in \phi$ *and* $k \in \{0, 1, 2, 3, 4, 5\}$ *where* ϕ *is upper semi-continuous and contractive modulus such that max* $\{\phi_k(t)\} \leq \phi(t)$ *for all* $t > 0$ *and* $\phi(t) < t$ *for* $t > 0$.

Lemma 3.3 [1]**.** *Let* $\phi_j \in \phi$ *and* $\{\beta_j\}$ *be a sequence of positive real numbers. If* $\beta_{j+1} \leq \phi(\beta_j)$ *for* $j \in N$, *then the sequence converges to* 0.

Now we prove our theorem on a Banach space.

Theorem 3.4. *Suppose in a Banach space* $(X, \|\cdot\|)$, *there are four mappings* G, J, H *and I holding the conditions*

(C1) $G(X) \subseteq H(X)$ *and* $I(X) \subseteq J(X)$

(C2) $\| G\alpha - I\beta \|^{2p} \leq [a\phi_0 \| J\alpha - H\beta \|^{2p} + (1 - \alpha)\max\{\phi_1(\| J\alpha - H\beta \|^{2p}),$

$$\phi_2(\| J\alpha, G\alpha \|^q \| H\beta - I\beta \|^{q'}), \phi_3(\| J\alpha - I\beta \|^r \| H\beta - G\alpha \|^{r'}),$$

$$\phi_4(\frac{1}{2}\| J\alpha - G\alpha \|^s \| H\beta - G\alpha \|^{s'}), \phi_5(\frac{1}{2}\| J\alpha - I\beta \|^l \| H\beta - I\beta \|^{l'})\}]$$

for all $\alpha, \beta \in X$, *where* $\phi_k \in \phi\, k = 0, 1, 2, 3, 4, 5, 0 \leq a \leq 1, 0 < p, q, q', r,$

$r's, s', l, l' \leq 1$ such that $2p = q + q' = r + r' = s + s' = l + l'$.

(C3) *The pair (G, J) satisfies CLR_J property or the pair (I, H) satisfies CLR_H property*

(C4) *the pair of mappings (G, J) and (I, H) are weakly compatible.*

Then the above mappings will be having unique common fixed point.

Proof. Begin with using the condition (C1), there is a point α_0 such that $G\alpha_0 = H\alpha_1$ for some $\alpha_1 \in X$. For this point α_1 there exists a point α_2 in X such that $I\alpha_1 = J\alpha_2$ and so on.

Continuing this process, it is possible to construct a sequence $\{\beta_j\}$ for $j = 0, 1, 2, 3, \ldots,$ in X such that $\beta_{2j} = G\alpha_{2j} = H\alpha_{2j+1}, \beta_{2j+1} = I\alpha_{2j+1} = J\alpha_{2j+2}$.

We now prove $\{\beta_j\}$ is a cauchy sequence.

Putting $\alpha = \alpha_{2j}$ and $\beta = \alpha_{2j+1}$

$$\| \beta_{2j} - \beta_{2j+1} \|^{2p} \leq a\phi_0(\| \beta_{2j-1} - \beta_{2j} \|^{2p})$$

$$+ (1 - a) \max \{\phi_1(\| \beta_{2j-1} - \beta_{2j} \|^{2p}), \phi_2(\| \beta_{2j-1} - \beta_{2j} \|^{q} \| \beta_{2j} - \beta_{2j+1} \|^{q}),$$

$$\phi_3(\| \beta_{2j-1} - \beta_{2j+1} \|^{r} \| \beta_{2j} - \beta_{2j} \|^{r}), \phi_4(\frac{1}{2} [\| \beta_{2j-1} - \beta_{2j} \|^{s} \| \beta_{2j} - \beta_{2j} \|^{s'}],$$

$$\phi_5(\frac{1}{2} [\| \beta_{2j-1} - \beta_{2j+1} \|^{l} \| \beta_{2j} - \beta_{2j+1} \|^{l'}])\}$$

Denote $\rho_j = \| \beta_j - \beta_{j+1} \|$

$$(\rho_{2j})^{2p} \leq a\phi_0(\rho_{2j-1})^{2p} + (1 - a) \max \{\phi_1(\rho_{2j-1})^{2p}, \phi_2(\rho_{2j-1})^{q} (\rho_{2j})^{q},$$

$$\phi_3(0), \phi_4(0), \phi_5(\frac{1}{2} [(\rho_{2j-1})^{l} + (\rho_{2j})^{l}] (\rho_{2j})^{i})\}$$

$$\leq a\phi_0(\rho_{2j-1})^{2p} + (1 - a) \max \{\phi_1(\rho_{2j-1})^{2p}, \phi_2(\rho_{2j-1})^{q} (\rho_{2j})^{q}, \phi_3(0), \phi_4(0),$$

$\phi_5(\frac{1}{2}[(\rho_{2j-1})^{l} + (\rho_{2j})^{l'}(\rho_{2j})^{l}(\rho_{2j})^{l'}])\}$

If $\rho_{2j} > \rho_{2j-1}$ then we have

$(\rho_{2j})^{2p} \leq a\phi_0(\rho_{2j})^{2p} + (1-a)\max\{\phi_1(\rho_{2j})^{2p}, \phi_2(\rho_{2j})^{q+q}, \phi_3(0), \phi_4(0),$

$\phi_5(\frac{1}{2}[(\rho_{2j})^{l+i} + (\rho_{2j})^{l+l'}])\}(\rho_{2j})^{2p} \leq a\phi_0(\rho_{2j})^{2p} + (1-a)\max\{\phi_1(\rho_{2j})^{2p},$

$\phi_2(\rho_{2j})^{2p}, \phi_3(0), \phi_4(0), \phi_5(\rho_{2j})^{2p}\}$

using Lemma (3.2)

$$(\rho_{2j})^{2p} \leq \phi((\rho_{2j})^{2p}) < (\rho_{2j})^{2p}$$

which is a contradiction.

Thus we must have $\rho_{2j} \leq \rho_{2j-1}$.

Then using this inequality the condition (C2) yields $\rho_{2j} \leq \phi(\rho_{2j-1}), \ldots$(i)

Similarly taking $\alpha = \alpha_{2j+2}$ and $\beta = \alpha_{2j+1}$ in (C2), we get

$\| \beta_{2j+1} - \beta_{2j+2} \|^{2p} \leq a\phi_0(\| \beta_{2j} - \beta_{2+1} \|^{2p})$

$+ (1-a)\max\{\phi_1(\| \beta_{2j} - \beta_{2j+1} \|^{2p}), \phi_2(\| \beta_{2j+1} - \beta_{2j+2} \|^{q}\| \beta_{2j} - \beta_{2j+1} \|^{q}),$

$\phi_3(\| \beta_{2j+1} - \beta_{2j+1} \|^{r}\| \beta_{2j} - \beta_{2j+1} \|^{r}), \phi_4(\frac{1}{2}[\| \beta_{2j+1} - \beta_{2j+2} \|^{s}\| \beta_{2j}$

$- \beta_{2j+2} \|^{s}]), \phi_5(\frac{1}{2}[\beta_{2j+2} - \beta_{2j+1} \|^{l}\| \beta_{2j} - \beta_{2j+1} \|^{l'}])\}$

$(\rho_{2j+1})^{2p} \leq a\phi_0(\rho_{2j})^{2p} + (1-a)\max\{\phi_1(\rho_{2j})^{2p}, \phi_2(\rho_{2j+1})^{q}(\rho_{2j})^{q'}, \phi_3(0),$

$\phi_4(\frac{1}{2}[(\rho_{2j+1})^{s}(\rho_{2j})^{s'} + (\rho_{2j+1})^{s'}]), \phi_5(0)\}$

$(\rho_{2j+1})^{2p} \leq a\phi_0(\rho_{2j})^{2p} + (1-a)\max\{\phi_1(\rho_{2j})^{2p}, \phi_2(\delta_{2j+1})^{q}(\rho_{2j})^{q'}, \phi_3(0),$

$\phi_4(\frac{1}{2}[(\rho_{2j+1})^{s}(\rho_{2j})^{s'} + (\rho_{2j+1})^{s'}(\rho_{2j+1})^{s}]), \phi_5(0)$

If $\rho_{2j+1} > \rho_{2j}$, then we have

$$(\rho_{2j+1})^{2p} \leq a\phi_0(\rho_{2j+1})^{2p} + (1-a)\max\{\phi_1(\rho_{2j+1})^{2p}, \phi_2(\rho_{2j+1})^{q+q},$$

$$\phi_3(0), \phi_4(\rho_{2j+1}), \phi_5(0)\}$$

since by Lemma 3.2

$$(\rho_{2j+1})^{2p} \leq \phi(\rho_{2j+1})^{2p} < (\rho_{2j+1})^{2p}$$

which is a contradiction.

Thus we must have $\rho_{2j+1} \leq \rho_{2j}$.

Again from (C2), we obtain $\rho_{2j+1} \leq \phi(\rho_{2j}), \ldots$(ii).

From (i) and (ii), in general

$$\rho_{j+1} \leq \phi(\rho_j), \text{ for } j = 0, 1, 2, 3, \ldots$$

On using Lemma 3.3, we get $\rho_j \to 0$ as $j \to \infty$.

This shows that

$$\rho_j \| \beta_j - \beta_{j+1} \| \to 0 \text{ as } j \to \infty.$$

Hence $\{\beta_j\}$ is a cauchy sequence.

Now X being a Banach space there exists a point $\mu \in X$ such that $\beta_j \to \mu$ as $j \to \infty$. Consequently the subsequences $\{G\alpha_{2j}\}, \{J\alpha_{2j+2}\}$, $\{H\alpha_{2j+2}\}$ and $\{I\alpha_{2j+1}\}$ of $\{\beta_j\}$ also converge to the same point $\mu \in X, \ldots$(iii)

Case (i)**:** Let us suppose that the pair (G, J) satisfies CLR_J property from the condition (C3). So there is a sequence $\{\alpha_j\}$ in X such that $\lim_{n\to\infty} G\alpha_j = \lim_{n\to\infty} J\alpha_j = J\mu$ for some $\mu \in X$.

Since $G(X) \subseteq H(X)$, for each $\{\alpha_j\} \subset X$ there is a sequence $\{\beta_j\} \subset X$ in such that $G\alpha_j = H\beta_j$.

Therefore $\lim_{j\to\infty} H\beta_j = \lim_{j\to\infty} G\alpha_j = \lim_{j\to\infty} J\alpha_j = J\mu$ where $\mu \in X, \ldots$(iv)

Now we prove that $I\beta_j = J\mu$ as $j \to \infty$.

Now in the inequality (C2) putting $\alpha = \alpha_j$, $\beta = \beta_j$

$\| G\alpha_j - I\beta_j \|^{2p} \leq [a\phi_0(\| J\alpha_j - H\beta_j \|^{2p}) + (1-a)\max\{\phi_1(\| J\alpha_j - H\beta_j \|^{2p}),$

$\phi_2(\| J\alpha_j - G\alpha_j \|^q \| H\beta_j - I\beta_j \|^{q'}),\ \phi_3(\| J\alpha_j - I\beta_j \|^r \| H\beta_j - G\alpha_j \|^{r'}),$

$\phi_4(\frac{1}{2}\| J\alpha_j - G\alpha_j \|^s \| H\beta - G\alpha_j \|^{s'}),\ \phi_5(\frac{1}{2}\| J\alpha_j - I\beta_j \|^l \| H\beta_j - \beta_j \|^{l'})\}]$

$\| J\mu - I\beta_j \|^{2p} \leq [a\phi_0(\| J\mu - J\mu \|) + (1-\alpha)\max\{\phi_1(\| J\mu - J\mu \|^{2p}),$

$\phi_2(\| J\mu - J\mu \|^q \| J\mu - I\beta_j \|^{q'}),\ \phi_3(\| J\mu - I\beta_j \|^r \| J\mu - J\mu \|^{r'}),$

$\phi_4(\| J\mu - J\mu \|^s \| J\mu - J\mu \|^{s'}),\ \phi_5(\frac{1}{2}\| J\mu - I\beta_j \|^{2p})$

$\| J\mu - I\beta_j \|^{2p} \leq [a\phi_0(0) + (1-a)\max\{\phi_1(0),\ \phi_2(0),\ \phi_3(0),\ \phi_4(0),$

$\phi_5(\frac{1}{2}\| J\mu - I\beta_j \|^{2p})\}]$

On using (iv) and Lemma 3.2, we get

$$\| J\mu - I\beta_j \|^{2p} < \phi(\| J\mu - I\beta_j \|^{2p}) < J\mu - I\beta_j \|^{2p}$$

which is a contradiction.

Hence $I\beta_j = J\mu$, …(v)

Therefore $H\beta_j = G\alpha_j = I\beta_j = J\alpha_j = J\mu$ as $j \to \infty$.

Since the pair (G, J) is weakly compatible and G and J commute at a point of coincidence, that is $G\alpha_j = J\alpha_j$ and this gives $GJ\alpha_j = JG\alpha_j$ and this implies $G\mu = J\mu$.

Now we show that $G\mu = \mu$.

Putting $\alpha = \mu$ and $\beta = \beta_{2j+1}$ in (C2), we get

$$\| G\mu = I\beta_{2j+1} \|^{2p} \le [a\phi_0(\| J\mu - H\beta_{2j+1} \|^{2p}) + (1-a)\max\{\phi_1(\| J\mu - H\beta_{2j+1} \|^{2p}),$$

$$\phi_2(\| J\mu - G\mu \|^q \| H\beta_{2j+1} - I\beta_{2j+1} \|^{q'}), \phi_3(\| J\mu - I\beta_{2j+1} \|^r \| H\beta_{2j+1} - G\mu \|^{r'}),$$

$$\phi_4(\frac{1}{2} \| J\mu - G\mu \|^s \| H\beta_{2j+1} - G\mu \|^{s'}),$$

$$\phi_5(\frac{1}{2} \| J\mu - I\beta_{2j+1} \|^l \| H\beta_{2j+1} - I\beta_{2j+1} \|^{l'})]\}$$

$$\| G\mu - \mu \|^{2p} \le [a\phi_0(\| G\mu - \mu \|^{2p}) + (1-a)\max\{\phi_1(\| G\mu - \mu \|^{2p}),$$

$$\phi_2(\| G\mu - G\mu \|^q \| \mu - \mu \|^{q'}), \phi_3(\| G\mu - I\mu \|^r \| \mu - G\mu \|^{r'}),$$

$$\phi_4(\frac{1}{2} \| G - G\mu \|^s \| \mu - G\mu \|^{s'}), \phi_5(\frac{1}{2} \| G\mu - \mu \|^l \| \mu - \mu \|^{l'})\}]$$

$$\| G\mu - \mu \|^{2p} \le [a\phi_0(\| G\mu - \mu \|^{2p}) + (1-a)\max\{\phi_1(\| G\mu - \mu \|^{2p}),$$

$$\phi_2(0), \phi_3(\| G\mu - I\mu \|^{2p}), \phi_4(\frac{1}{2} \| G - G\mu \|^{2p}), \phi_5(0)\}].$$

On using (iii) and Lemma 3.2

$$\| G\mu - \mu \|^{2p} \le \phi(\| G\mu - \mu \|^{2p}) < \| G\mu - \mu \|^{2p}$$

which is a contradiction.

Hence $G\mu = \mu, \ldots$(vi)

which implies $G\mu = J\mu = \mu, \ldots$(vii)

Again since the pair (I, H) is weakly compatible, I and H commute at coincident point, that is $I\beta_j = H\beta_j$ this gives $IH\beta_j = HI\beta_j$ and this implies $I\mu = H\mu$.

Now we show that $I\mu = \mu$.

Putting $\alpha = \mu$ and $\beta = \mu$ in (C2), we get

$$\| G\mu - I\mu \|^{2p} \le [a\phi_0(\| J\mu - H\mu \|^{2p}) + (1-a)\max\{\phi_1(\| J\mu - H\mu \|^{2p}),$$

$\phi_2(\| J\mu - G\mu \|^q \| H\mu - I\mu \|^{q'}),\ \phi_3(\| J\mu - I\mu \|^r \| H\mu - G\mu \|^{r'}),$

$\phi_4(\frac{1}{2}\| J\mu - I\mu \|^s \| H\mu - G\mu \|^{s'}),\ \phi_5(\frac{1}{2}\| J\mu - I\mu \|^l \| H\mu - I\mu \|^{l'})\}],$

$\| \mu - I\mu \|^{2p} \leq [a\phi_0(\| \mu - I\mu \|^{2p}) + (1-a)\max\{\phi_1(\| \mu - I\mu \|^{2p}),$

$\phi_2(\| \mu - \mu \|^q \| I\mu - I\mu \|^{q'}),\ \phi_3(\| \mu - I\mu \|^r \| I\mu - \mu \|^{r'}),$

$\phi_4(\frac{1}{2}\| \mu - \mu \|^s \| I\mu - \mu \|^{s'}),\ \phi_5(\frac{1}{2}\| \mu - I\mu \|^l \| I\mu - I\mu \|^{l'})\}],$

$\| \mu - I\mu \|^{2p} \leq [a\phi_0(\| \mu - I\mu \|^{2p}) + (1-a)\max\{\phi_1(\| \mu - I\mu \|^{2p}),$

$\phi_2(0),\ \phi_3(\| \mu - \mu I \|^{2p}),\ \phi_4(0),\ \phi_5(0)\}]$

On using Lemma 3.2

$$\| \mu - I\mu \|^{2p} \leq \phi(\| \mu - I\mu \|^{2p}) < \| \mu - I\mu \|^{2p}$$

which is a contradiction.

Hence $I\mu = \mu$.

Therefore $I\mu = H\mu = \mu, \ldots$(viii)

From (vii) and (viii), we get

$$G\mu = J\mu = I\mu = H\mu = \mu.$$

Hence μ is a common fixed point of the four mappings.

For Uniqueness:

Suppose μ and $\mu^*(\mu \neq \mu^*)$ are common fixed points of G, H, I and J, then substitute $\alpha = \mu$ and $\beta = \mu^*$ in the inequality (C2)

$\| G\mu - I\mu^* \|^{2p} \leq [a\phi_0(\| J\mu - H\mu^* \|^{2p} + (1-a)\max\{\phi_1(\| J\mu - H\mu^* \|^{2p}),$

$\phi_2(\| J\mu - G\mu \|^q \| H\mu^* - I\mu^* \|^{q'}),\ \phi_3(\| J\mu - I\mu^* \|^r \| H\mu^* - G\mu \|^{r'}),$

$$\phi_4(\frac{1}{2}\| J\mu - G\mu \|^s \| H\mu^* - G\mu \|^{s'}),\ \phi_5\ \frac{1}{2}(\| J\mu - I\mu^* \|^l \| H\mu^* - I\mu^* \|^{l'})\}]$$

$$\| \mu - \mu^* \|^{2p} \leq [a\phi_0(\| \mu - \mu^* \|^{2p}) + (1 - a)\max\{\phi_1(\mu - \mu^* \|^{2p}),\ \phi_2(0),$$

$$\phi_3(\| \mu - \mu^* \|^r \| \mu^* - \mu \|^{r'}),\ \phi_4(0),\ \phi_5(0)\}]$$

since by Lemma 3.2

$$\| \mu - \mu^* \|^{2p} \leq \phi(\| \mu - \mu^* \|^{2p}) < \| \mu - \mu^* \|^{2p}, \text{ a contradiction.}$$

Therefore $\mu = \mu^*$.

This proves the uniqueness of the common fixed point of four mappings G, H, J and I.

Again by considering the pair (I, H) satisfying CLR_H property we can prove the theorem in case (ii).

Now we discuss an illustration to support our result.

4. Example

Suppose $X = [0, 1]$ with $\| \alpha - \beta \| = | \alpha - \beta | \forall \alpha, \beta \in X$.

Define

$$G(\alpha) = I(\alpha) = \frac{1 - \alpha}{5} \text{ if } 0 \leq \alpha \leq 1 \text{ and}$$

$$J(\alpha) = H(a) = \begin{cases} 1 - 5\alpha & \text{if } 0 \leq \alpha \leq \frac{1}{6} \\ \frac{1 - \alpha}{2} & \text{if } \frac{1}{6} < \alpha \leq 1. \end{cases}$$

Then $G(X) = I(X) = \left[0, \frac{1}{5}\right]$ while $J(X) = H(X) = \left[\frac{1}{6}, 1\right] \cup \left(0, \frac{5}{12}\right]$.

Hence the condition (C1) is satisfied.

Take a sequence $\{\alpha_k\}$ as $a_k = \frac{1}{6} - \frac{1}{k}$ for $k > 0$.

Now

$$G\alpha_k = G\left(\frac{1}{6} - \frac{1}{k}\right) = \frac{1 - \frac{1}{6} + \frac{1}{k}}{5} = \frac{1}{6} + \frac{1}{5k} = \frac{1}{6} \text{ as } k \to \infty \text{ and}$$

$$J\alpha_k = J\left(\frac{1}{6} - \frac{1}{k}\right) = 1 - 5\left(\frac{1}{6} - \frac{1}{k}\right) = \frac{1}{6} + \frac{5}{k} = \frac{1}{6} \text{ as } k \to \infty.$$

Also

$$G\alpha_k = J\alpha_k = \frac{1}{6} = J\left(\frac{1}{6}\right) \text{ as } k \to \infty \text{ and}$$

$$I\alpha_k = H\alpha_k = \frac{1}{6} = H\left(\frac{1}{6}\right) \text{ where } \frac{1}{6} \in X, \text{ as } k \to \infty.$$

Therefore the pairs (G, J) and (I, H) are satisfying CLR_J and CLR_H properties.

Further $G\left(\frac{1}{6}\right) = \frac{1}{6}$ and $J\left(\frac{1}{6}\right) = \frac{1}{6}$ which implies $G\left(\frac{1}{6}\right) = J\left(\frac{1}{6}\right)$ and $H\left(\frac{1}{6}\right) = \frac{1}{6}$ and $I\left(\frac{1}{6}\right) = \frac{1}{6}$ which implies $H\left(\frac{1}{2}\right) = J\left(\frac{1}{6}\right)$.

Further $GJ\left(\frac{1}{6}\right) = G\left(\frac{1}{6}\right) = \frac{1 - \frac{1}{6}}{5} = \frac{1}{6}$ and $IH\left(\frac{1}{6}\right) = I\left(1 - \frac{5}{6}\right) = I\left(\frac{1}{6}\right)$ $= \frac{1 - \frac{1}{6}}{5} = \frac{1}{6}$.

This implies $GJ\left(\frac{1}{6}\right) = JG\left(\frac{1}{6}\right)$ and $IH\left(\frac{1}{6}\right) = IH\left(\frac{1}{6}\right)$ which proves $(G, J), (I, H)$ are weakly compatible mappings.

We now establish that the mappings G, H, I and J satisfy the condition (C2).

Case (i):

If $\alpha, \beta \in \left[0, \frac{1}{6}\right]$, we define $\| G\alpha - I\beta \| = | G\alpha - I\beta |$.

Putting $\alpha = \frac{1}{8}$ and $\beta = \frac{1}{9}$, the inequality (C2) gives

$$\| G\left(\frac{1}{8}\right) - I\left(\frac{1}{9}\right) \|^{2p}$$

$$\leq \Bigg[a\phi_0\left(\| J\left(\frac{1}{8}\right) - H\left(\frac{1}{9}\right) \|^{2p}\right) + (1-a)\max\Bigg\{\phi_1\left(\| J\left(\frac{1}{8}\right) - H\left(\frac{1}{9}\right) \|^{2p}\right),$$

$$\phi_2\left(\| J\left(\frac{1}{8}\right) - G\left(\frac{1}{8}\right) \|^{q} \| H\left(\frac{1}{9}\right) - G\left(\frac{1}{8}\right) \|^{q'}\right),$$

$$\phi_3\left(\| J\left(\frac{1}{8}\right) - I\left(\frac{1}{9}\right) \|^{r} \| H\left(\frac{1}{9}\right) - G\left(\frac{1}{8}\right) \|^{r'}\right),$$

$$\phi_4\left(\frac{1}{2}\| J\left(\frac{1}{8}\right) - G\left(\frac{1}{8}\right) \|^{s} \| H\left(\frac{1}{9}\right) - G\left(\frac{1}{8}\right) \|^{s'}\right),$$

$$\phi_5\left(\frac{1}{2}\| J\left(\frac{1}{8}\right) - I\left(\frac{1}{9}\right) \|^{l} \| H\left(\frac{1}{9}\right) - I\left(\frac{1}{9}\right) \|^{l'}\right)\Bigg\}\Bigg]$$

For $a = \frac{1}{2}$ and taking $p = p' = q = q' = r = r' = s = s' = l = l' = \frac{1}{2}$

$$\| 0.002 \| \leq \left[\frac{1}{2}\phi_0(0.002) + \left(1 - \frac{1}{2}\right)\max\begin{Bmatrix}\phi_1(0.069),\ \phi_2(0.2244),\ \phi_3(0.2264), \\ \phi_4(0.1149),\ \phi_5(0.1149)\end{Bmatrix}\right]$$

$$| 0.002 | < | 0.1142 |$$

Case (ii): If $\alpha, \beta \in \left(\frac{1}{6}, 1\right]$, we define $\| G\alpha - I\beta \| = | G\alpha - I\beta |$.

Putting $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{2}$, then the inequality (C2) gives

$$\| G\left(\frac{1}{3}\right) - I\left(\frac{1}{2}\right) \|^{2p} \leq \Bigg[a\phi_0\left(\| J\left(\frac{1}{3}\right) - H\left(\frac{1}{2}\right) \|^{2p}\right) + (1-a)\max$$

$$\phi_1\left(\| J\left(\frac{1}{3}\right) - H\left(\frac{1}{2}\right) \|^{2p}\right), \phi_2\left(\| J\left(\frac{1}{2}\right) - G\left(\frac{1}{3}\right) \|^{q} \| H\left(\frac{1}{2}\right) - I\left(\frac{1}{2}\right) \|^{q'}\right),$$

$$\phi_3\left(\| J\left(\frac{1}{3}\right) - I\left(\frac{1}{2}\right) \|^{r} \| H\left(\frac{1}{2}\right) - G\left(\frac{1}{3}\right) \|^{r'}\right), \phi_4\left(\frac{1}{2}\| J\left(\frac{1}{3}\right) - G\left(\frac{1}{2}\right) \|^{s} \| H\left(\frac{1}{2}\right)\right.$$

$$\left. - G\left(\frac{1}{3}\right) \|^{s'}\right), \phi_5\left(\frac{1}{2}\| J\left(\frac{1}{2}\right) - I\left(\frac{1}{2}\right) \|^{l} \| \left(\frac{1}{2}\right) - I\left(\frac{1}{2}\right) \|^{l'}\right)\Bigg\}\Bigg]$$

for $a = \frac{1}{2}$ and taking $p = p' = q = q' = r = r' = s = s' = l = l' = \frac{1}{2}$

$$\| 0.033 \| \leq \left[\frac{1}{2}\phi_0(0.083) + \left(1 - \frac{1}{2}\right)\max\left\{\begin{matrix}\phi_1(0.083),\ \phi_2(0.1729),\ \phi_3(0.186), \\ \phi_4(0.0499),\ \phi_5(0.0932)\end{matrix}\right\}\right]$$

$$| 0.033 | < | 0.1345 |$$

Hence the condition (C2) is satisfied in both the cases (i) and (ii).

Further it can be observed that $\frac{1}{6}$ is unique common fixed point of the four mappings G, J, H and I.

5. Conclusion

This paper aimed on a Banach space to generate a common fixed point theorem using common limit range property and weakly compatible mappings. Further an example is discussed to validate our theorem.

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