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ON DECOMPOSITIONS OF CONTINUITY AND SOME WEAKER FORMS OF CONTINUITY VIA IDEALIZATION

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Abstract

In this paper, we give decompositions of continuity and some weaker forms of continuity via idealization using the concepts of A_{IS}- sets, B_{1IS}- sets, B_{2IS}- sets, B_{3IS}- sets, αA_{IS}- sets, αC_{IS}- sets and WLC_{IS}- sets.

1. Introduction

Ideal in topological spaces have been considered since 1966 by Kuratowski [7] and Vaidyanathaswamy [14]. After several decades, in 1990, Jankovic and Hammlet [4] investigated the topological ideals which is the generalization of general topology. Whereas in 2010, Khan and Noiri [5] introduced and studied the concept of semi local functions. In 2014, Shanthi and Ramesh Kumar [11] introduced semi-I_S-open sets, pre-I_S-open sets and $\alpha - I_S$-open sets. In this paper we introduce the notions of A_{IS}-sets, B_{1IS}-sets, B_{2IS}-sets, B_{3IS}-sets, αA_{IS}-sets, αC_{IS}-sets and WLC_{IS}-sets to

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obtain decomposition of some weaker forms of continuity. Let (X, τ) be a topological space and I is an ideal of subset of X. An ideal I on a topological space (X, τ) is a collection of nonempty subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(\cdot)^* : \wp(X) \to \wp(X)$, called the local function of A with respect to τ and I, is defined as follows: for $A \subset X$, $A^*(I, \tau) = \{x \in X / U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau / x \in U\}$ (Kuratowski 1966). A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(I, \tau)$, called the $*$-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(I, \tau)$ (Vaidyanathaswamy, 1945). When there is no chance for confusion, we will simply write A^* for $A^*(I, \tau)$ and τ^* or $\tau^*(I)$ for $\tau^*(I, \tau)$. If I is an ideal on X, then (X, τ, I) is called an ideal space. $\beta = \{G - A / G \in \tau, A \in I\}$ is a basis for τ^* (Jankovic and Hamlett, 1992). If $A \subset X$, $cl(A)$ and $\text{int}(A)$ will respectively denote the closure and the interior of A in (X, τ) and $\text{int}^*(A)$ will denote the interior of A in (X, τ^*).

Definition 1.1. Let (X, τ) be a topological space. A subset A of X is said to be semi open [8] if there exists an open set U in X such that $U \subset A \subset cl(U)$. The complement of a semi open set is said to be semi-closed. The collection of semi open (resp. semi closed) sets in X is denoted by SO(X) (resp. SC(X)). The semi closure of A in (X, τ) is denoted by the intersection of of all semi closed sets containing A and is denoted by $scl(A)$.

Definition 1.2. For $A \subset X, A_*(I, \tau) = \{x \in X / U \cap A \notin I$ for very $U \in SO(X, x)\}$ is called the semi-local function [5] of A with respect to I and τ, where $SO(X, x) = \{U \in SO(X) : x \in U\}$. We simply write A_* instead of $A_*(I, \tau)$. It is given in [1] that $\tau^{*s}(I)$ is a topology on X, generated by the sub basis $\{U - E : U \in SO(X)$ and $E \in I\}$ or equivalently $\tau^{*s}(I) = \{U \subset X : cl^{*s}(X - U) = X - U\}$. The closure operator cl^{*s} for a topology $\tau^{*s}(I)$ is defined as follows: for $A \subseteq X$, $cl^{*s}(A) = A \cup A_*$ and $\text{int}^{*s}(A)$ denote the interior of the set A in (X, τ^{*s}, I). It is known that $\tau \subset \tau^*(I) \subset \tau^{*s}(I)$. A subset A of (X, τ, I) is called semi-$*$-perfect [6] if $A = A_*$, $*$-semi dense in itself [6] (resp. semi-$*$-closed [6]) if $A \subset A_*$ (resp. $A_* \subset A$).

Lemma 1.3 [5]. *Let (X, τ, I) be an ideal space and $A, B \subset X$. Then for the semi-local function the following properties hold*:

(i) *If $A \subset B$, then $A_* \subset B_*$.*

(ii) *If $U \in \tau$, then $U \cap A_* \subset (U \cap A)_*$.*

Definition 1.4. A subset A of a topological space X is said to be

(i) α-open [10] if $A \subset \text{int}(cl(\text{int}(A)))$.

(ii) pre-open [9] if $A \subset \text{int}(cl(A))$.

(iii) semi-open [8] if $A \subset cl(\text{int}(A))$.

Definition 1.5. A subset A of an ideal space (X, τ, I) is said to be

(i) $\alpha - I$-open [3] if $A \subset \text{int}(cl^*(\text{int}(A)))$.

(ii) pre-I-open [2] if $A \subset \text{int}(cl^*(A))$.

(iii) semi-I-open [3] if $A \subset cl^*(\text{int}(A))$.

Definition 1.6. A subset A of an ideal space (X, τ, I) is said to be

(i) $\alpha - I_S$-open [11] if $A \subset \text{int}(cl^{*s}(\text{int}(A)))$.

(ii) Pre-I_S-open [11] if $A \subset \text{int}(cl^{*s}(A))$.

(iii) Semi-I_S-open [11] if $A \subset cl^{*s}(\text{int}(A))$.

(iv) $\alpha^* - I_S$-set [11] if $\text{int}(cl^{*s}(\text{int}(A))) = \text{int}(A)$.

(v) C_{IS}-set [11] if $A = U \cap V$, where $U \in \tau$ and V is an $\alpha^* - I_S$-set.

(vi) I_S - locally closed set [13] if $A = U \cap V$, where $U \in \tau$ and $V_* = V$.

The family of all $\alpha - I_S$ - open (resp. semi - I_S - open, pre - I_S - open) sets in an ideal space (X, τ, I) is denoted by $\alpha ISO(X)$ (*resp.* $SISO(X)$, $PISO(X)$).

Lemma 1.7 [11]**.** *Let* (X, τ, I) *be an ideal space and* $A \subset X$. *Then the following conditions are equivalent*:

(i) *A is open*;

(ii) *A is* $\alpha - I_S$ - *open and a* C_{IS} - *set*.

Lemma1.8 [12]**.** *Let* (X, τ, I) *be an ideal space and* $A \subset X$. *If U is open in* (X, τ, I), *then* $U \cap cl^{*s}(A) \subset cl^{*s}(U \cap A)$.

Lemma 1.9 [12]**.** *Let* (X, τ, I) *be an ideal space*

(i) *If* $V \in SISO(X)$ *and* $A \in \alpha ISO(X)$, *then* $V \cap A \in SISO(X)$

(ii) *If* $V \in PISO(X)$ *and* $A \in \alpha ISO(X)$, *then* $V \cap A \in PISO(X)$.

Lemma 1.10 [12]**.** *Let* (X, τ, I) *be an ideal space. A subset A of X is* $\alpha - I_S$ - *open if and only if it is semi* - I_S - *open and pre* - I_S - *open*.

2. A_{IS} - sets and αA_{IS} - sets

Definition 2.1. A subset *A* of an ideal space (X, τ, I) is called

(i) an A_{IS} - set if $A = U \cap V$, where *U* is open and $cl^{*s}(\text{int}(V)) = V$.

(ii) a B_{1IS} - set if $A = U \cap V$, where U is $\alpha - I_S$ - open and $cl^{*s}(\text{int}(V)) = X$.

(iii) a B_{2IS} - set if $A = U \cap V$, where *U* is $\alpha - I_S$ - open and $cl^{*s}(V) = X$.

(iv) an αA_{IS} - set if $A = U \cap V$, where U is $\alpha - I_S$ - open and $cl^{*s}(\text{int}(V)) = V$.

(v) an αC_{IS} - set if $A = U \cap V$, where U is $\alpha - I_S$ - open and $\text{int}(cl^{*s}(\text{int}(V))) \subset V$.

(vi) a WLC_{IS} - set if $A = U \cap V,$ where U is open and $cl^{*s}(V) = V.$

Proposition 2.1. *Let* (X, τ, I) *be an ideal space and* $A \subset X.$ *The following properties hold*:

(i) *If A is an* A_{IS} *- set, then A is a* C_{IS} *- set.*

(ii) *If A is a* B_{1IS} *- set, then A is a* B_{2IS} *- set.*

(iii) *If A is an* αA_{IS} *- set, then A is a* αC_{IS} *- set.*

(iv) *If A is an* A_{IS} *- set, then A is an* αA_{IS} *- set.*

(v) *If A is an* I_s *- locally closed set, then A is a* WLC_{IS} *- set.*

Proof. Obvious.

Remark 2.1. Converse of the Proposition 2.1 need not be true as seen from the following examples.

Example 2.1. Let $X = \{a, b, c, d\}, \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\phi, \{b\}, \{c\}, \{b, c\}\}.$ Then (i) $A = \{b, c, d\}$ is a C_{IS} - set but it is not an A_{IS} - set.

(ii) $A = \{a, b, c\}$ is a B_{2IS} - set but it is not a B_{1IS} - set.

(iii) $A = \{b, d\}$ is a αC_{IS} - set but it is not an αA_{IS} - set.

(iv) $A = \{b, c\}$ is a WLC_{IS} - set but it is not a I_S - locally closed set.

Example 2.2. Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{a, c\}, X\}$ and $I = \{\phi\}.$ Then $A = \{a, b\}$ is an αA_{IS} - set but it is not an A_{IS} - set.

Theorem 2.1. *Let* (X, τ, I) *be an ideal space and* $A \subset X.$ *Then the following conditions are equivalent*:

(i) *A is open*;

(ii) *A is* $\alpha - I_s$ *- open and an* A_{IS} *- set.*

Proof. (i) $\Rightarrow$ (ii) If A is open then $A = \text{int}(A) \subset cl^{*s}(\text{int}(A)) \subset \text{int}(cl^{*s}(\text{int}(A))).$

Hence A is $\alpha - I_s$-open. Since $A = A \cap X$ where A is an open set and $cl^{*s}(\text{int}(X)) = X$, A is an A_{IS}-set.

(ii) $\Rightarrow$ (i) By Lemma 1.7, [Proposition 4.16 of [11]] A is $\alpha - I_s$-open and a C_{IS}-set if and only if A is open. Thus applying Proposition 2.1, A is a C_{IS}-set. Therefore, it follows that A is open.

Proposition 2.2. *A subset A is an A_{IS}-set in an ideal space (X, τ, I) if and only if it is Semi-I_s-open and a WLC_{IS}-set.*

Proof. Necessity: Let A be an A_{IS}-set. Then $A = U \cap V$, where $U \in \tau$ and $cl^{*s}(\text{int}(V)) = V$. By Lemma 1.8, we have $A = U \cap cl^{*s}(\text{int}(V)) \subset cl^{*s}(U \cap \text{int}(V)) = cl^{*s}(\text{int}(U \cap V)) = cl^{*s}(\text{int}(A))$. This shows that A is semi -I_S-open. Moreover, $cl^{*s}(V) = cl^{*s}(cl^{*s}(\text{int}(V))) = cl^{*s}(\text{int}(V)) = V$ and hence $A = U \cap cl^{*s}(V)$. Therefore, A is a WLC_{IS}-set.

Sufficiency: Suppose that A is Semi-I_S-open and a WLC_{IS}-set. Then $A = U \cap V$, where U is open and. $cl^{*s}(V) = V$. Since $A = U \cap V$, $A \subset U$, $A \subset V$ and hence $A \subset U \cap cl^{*s}(A) \subset U \cap cl^{*s}(V) = U \cap V = A$. Therefore, we have $A = U \cap cl^{*s}(A)$. Next, since A is Semi-I_s-open, we have $cl^{*s}(\text{int}(A)) \supset A$ and $cl^{*s}(\text{int}(A)) \supset cl^{*s}(A)$. Therefore, we obtain $cl^{*s}(A) \supset cl^{*s}(\text{int}(cl^{*s}(A))) \supset cl^{*s}(\text{int}(A)) \supset cl^{*s}(A)$ and hence $cl^{*s}(\text{int}(cl^{*s}(A))) = cl^{*s}(A)$. This shows that A is an A_{IS}-set.

Proposition 2.3. *For an ideal space (X, τ, I), every αA_{IS}-set is semi -I_S-open.*

Proof. Let A be an αA_{IS}-set in (X, τ, I). Then $A = U \cap V$, where U is $\alpha - I_S$-open and $cl^{*s}(\text{int}(V)) = V$. Therefore V is semi-I_S-open. By Lemma 1.9 (i), A is a semi-I_S-open set.

Theorem 2.2. *Let (X, τ, I) be an ideal space and $A \subset X$. Then A is an A_{IS}-set if and only if A is an αA_{IS}-set and a WLC_{IS}-set.*

Proof. Necessity: By Proposition 2.1, every A_{IS} - set is an αA_{IS} - set. By Proposition 2.2, A is a WLC_{IS} - set.

Sufficiency: Let A be an αA_{IS} - set and a WLC_{IS} - set in (X, τ, I). By Proposition 2.3, A is also Semi - I_S - open. Thus it follows from Proposition 2.2 that A is an A_{IS} - set.

Proposition 2.4. *For an ideal space* (X, τ, I), *every* B_{2IS} *- set is pre -* I_S *- open.*

Proof. Let A be a B_{2IS} - set. Then $A = U \cap V$, where U is $\alpha - I_S$ - open and $cl^{*s}(V) = X$ and hence V is pre - I_S - open. By Lemma 1.9 (ii), A is a pre - I_S - open set.

Theorem 2.3. *Let* (X, τ, I) *be an ideal space and* $A \subset X$. *Then A is* $\alpha - I_S$ *- open if and only if A is a* B_{2IS} *- set and an* αA_{IS} *- set.*

Proof. The necessity is obvious.

Sufficiency: Let A be a B_{2IS} - set and an αA_{IS} - set. By Proposition 2.4, A is pre - I_S - open. By Proposition 2.3, A is also semi - I_S - open. By Lemma 1.10, A is $\alpha - I_S$ - open.

Theorem 2.4. *Let* (X, τ, I) *be an ideal space and* $A \subset X$. *Then the following conditions are equivalent*:

(i) *A is open*;

(ii) *A is a* B_{2IS} *- set, an* αA_{IS} *- set and a* WLC_{IS} *- set.*

Proof. A is open if and only if A is $\alpha - I_s$ - open and an A_{IS} - set by Theorem 2.1. By Theorem 2.2, A is A_{IS} - set if and only if A is an αA_{IS} - set and a WLC_{IS} - set. Thus it follows from the Theorem 2.3 that A is open if and only if A is a B_{2IS} - set, an αA_{IS} - set and a WLC_{IS} - set.

3. Decompositions of Continuity

Definition 3.1. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be $\alpha - I$ - continuous [3] (resp. semi - I - continuous [3], pre - I - continuous [2]) if for every $V \in \sigma, f^{-1}(V)$ is an $\alpha - I$-open set (resp. semi-I-open set, pre-I-open set) of (X, τ, I).

Definition 3.2. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be $\alpha - I_S$-continuous [11] (resp. semi-I_S-continuous [11], pre-I_S-continuous [11]) if for every $V \in \sigma, f^{-1}(V)$ is an $\alpha - I_S$-open set (resp. semi-I_S-open set, pre-I_S-open set) of (X, τ, I).

Definition 3.3. A function $f : (X, \tau, I) \to (Y, \sigma)$ is A_{IS}-continuous, (resp. B_{1IS}-continuous, B_{2IS}-continuous, αA_{IS}-continuous, αC_{IS}-continuous, weakly $-I_S - LC$-continuous) if for every $V \in \sigma, f^{-1}(V)$ is an A_{IS}-set (resp. a B_{1IS}-set, a B_{2IS}-set, an αA_{IS}-set, an αC_{IS}-set, a WLC_{IS}-set) of (X, τ, I).

Theorem 3.1. *Let (X, τ, I) be an ideal space. For a function $f : (X, \tau, I) \to (Y, \sigma)$ the following conditions are equivalent*:

(i) *f is $\alpha - I_S$-continuous*;

(ii) *f is semi-I_S-continuous and pre-I_S-continuous*;

(iii) *f is B_{2IS}-continuous and αA_{IS}-continuous.*

Proof. This is an immediate consequence of Lemma 1.10 and Theorem 2.3.

Theorem 3.2. *Let (X, τ, I) be an ideal space. For a function $f : (X, \tau, I) \to (Y, \sigma)$ the following conditions are equivalent*:

(i) *f is A_{IS}-continuous*;

(ii) *f is semi-I_S-continuous and weakly $-I_S - LC$-continuous*;

(iii) *f is αA_{IS}- continuous and weakly $-I_S - LC$-continuous.*

Proof. This is an immediate consequence of Proposition 2.2 and Theorem 2.2.

Theorem 3.3. *Let* (X, τ, I) *be an ideal space. For a function* $f : (X, \tau, I) \to (Y, \sigma)$ *the following conditions are equivalent*:

(i) *f is continuous*;

(ii) *f is* $\alpha - I_S$*- continuous and* A_{IS}*- continuous*;

(iii) *f* is B_{2IS}*- continuous*, αA_{IS}*- continuous and weakly* $-I_S - LC$*- continuous*.

Proof. This is an immediate consequence of Theorem 2.1 and Theorem 2.4.

References

[1] M. E. Abd El-Monsef, E. F. Lashien and A. A. Nasef, Some topological operators via ideals, Kyungpook Math. J., 32(2) (1992), 273-284.

[2] J. Dontchev, Idealization of Ganster-Reilly decomposition theorems, http://arxiv.org/abs/Math.GN/9901017,5 Jan.1999

[3] E. Hatir and T. Noiri, On decompositions of continuity via idealization, Acta. Math. Hunger. 96 (2002), 341-349.

[3] D. Jankovic and T. R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly, 97 (1990), 295-310.

[4] M. Khan and T. Noiri, Semi-local functions in ideal topological spaces, J. Adv. Res. Pure Math. 2 (2010), 36-42.

[5] M. Khan and T. Noiri, On *g-I* closed sets in ideal topological spaces, J. Adv. Stud. in Top.,(2010), 29-33.

[6] K. Kuratowski, Topology, Vol.I, Academicpress, New York, 1966.

[7] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.

[8] A. S. Mashhour, M. E. Abd. El-Monsef and S. N. El-deeb, On pre-continuous and weak pre continuous mappings, Pro. Math. Phys. Soc. Egypt, 53 (1982), 47-53.

[9] O. Njasted, On some nearly open sets, Pacific J. Math. 15 (1965), 961-970.

[10] R. Shanthi and M. Rameshkumar, A decomposition of continuity in ideal topological spaces by using semi-local functions, Asian J. Math. Appl. (2014), 1-11.

[11] R. Shanthi and M. Rameshkumar, On $\alpha - I_s$- open sets and $\alpha - I_s$- continuous functions, J. Math. Comput. Sci. 5(5) (2015), 615-625.

[12] R. Shanthi and M. Ramesh Kumar, f_{Is} and Regular -I_s- closed sets in Ideal Topological Spaces, Int. J. Pure and Appl. Math 113(13) (2017), 163-171.

[13] R. Vaidyanathaswamy, Set Topology, Chelsea Publishing Company, 1960.