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ON DRASTIC SUM AND DRASTIC PRODUCT OF TWO FUZZY GRAPHS

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Abstract

In this paper, drastic product of two fuzzy graphs is defined and some of its properties are studied with examples. Effective property and regular property of drastic product of two fuzzy graphs are studied and degree of vertex in drastic product of two fuzzy graphs is given. Truncations of drastic sum of two fuzzy graphs are discussed.

1. Introduction

Azriel Rosenfeld introduced and studied certain properties of fuzzy graphs in 1975 [9]. Fuzzy graphs have vast range of applications. J. N. Mordeson and C. S. Peng introduced operations on fuzzy graphs and studied some of the operations and their properties [3]. Regular properties of fuzzy graphs [7] and properties of truncations of fuzzy graphs are given by A. Nagoorgani and K. Radha [8].

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1. Preliminaries

Definition 1.1 [9]**.** A fuzzy graph G is a pair of functions $G : (\sigma, \mu)$ where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ satisfying the condition $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$. The underlying crisp graph of $G : (\sigma, \mu)$ is denoted by $G^*(V, E)$ where $E \subseteq V \times V$.

Definition 1.2 [9]**.** The degree of a vertex u is defined as $d_G(u) = \sum_{u \neq v} \mu(uv)$. This can also be expressed as $d_G(u) = \sum_{uv \in E} \mu(uv)$.

Definition 1.3 [7]**.** Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. If $d_G(v) = k$ for all $v \in V$, that is, if each vertex has same degree k, then G is said to be a regular fuzzy graph of degree k or a k-regular fuzzy graph.

Definition 1.4 [9]**.** Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. A fuzzy graph $H : (\alpha, \beta)$ on $H^*(V', E')$ is said to a fuzzy subgraph of $G : (\sigma, \mu)$ if it satisfies $\alpha(u) \leq \sigma(u)$ for all u in V' and $\beta(uv) \leq \mu(uv)$ for all uv in E'.

Definition 1.5 [4]**.** The lower and upper truncations of σ at level t, $0 < t \leq 1$, are the fuzzy subsets $\sigma_{(t)}$ and $\sigma^{(t)}$ defined respectively by,

$$\sigma_{(t)}(u) = \begin{cases} \sigma(u), & \text{if } u \in \sigma^t \\ 0, & \text{if } u \notin \sigma^t \end{cases} \text{ and } \sigma^{(t)}(u) = \begin{cases} t, & \text{if } u \in \sigma^t \\ \sigma(u), & \text{if } u \notin \sigma^t. \end{cases}$$

Definition 1.6 [5]**.** Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying crisp graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. The drastic sum of G_1 and G_2 is a fuzzy graph $(\sigma_1 \dot{\cup} \sigma_2, \mu_1 \dot{\cup} \mu_2)$ on $(V_1 \cup V_2, E_1 \cup E_2)$ defined by

$$(\sigma_1 \dot{\cup} \sigma_2)(u) = \begin{cases} \sigma_1(u), & \text{if } u \in V_1 \text{ and } u \notin V_2 \\ \sigma_2(u), & \text{if } u \in V_2 \text{ and } u \notin V_1 \\ 1, & \text{if } u \in V_1 \cap V_2 \end{cases}$$

$$(\mu_1 \dot{\cup} \mu_2)(uv) = \begin{cases} \mu_1(uv), \text{ if } uv \in E_1 \text{ and } uv \notin E_2 \\ \mu_1(u), \text{ if } uv \in E_1 \text{ and } uv \notin E_2 \\ 1, \text{ if } uv \in E_1 \cap E_2. \end{cases}$$

Definition 1.7 [6]**.** Drastic product of two fuzzy sets A_1 and A_2 is defined as

$$\mu_{A_1 \dot{\cap} A_2}(x) = \begin{cases} \mu_{A_1}(x), \text{ if } \mu_{A_2}(x) = 1 \\ \mu_{A_2}(x), \text{ if } \mu_{A_1}(x) = 1 \\ 0, \text{ if } \mu_{A_1}(x), \mu_{A_2}(x) < 1. \end{cases}$$

Definition 1.8 [2]**.** A fuzzy set μ_A of A is called normalized fuzzy set if $\mu_A(x) = 1$.

2. Drastic Product of Two Fuzzy Graphs

Definition 2.1. Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying crisp graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. Define $\sigma_1 \dot{\cap} \sigma_2$ and $\mu_1 \dot{\cap} \mu_2$ on $V_1 \cap V_2$ and $E_1 \cap E_2$ respectively by

$$(\sigma_1 \dot{\cap} \sigma_2)(u) = \begin{cases} \sigma_1(u), \text{ if } \sigma_2(u) = 1 \\ \sigma_2(u), \text{ if } \sigma_1(u) = 1 \\ 0, \text{ if } \sigma_1(u), \sigma_2(u) < 1 \end{cases}$$

and

$$(\mu_1 \dot{\cap} \mu_2)(uv) = \begin{cases} \mu_1(uv), \text{ if } \mu_2(uv) = 1 \\ \mu_2(uv), \text{ if } \mu_1(uv) = 1 \\ 0, \text{ if } \mu_1(uv), \mu_2(uv) < 1 \end{cases}$$

Now we have to prove that the drastic product of two fuzzy graphs is also a fuzzy graph. For that we have to prove that $(\mu_1 \dot{\cap} \mu_2)(uv) \le (\sigma_1 \dot{\cap} \sigma_2)(u) \wedge (\sigma_1 \dot{\cap} \sigma_2)(v)$ for all $uv \in E$ and $u, v \in V$. Consider three cases (1) $\mu_1(uv) = 1$, (2) $\mu_2(uv) = 1$ and (3) $\mu_1(uv), \mu_2(uv) < 1$.

Case 1. $\mu_1(uv) = 1$.

$$(\mu_1 \dot{\cap} \mu_2)(uv) = \mu_2(uv).$$

Since $\mu_1(uv)=1$, we have $\sigma_1(u)-1, \sigma_1(v)=1$ and therefore $(\sigma_1 \dot{\cap} \sigma_2)(u)=\sigma_2(u)$ and $(\sigma_1 \dot{\cap} \sigma_2)(v)=\sigma_2(v)$. $(\mu_1 \dot{\cap} \mu_2)(uv)=\mu_2(uv) \leq \sigma_2(u) \wedge \sigma_2(v)=(\sigma_1 \dot{\cap} \sigma_2)(u) \wedge (\sigma_1 \dot{\cap} \sigma_2)(v)$.

Case 2. $\mu_2(uv)=1$. The proof is same as in case (1).

Case 3. $\mu_1(uv), \mu_2(uv)<1$. $(\mu_1 \dot{\cap} \mu_2)(uv)=0$.

Therefore, $(\mu_1 \dot{\cap} \mu_2)(uv) \leq (\sigma_1 \dot{\cap} \sigma_2)(u) \wedge (\sigma_1 \dot{\cap} \sigma_2)(v)$.

Hence, $(\sigma_1 \dot{\cap} \sigma_2, \mu_1 \dot{\cap} \mu_2)$ is a fuzzy graph on $(V_1 \cap V_2, E_1 \cap E_2)$. This is called the drastic product of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$.

Remark. The drastic product of two fuzzy sets can be defined only when $\sigma_1, \sigma_2, \mu_1$ and μ_2 are normalized fuzzy sets.

Example 2.2

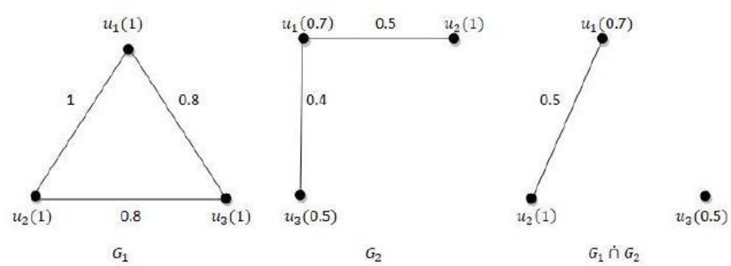

Figure 2.1.

In this example, G_1 and G_2 are two fuzzy graphs. Here, $\sigma_1(u_i)=1, i=1, 2, 3$ and $\mu_1(u_1u_2)=1$. Therefore, $(\sigma_1 \dot{\cap} \sigma_2)(u_i)=\sigma_2(u_i)$ for all u_i and $(\mu_1 \dot{\cap} \mu_2)(u_1u_2)=\mu_2(u_1u_2)$.

Example 2.3.

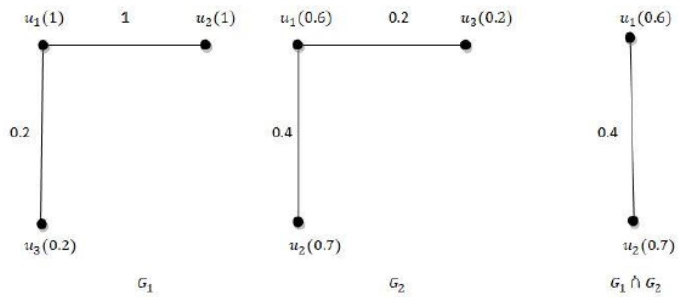

Figure 2.2.

In this graph, $\sigma_1(u_i) = 1,\ i = 1,\ 2$ and $\mu_1(u_1u_2) = 1$. Therefore, $(\sigma_1 \dot{\cap} \sigma_2)(u_i) = \sigma_2(u_i)$ for $i = 1,\ 2$ and $(\mu_1 \dot{\cap} \mu_2)(u_1u_2) = \mu_2(u_1u_2)$.

3. Degrees of Vertices of Drastic Product of Two Fuzzy Graphs

In this section, we obtain the formulae for finding the degrees of vertices of two fuzzy graphs in terms of degrees of the two fuzzy graphs.

Theorem 3.1. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs. The degree of vertex in the drastic product of two fuzzy graphs is given by*

$$d_{G_1 \dot{\cap} G_2}(u) = \sum_{\mu_1(uv)=1} \mu_2(uv) + \sum_{\mu_2(uv)=1} \mu_1(uv) - m,$$

where m is the number of edges in $E_1 \cap E_2$ such that $\mu_1(uv) = \mu_2(uv) = 1$.

Proof. Let G_1 and G_2 be two fuzzy graphs with underlying crisp graphs G_1^* and G_2^*. By definition,

$$d_{G_1 \dot{\cap} G_2}(u) = \sum_{uv \in E_1 \cap E_2} (\mu_1 \dot{\cap} \mu_2)(uv)$$

$$= \sum_{\mu_1(uv)=1} \mu_2(uv) + \sum_{\mu_2(uv)=1} \mu_1(uv) - \sum_{\mu_1(uv)=\mu_2(uv)=1} 1.$$

Since the value 1 corresponding to $\mu_1(uv) - \mu_2(uv) = 1$ appears in both of the first two sums,

$$d_{G_1 \dot{\cap} G_2}(u) = \sum_{\mu_1(uv)=1} \mu_2(uv) + \sum_{\mu_2(uv)=1} \mu_1(uv) - m.$$

Theorem 3.2. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying crisp graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. Let $u \in V_1 \cap V_2$. If $\mu_1(uv) = 1$ and $\mu_2(uv) < 1$ for every edge $uv \in E_1 \cap E_2$, $d_{G_1 \dot{\cap} G_2}(u) = d_{G_1 \cap G_2}(u)$.*

Proof. Since $\mu_1(uv) = 1$ and $\mu_2(uv) < 1$ for every edge $uv \in E_1 \cap E_2$,

$$d_{G_1 \dot{\cap} G_2}(u) = \sum_{\mu_1(uv)=1} \mu_2(uv) = \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \wedge \mu_2 = d_{G_1 \cap G_2}(u).$$

Theorem 3.3. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying crisp graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. Let $u \in V_1 \cap V_2$. If $\mu_i(uv) < 1$, $i = 1, 2$ for every edge $uv \in E_1 \cap E_2$, then $d_{G_1 \dot{\cap} G_2}(u) = 0$ or u is an isolated vertex.*

Proof. Let $\mu_i(uv) < 1$, $i = 1, 2$ for every edge $uv \in E_1 \cap E_2$. Then, $(\mu_1 \dot{\cap} \mu_2)(uv) = 0$ for every edge $uv \in E_1 \cap E_2$. Therefore, $d_{G_1 \dot{\cap} G_2}(u) = 0$.

Example 3.2. In example 2.2, $\mu_1(u_1 u_2) = 1$. Therefore degree of the vertices u_1 and u_2 is $d_{G_2}(u_1)$ and $d_{G_2}(u_2)$ respectively.

4. Regular Properties of Drastic Product of Two Fuzzy Graphs

In this section we study some regular properties of drastic product of two fuzzy graphs.

Theorem 4.1. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying crisp graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. Let $\mu_1(e) = 1$ and $\mu_2(e) < 1$ for every edge $uv \in E_1 \cap E_2$. Then $G_1 \dot{\cap} G_2$ is regular fuzzy graph if and only if $G_1 \cap G_2$ is a regular fuzzy graph.*

Proof. By Theorem 3.2, if $u \in V_1 \cap V_2$ with $\mu_1(uv) = 1$ and $\mu_2(uv) = 1$ for every edge $uv \in E_1 \cap E_2$, then $d_{G_1 \dot{\cap} G_2}(u) = d_{G_1 \cap G_2}(u)$.

Therefore $G_1 \dot{\cap} G_2$ is regular fuzzy graph if and only if $G_1 \cap G_2$ is a regular fuzzy graph.

5. Truncations on Drastic Sum of Two Fuzzy Graphs

Theorem 5.1. *$G_{1(t)} \dot{\cup} G_{2(t)}$ is a fuzzy subgraph of $(G_1 \dot{\cup} G_2)_{(t)}$.*

Proof. First we prove that $(\sigma_1 \dot{\cup} \sigma_2)_{(t)} \geq \sigma_{1(t)} \dot{\cup} \sigma_{2(t)}$.

If $u \in V_1 - V_2 (\sigma_1 \dot{\cup} \sigma_2)_{(t)}(u) = \sigma_{1(t)}(u) = (\sigma_{1(t)} \dot{\cup} \sigma_{2(t)})_{(t)})(u)$.

If $u \in V_2$, $(\sigma_1 \dot{\cup} \sigma_2)_{(t)}(u) = \sigma_{2(t)}(u) = (\sigma_{1(t)} \dot{\cup} \sigma_{2(t)})_{(t)})(u)$.

Let $u \in V_1 \cap V_2$. Then $(\sigma_1 \dot{\cup} \sigma_2)(u) = 1$ gives $(\sigma_1 \dot{\cup} \sigma_2)_{(t)}(u) = 1$.

Hence $(\sigma_1 \dot{\cup} \sigma_2)_{(t)} \geq \sigma_{1(t)} \dot{\cup} \sigma_{2(t)}$.

Next we prove that $\mu_{1(t)} \dot{\cup} \mu_{2(t)} \leq (\mu_1 \dot{\cup} \mu_2)_{(t)}$.

For this, we consider the following three cases:

Case 1. $u \in E_1 \cap E_2$ with either $\mu_1(uv) \geq t$ or $\mu_2(uv) \geq t$ but not both.

Suppose that $\mu_1(uv) \geq t$. Then $\mu_2(uv) \geq t$. So $\mu_{1(t)}(uv) = \mu_1(uv)$ and $\mu_{2(t)}(uv) = 0$.

Hence the edge uv will be in $G_{1(t)} \dot{\cup} G_{2(t)}$ with $(\mu_{1(t)} \dot{\cup} \mu_{2(t)})(uv) = \mu_1(uv)$.

Since $uv \in E_1 \cap E_2$, $(\mu_1 \dot{\cup} \mu_2)(uv) = 1 \Rightarrow (\mu_1 \dot{\cup} \mu_2)_{(t)}(uv) = 1$.

Therefore $(\mu_{1(t)} \dot{\cup} \mu_{2(t)})(uv) \leq (\mu_1 \dot{\cup} \mu_2)_{(t)}(uv)$. The proof is similar if $\mu_2(uv) \geq t$.

Case 2. $uv \in E_1 \cap E_2$ with either $\mu_1(uv) < t$, $\mu_2(uv) < t$ or $\mu_1(uv) \geq t$, $\mu_2(uv) \geq t$. Since $uv \in E_1 \cap E_2$, $(\mu_1 \dot{\cup} \mu_2)_{(t)}(uv) = 1$.

If $\mu_1(uv) < t$, then $\mu_{1(t)}(uv) = 0$ and if $\mu_2(uv) < t$, then $\mu_{2(t)}(uv) = 0$.

So $(\mu_{1(t)} \dot{\cup} \mu_{2(t)})(uv) = 0$. If $\mu_1(uv) \geq t$, then $\mu_{1(t)}(uv) = \mu_1(uv) > 0$ and if $\mu_2(uv) \geq t$, then $\mu_{2(t)}(uv) = \mu_2(uv) > 0$.

So $(\mu_{1(t)} \dot{\cup} \mu_{2(t)})(uv) = 1$.

Hence $(\mu_1 \dot{\cup} \mu_2)_{(t)} \geq \mu_{1(t)} \dot{\cup} \mu_{2(t)}$.

Case 3. $uv \in E_1$ or $uv \in E_2$ but not both.

If $uv \in E_1$, $(\mu_1 \dot{\cup} \mu_2)(uv) = \mu_1(uv)$ and if $uv \in E_2$, $(\mu_1 \dot{\cup} \mu_2)(uv) = \mu_2(uv)$.

Hence $(\mu_1 \dot{\cup} \mu_2)_{(t)}(uv) = \mu_{i(t)}(uv) = (\mu_{1(t)} \dot{\cup} \mu_{2(t)})(uv)$, for $i = 1, 2$.

From the above three cases, we get $\mu_{1(t)} \dot{\cup} \mu_{2(t)} \leq (\mu_1 \dot{\cup} \mu_2)_{(t)}$.

Hence $G_{1(t)} \dot{\cup} G_{2(t)}$ is a fuzzy sub graph of $(G_1 \dot{\cup} G_2)_{(t)}$.

Theorem 5.2. $(G_1 \dot{\cup} G_2)^{(t)}$ *is a fuzzy subgraph of* $G_1^{(t)} \dot{\cup} G_2^{(t)}$.

Proof. Let G_1 and G_2 be two fuzzy graphs.

First let us prove that $\sigma_1^{(t)} \dot{\cup} \sigma_2^{(t)} \geq (\sigma_1 \dot{\cup} \sigma_2)^{(t)}$. For that consider three cases.

Case 1. Let $u \in V_1 - V_2$.

$$(\sigma_1 \dot{\cup} \sigma_2)(u) = \sigma_1(u) \cdot (\sigma_1 \dot{\cup} \sigma_2)^{(t)}(u) = \sigma_1^{(t)}(u) = \sigma_1^{(t)}(u) \dot{\cup} \sigma_2^{(t)}(u).$$

Case 2. Let $u \in V_2 - V_1$.

$$(\sigma_1 \dot{\cup} \sigma_2)(u) = \sigma_2(u) \cdot (\sigma_1 \dot{\cup} \sigma_2)^{(t)}(u) = \sigma_2^{(t)}(u) = \sigma_1^{(t)}(u) \dot{\cup} \sigma_2^{(t)}(u).$$

Case 3. Let $u \in V_1 \cap V_2$.

$$(\sigma_1 \dot{\cup} \sigma_2)(u) = 1.\ (\sigma_1 \dot{\cup} \sigma_2)^{(t)}(u) = t.\ \sigma_2^{(t)}(u) \dot{\cup} \sigma_2^{(t)}(u) = 1.$$

Let us prove that $\mu_1^{(t)} \dot{\cup} \mu_2^{(t)} \geq (\mu_1 \dot{\cup} \mu_2)^{(t)}$.

Case 1. Let $uv \in E_1 - E_2$.

$$(\mu_1 \dot{\cup} \mu_2)(uv) = \mu_1(uv) \cdot (\mu_1 \dot{\cup} \mu_2)^{(t)}(uv) = \mu_1^{(t)}(uv) = \mu_1^{(t)}(uv) \dot{\cup} \mu_2^{(t)}(uv).$$

Case 2. Let $uv \in E_2 - E_1$.

$$(\mu_1 \dot{\cup} \mu_2)(uv) = \mu_2(uv) \cdot (\mu_1 \dot{\cup} \mu_2)^{(t)}(uv) = \mu_2^{(t)}(uv) = \mu_1^{(t)}(uv) \dot{\cup} \mu_2^{(t)}(uv).$$

Case 3. Let $uv \in E_1 \cap E_2$.

$$(\mu_1 \dot{\cup} \mu_2)(uv) = 1 \cdot (\mu_1 \dot{\cup} \mu_2)^{(t)}(uv) = t.\ \mu_1^{(t)}(uv) \dot{\cup} \mu_2^{(t)}(uv) = 1.$$

Conclusion

In this paper, the drastic product of two fuzzy graphs is introduced and certain properties are studied with examples. The regular property of drastic product of fuzzy graphs is studied. A formula for finding the degrees of vertices in drastic product of fuzzy graphs is given. Some properties of truncations of drastic sum of two fuzzy graphs are discussed.

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