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MODELING OF HARVESTING OF PREY-PREDATOR FISHERY IN THE PRESENCE OF TOXICITY WITH A MODIFIED CATCH RATE FUNCTION

SUDIPTA SARKAR, ASHIS KUMAR SARKAR and KRIPASINDHU CHAUDHURI

Department of Mathematics
Heritage Institute of Technology, Anandapur
Kolkata-700107, West Bengal, India
Email: sudipta.jumath@gmail.com

Department of Mathematics
Jadavpur University
Kolkata-700032, India
Email: aksarkar.jumath@gmail.com

Department of Mathematics
Jadavpur University
Kolkata-700032, India
Email: k s chaudhuri@yahoo.com

Abstract

In this paper we consider a non-linear model to analyse the dynamics of a prey predator fishery resource system in an aquatic environment in which both the species are infected by the toxicants released by some other species subjected to bio-economic combined harvesting. Bio-economic harvesting of prey predator fish species where each of species is affected by the toxicants released from other resources is discussed here by using modified catch rate function. The equilibria, stability, bionomic equilibrium and optimal harvesting policy by using Pontryagin's maximal principle have been established. We have derived that the dynamical behavior of the fish species will be much sensitive to the system parameters and their initial population densities. Some numerical simulations and the corresponding solution curves are cited to illustrate the theoretical results of the proposed model. Finally, the existence of limit cycle is shown here.

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1. Introduction

Bio-economic modeling is concerned with scientific management of the exploitation of renewable resources like fisheries, foresties, wild life management etc. The growing human requires of more food and more energy have led to increased exploitation of these resources. Besides these there are so many reasons of exploitation of renewable resources, as for example civilization, industrial purpose etc. In bio-economic modeling our objective is to find out the way how to manage the exploitation and extinction of renewable resources. Recently researchers are interested in harvesting of multispecies fisheries concerning this problem. Overfishing is not only the cause of extinction of fish population but also there are many other reasons, like toxicant substances, intra and inter species competition among the fish species etc. But as with the growing human needs industries are also producing a huge amount of toxicants part of which are accumulated in marine water, the species living in that marina environment will be highly affected by these toxicants. Our main objective is to study the effect of toxicity among fish species released by each of them and comes from other sources, like industries, agricultural field etc.

The effect of toxicants on biological communities have become a major environmental problem in the recent decades. Mathematical modeling related with such ecotoxicological problems were started with the studies of Hallam and Clark [6], Hallam et al. [8], Hallam and De Luna [9], De luna and Hallam [2], Freedman and Shukla [5] and others. Some other mathematical studies in this field were carried out by Chattopadhyay [1], Shukla and Dubey [17], Mukhopadhyay et al. [13], Dubey and Hussain [4], Shukla et al. [16] etc. Most of these models deal with general single species or two species biological communities without any special significance on either terrestrial or aquatic environment.

In recent times researchers are taking initiative in the ecotoxicological effects of toxicants released by the marine biological species themselves. The toxin released by one species not only affects that species but also may affect the growth of other species. Maynard Smith [12] investigated the effects of toxic substances in two species. Maynard Smith [12] proposed the effects of toxicants in two species Lotka-Volterra competitive model by considering the fact that each species releases a substance toxic to the other only when the other is present. Chattopadhyay [1] analysed the stability properties of the above system, although the study contains the aw of ignoring an important delay factor in the system. In reality, a species requires some time to achieve a level of maturity for producing the toxicants. The model of Chattopadhyay [1] was revised by Mukhopadhyay et al. [13] by taking the delay factor into consideration.

However, the effects of toxicants on the fish species have become problems of major environmental concern. Mathematical modeling in dealing with such problems were started with the studies of Kar and Chaudhuri [10], Kar, Pahari and Chaudhuri [11] etc. Most of the models deal with general single species or two species fishery models without any special emphasis on aquatic environments.

In recent times researchers are taking initiative in the ecotoxicological effects of toxicants released by the marine biological species themselves. The toxin released by one species not only affects that species but also may affect the growth of other species. The idea of Maynard Smith [12] was expanded further by Kar and Chaudhuri [10] to a two species competing fish species model which are commercially exploited. Haque and Sarwardi [7] also developed a model to study the effect of toxicity on a harvested fishery model.

In this paper we mainly discuss on fishery model as fish is the one of the most important renewable resources of the ecological system. From the literature discussed above and to the best of our knowledge, in this paper we have proposed a mathematical model to analyse the dynamics of a fishery resource system in an aquatic habitat subjected to harvesting of the both fish population where the growth of the both species is affected by some toxicants released by the other species. In this model we propose that both the fish species are harvested using modified catch rate function proposed by Sarkar, Sarkar and Chaudhuri [15]. The present model is organized as follows: The model formulation are proposed in Section 2. The equilibria and their feasibility are discussed in Section 3. The local and global stability analysis of the system are discussed in Sections 4 and 5. The bionomic equilibria and optimal harvesting policy of the present model are analysed. We derive the conditions for the existence of biological and bionomic equilibrium and study their stability behaviour in Sections 6 and 7 respectively. Numerical results are shown in Section 8. Existence of limit cycle is discussed in Section 9 while a final discussion and interpretation of the current study in ecological terms appear in the Section 10.

2. Formulation of the Problem

We consider the following dynamical system as a simple prey-predator interaction model

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \alpha xy$$

$$\frac{dy}{dt} = -sy + \beta xy \tag{1}$$

Where $x = x(t)$ is the biomass density of prey population at time t; $y = y(t)$ is the biomass density of predator population at time t; r is the maximum specific growth rate of the prey population; s is the relative rate at which the predators die out in absence of prey. The model assumes that the prey reproduction is influenced by the predators only while the predator reproduction is limited by the amount of prey caught. In absence of predators, the prey population grows with a relative rate r while in absence of prey, the predators die out exponentially with a relative rate s. The biomass of the prey consumed by the predator per unit time is given by αx which is interpreted as the trophic function or the predator's functional response to the prey population density. A fraction $\frac{\beta}{\alpha}(0 < \beta < \alpha \leq 1)$ of the energy consumed with this biomass goes into predator reproduction while the rest of the energy is used to sustain metabolism and hunting activity of predators. Here k is the environmental carrying capacity of prey population. In the Lotka-Voltera prey -predator model, the prey species grows exponentially up to infinity in the absence of predator species. We modify the first term of Lotka-Voltera prey-predator model as $rx\left(1 - \frac{x}{k}\right)$, where k=environmental carrying capacity of the prey species. Also here both the prey and predator are subjected to a combined harvesting effort E.

The catch rate function usually taken in the beginning of fishery models is of the form $h = qEx$. This is based on the CPUE (catch-per-unit-effort) hypothesis [Clark 1976]. Later on, it is revised in the functional form of $h = \frac{qEx}{bE + lx}$.

Here we assume that the fisherman search randomly in a given area effectively which is a function of the effort level to harvest the fish resource by the fisherman. We rename this concept as a searching efficiency for the area of discovery. The capture rate of fish resource dependent on how effectively (efficiently) the effort level is used in presence of other fisherman. On the basis of the above hypothesis accordingly we modify the catch rate function as a function of the fish resource population being captured for different effort levels in the form $h = \frac{qEx}{b + E}$ where E denotes the harvesting effort, q (constant) the catch ability coefficient and b is a positive constant.

Keeping these in view, the dynamics of prey-predator fish populations may be governed by the following autonomous system of differential equations

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \alpha xy - \frac{q_1 Ex}{b_1 + E}$$
$$\frac{dy}{dt} = -sy + \beta xy - \frac{q_2 Ey}{b_2 + E} \tag{2}$$

where q_1, q_2 represent the catch ability coefficient of the two species and b_1, b_2 are suitable constants.

The above mentioned model is now extended to the following one after incorporation of toxic effect.

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \alpha xy - \gamma_1 x^3 - \frac{q_1 Ex}{b_1 + E}$$
$$\frac{dy}{dt} = -sy + \beta xy - \gamma_2 y^2 - \frac{q_2 Ey}{b_2 + E} \tag{3}$$

Here the new parameters $\gamma_1 \cdot \gamma_2$ are the coefficients of toxicity. All the parameters in this model $r, s, \alpha, \beta, \gamma_1 \cdot \gamma_2, k, L$ are positive constants. The term $\gamma_1 x^3$ comes directly through the infection of the prey species by some external toxic substances such as industrial wastes. Here we observe that $\frac{d(\gamma_1 x^3)}{dx} = 3\gamma_1 x^2 > 0$ and $\frac{d^2(\gamma_1 x^3)}{dx} = 6\gamma_1 x > 0$. Therefore, there is an accelerated growth in the production of the toxicants to the density of the prey species as more and more of the species consume the infected food. Here γ_1 is known as the coefficient of toxicity to the prey species. Similar is the case for predator species except the effect of toxicity on the predator species being less than on the preys is taken as $\gamma_2 y^2$. Here γ_2 is the coefficient of toxicity to the predator species.

3. The Steady States

The possible steady states of the dynamical system of equation (3) are $S_0(0, 0)$, $S_1(x_1, 0)$ and $S^*(x^*, y^*)$, where x_1 $\dfrac{-\frac{r}{k} + \sqrt{\left(\frac{r}{k}\right)^2 + 4\gamma_1\left(r - \frac{q_1 E}{b_1 + E}\right)}}{2\gamma_1}$

which exists if $\frac{r}{q_1} > \frac{E}{b_1 + E}$ and $S^*(x^*, y^*)$ where

$$\frac{dx}{dt} = x\left(1 - \frac{x^*}{k}\right) - \alpha y^* - \gamma_1 x^{*2} - \frac{q_1 E}{b_1 + E}$$

$$\frac{dy}{dt} = -s + \beta x^* - \gamma_2 y^* - \frac{q_2 E}{b_2 + E} \tag{4}$$

Now, $$x^* = \frac{-\left(\frac{r}{k} + \frac{\alpha\beta}{\gamma_2}\right) + \sqrt{\left(\frac{r}{k} + \frac{\alpha\beta}{\gamma_2}\right)^2 + 4\gamma_1\left(r + \frac{s\alpha}{\gamma_2} + \frac{\alpha q_2 E}{\gamma_2(b_2 + E)} - \frac{q_1 E}{b_1 + E}\right)}}{2\gamma_1}$$

which is positive when $\frac{\alpha q_2}{(b_2 + E)} > \frac{q_1\gamma_2}{(b_1 + E)}$ and $y^* = \dfrac{\beta x^* - s - \frac{q_2 E}{b_2 + E}}{\gamma_2}$

which is positive when $x^* > 0$ and $\beta x^* > s + \frac{q_2 E}{b_2 + E}$.

4. Local Stability

We shall now investigate the local behaviour of the model (3) around each of the above mentioned steady states. The variational matrix of the system of equations (3) is defined as follows

$$\begin{vmatrix} r - \frac{2rx}{k} - \alpha y - 3\gamma_1 x^2 - \frac{q_1 E}{b_1 + E} & -\alpha x \\ \beta y & -s + \beta x - 2\gamma_2 y - \frac{q_2 E}{b_2 + E} \end{vmatrix}$$

For $S_0(0, 0)$, the characteristic equation is

$$\begin{vmatrix} r - \frac{q_1 E}{b_1 + E} - \lambda & 0 \\ 0 & -s - \frac{q_2 E}{b_2 + E} - \lambda \end{vmatrix} = 0.$$

The eigenvalue of the variational matrix are $r - \frac{q_1 E}{b_1 + E}$ and $-s - \frac{q_2 E}{b_2 + E}$. Roots of this equation $\lambda_1 = r - \frac{q_1 E}{b_1 + E}$ and $\lambda_2 = -s - \frac{q_2 E}{b_2 + E}$. Here $\lambda_1 < 0$ if $\frac{r}{q_1} < \frac{E}{b_1 + E}$ and $\lambda_2 < 0$. So λ_1 and λ_2 are both real and negative when $E > \frac{b_1 r}{q_1 - r}$.

So, the trivial steady state $S_0(0, 0)$ is a stable node. When $E > \frac{b_1 r}{q_1 - r}$, then $\lambda_1 > 0$ and $\lambda_2 < 0$ and $S_0(0, 0)$ is a saddle point.

For $S_1(x_1, 0)$, the characteristic equation is

$$\begin{vmatrix} \frac{-rx}{k} - \gamma_1 x_1^2 - \lambda & -\alpha x_1 \\ 0 & -s - \beta x_1 + \frac{q_2 E}{b_2 + E} - \lambda \end{vmatrix} = 0.$$

Here $\lambda_1 = -\left(\frac{rx_1}{k} + \gamma_1 x_1^2\right) < 0$ and $\lambda_2 = -\left(s - \beta x_1 + \frac{q_2 E}{b_2 + E}\right)$. Here

$\lambda_1 < 0$ and hence the steady state $S_1(x_1, 0)$, is a stable node when $\lambda_2 < 0 \Rightarrow \beta x_1 < s + \frac{q_2 E}{b + 2 + E}$ The interior equilibrium (x^*, y^*) is the solution of the following system of equations

$$r\left(1 - \frac{x^*}{k}\right) - \alpha y^* - \gamma_1 x^{*2} - \frac{q_1 E}{b_1 + E} = 0$$

$$-s + \beta x^* - \gamma_2 y^* - \frac{q_2 E}{b_2 + E} = 0. \tag{5}$$

The variational matrix of the system of equations (3) around S^* is

$$J^* = \begin{vmatrix} \frac{-rx^*}{k} - 2\gamma_1 x^{*2} & -\alpha x^* \\ \beta y^* & -\gamma_2 y^* \end{vmatrix}$$

So the characteristic equation of the above matrix J^* is

$$\begin{vmatrix} \frac{-rx^*}{k} - 2\gamma_1 x^{*2} - \lambda & -\alpha x^* \\ \beta y^* & -\gamma_2 y^* - \lambda \end{vmatrix} = 0$$

$$\text{or } \lambda^2 + \left(\frac{rx^*}{k} + 2\gamma_1 x^{*2} + \gamma_2 y^*\right)\lambda + \left(\frac{rx^*}{k} + 2\gamma_1 x^{*2}\right)\gamma_2 y^* + \alpha\beta x^* y^* = 0. \tag{6}$$

Here the sum of the roots $= -b = -\left(\frac{rx^*}{k} + 2\gamma_1 x^{*2} + \gamma_2 y^*\right) < 0$ and the product of the roots $= c = \left(\frac{rx^*}{k} + 2\gamma_1 x^{*2} + \right)\gamma_2 y^* + \alpha\beta x^* y^* > 0.$ In absence of toxicity, we have $\gamma_1 = \gamma_2 = 0$ and then $c = \alpha\beta x^* y^*$ which is positive for all positive values of α, β, x^*, y^*.

So the roots of the quadratic equation are either real and negative or complex conjugates with negative real parts. Hence, the steady state S^* is either a locally stable node or a locally stable focus. In presence of toxicity, $b, c > 0.$ So both the roots of the equation (6) are either real and negative or complex conjugates with negative real parts. Hence the non-trivial steady state $S^*(x^*, y^*)$ is either a locally stable node or a locally stable focus in the presence or absence of toxicity.

Hence the local stability of the system is not directly dependent on the intensities of the toxicants provided as x^* and y^* are both positive which need to satisfy the relation $\frac{\alpha q_2}{(b_2 + E)} > \frac{q_1 \gamma_2}{(b_1 + E)}$. So for the stability of the dynamical system it is necessary that the population density of the both the species must be positive which means that as the increasing effects of toxicity gradually the population density of both the species will decline which in turn will affect the stability of the system and finally both the species will tend to annihilation of the species.

5. Global Stability

In this section, we consider the global stability of the system of equation (2) by constructing a suitable Lyapunov function

$$V(x, y) = \left[(x - x^*) - x^* \log \frac{x}{x^*} + h(y - y^*) - y^* \log(\frac{y}{y^*}) \right] \tag{7}$$

where h is a suitable constant to be determined in the subsequent steps. It can be easily verified that the function V is zero at the equilibrium point (x^*, y^*) and is positive for all other values of x, y. The time derivative of V along the trajectories of equation (3) is

$$\frac{dv}{dt} = \frac{x - x^*}{x} \frac{dx}{dt} + h \frac{y - y^*}{y} \frac{dy}{dt} = (x - x^*) \left[r(1 - \frac{x}{k}) - \alpha y - \gamma_1 x^2 - \frac{q_1 E}{b_1 + E} \right]$$

$$+ h(y - y^*) \left[- s + \beta x - \gamma_2 y - \frac{q_2 E}{b_2 + E} \right] \tag{8}$$

Also, we have the set of the equilibrium equations

$$r\left(1 - \frac{x^*}{k}\right) - \alpha y^* - \gamma_1 x^{*2} - \frac{q_1 E}{b_1 + E} = 0$$

$$-s+\beta x^{*}-\gamma_2 y^{*}-\frac{q_2 E}{b_2+E}=0 \tag{9}$$

corresponding to the steady state $S_2(x^{*}, y^{*})$.

We can write equation (7) together with the above two equations in the form

$$\frac{dv}{dt}=(x-x^{*})\left[r(1-\frac{x}{k})-\alpha y-\gamma_1 x^2-\frac{q_1 E}{b_1+E}-r(1-\frac{x^{*}}{k})+\alpha y^{*}+\gamma_1 x^{*2}+\frac{q_1 E}{b_1+E}\right]$$

$$+h(y-y^{*})\left[-s+\beta x-\gamma_1 y-\frac{q_2 E}{b_2+E}+s-\beta x^{*}+\gamma_2 y^{*}+\frac{q_2 E}{b_2+E}\right]$$

$$=-\left[(x-x^{*})^2\left(\frac{r}{k}+\gamma_1(x+x^{*})\right)+(x-x^{*})(y-y^{*})(\alpha-h\beta)+(y-y^{*})^2 h\gamma_2\right] \tag{10}$$

If we choose $h=\frac{\alpha}{\beta}$

$$\frac{dv}{dt}=-\left[(x-x^{*})^2\left(\frac{r}{k}+\gamma_1(x-x^{*})\right)+(y-y^{*})^2\frac{\alpha}{\beta}\gamma_2\right]<0$$

Now since $\frac{dv}{dt}$ is negative semi definite in some neighbourhood of (x^{*}, y^{*}), the interior equilibrium point (x^{*}, y^{*}) is globally asymptotically stable.

6. Bionomic Equilibrium

The term bionomic equilibrium is an amalgamation of the ideas of biological equilibrium and economic equilibrium. As we already know, a biological equilibrium is given by $\dot{x}=0=\dot{y}$. In the fishery literature, the bionomic equilibrium is said to be obtained when the total revenue earned by selling the harvested biomass (TR) equals the total cost for the effort employed to harvesting (TC). In such a case, the economic rent is completely dissipated. Let C be the constant fishing cost per unit effort, p_1, constant price per unit biomass of the prey species, p_2, constant price per unit biomass of the predator species.

The economic rent (net revenue) at any time is given by

$$\pi(x,\ y,\ E,\ t) = \left(\frac{p_1 q_1 x}{b_1 + E} + \frac{p_2 q_2 y}{b_2 + E} - C\right) E \tag{11}$$

Although the harvesting cost per unit effort is not a constant, we take it to be a constant, for the sake of simplicity. Now,

$$\dot{x}_1 = 0 \Rightarrow r\left(1 - \frac{x}{k}\right) - \alpha y - \gamma_1 x^2 - \frac{q_1 E}{b_1 + E} \tag{12}$$

From the equation (12), we get

$$E = \frac{\alpha y b_1 + \gamma_1 x^2 b_1 - r\left(1 - \frac{x}{k}\right) b_1}{r\left(1 - \frac{x}{k}\right) - \alpha y - \gamma_1 x^2 - q_1} \tag{13}$$

Thus, E is positive when $\alpha y + \gamma_1 x^2 < r\left(1 - \frac{x}{k}\right) < \alpha y + \gamma_1 x^2 + q_1$.

Again,

$$\dot{y}_1 = 0 \Rightarrow -s + \beta x - \gamma_2 y - \frac{q_2 E}{b_2 + E}. \tag{14}$$

From the equation (14), we get

$$E = \frac{b_2 (s + \gamma_2 y - \beta x)}{\beta x - q_2 - s - \gamma_2 y} \tag{15}$$

So, E is positive when $\beta x - q_2 < s + \gamma_2 y < \beta x$.

Hence the non-trivial equilibrium solution occurs at the point on the curve

$$\frac{\alpha y b_1 + \gamma_1 x^2 b_1 - r\left(1 - \frac{x}{k}\right) b_1}{r\left(1 - \frac{x}{k}\right) - \alpha y - \gamma_1 x^2 - q_1} = \frac{b_2 (s + \gamma_2 y - \beta x)}{\beta x - q_2 - s - \gamma_2 y} \tag{16}$$

where $0 \le x \le k$

The bionomic equilibrium of the open access fishery is found by Equation

(16) along with the condition

$$\pi = TR - TC$$

$$= \left(\frac{p_1 q_1 x}{b_1 + E} + \frac{p_2 q_2 y}{b_2 + E} - C\right) E$$

$$\Rightarrow \left(\frac{p_1 q_1 x}{b_1 + E} + \frac{p_2 q_2 y}{b_2 + E} - C\right) = 0. \tag{17}$$

7. Optimal Harvesting Policy

The fundamental problem for finding out an optimal policy in a economic (commercial fishery) is to determine the optimal tradeoff between the current and future harvests. The present value $\Im$ of a continuous time-stream of revenues is given by

$$\Im = \int_0^\infty \pi(x,\ y,\ E,\ t) e^{-\delta t} dt \tag{18}$$

where $\pi(x,\ y,\ E,\ t) = \frac{p_1 q_1 x E}{b_1 + E} + \frac{p_2 q_2 y E}{b_2 + E} - CE$ and δ denotes the instantaneous annual rate of discount, C is the cost of fishing per unit effort, $p_1,\ p_2$ are the price per unit biomass of x and y species respectively. Our problem is to maximize $\Im$ subject to the state equation (3) by invoking Pontryagin's Maximum Principle [14]. The control variable $E(t)$ is subjected to the constraints so that $V_t = [0,\ E_{\max}]$ is the control set where $E_{\max}$ is a feasible upper limit for the harvesting effort.

The Hamiltonian for the problem is given by

$$H = \left(\frac{p_1 q_1 x}{b_1 + E} + \frac{p_2 q_2 y}{b_2 + E} - C\right) E e^{-\delta t} + \lambda_1 \left[rx\left(1 - \frac{x}{k}\right) - \alpha xy - \gamma_1 x^3 - \frac{q_1 E x}{b_1 + E} \right]$$
$$+ \lambda_2 \left[- sy + \beta xy - \gamma_2 y^2 - \frac{q_2 E y}{b_2 + E} \right] \tag{19}$$

where $\lambda_1(t),\ \lambda_2(t)$ are the adjoint variables. The adjoint equations are

$$\frac{d\lambda_1}{dt} = \frac{\partial H}{\partial x}$$

$$= -\left[\frac{p_1 q_1}{b_1 + E} Ee^{-\delta t} + \lambda_1\left(r - \frac{2rx}{k} - \alpha y = 3\gamma_1 x^2 - \frac{q_1 E}{b_1 + E}\right) + \lambda_2(\beta y)\right] \quad (20)$$

and

$$\frac{d\lambda_2}{dt} = \frac{\partial H}{\partial y}$$

$$= -\left[\frac{p_2 q_2}{b_2 + E} Ee^{-\delta t} + \lambda_1(-\alpha x) + \lambda_2\left(-s + \beta x - 2\gamma_2 y - \frac{q_2 E}{b_2 + E}\right)\right] \quad (21)$$

Our aim is to find an optimal equilibrium solution of the problem so that we may take

$$E = \frac{\alpha y b_1 + \gamma_1 x^2 b_1 - r\left(1 - \frac{x}{k}\right)b_1}{r\left(1 - \frac{x}{k}\right) - \alpha y - \gamma_1 x^2 - q_1} = \frac{b_2(s + \gamma_2 y - \beta x)}{\beta x - q_2 - s - \gamma_2 y}. \quad (22)$$

By using equation (22), equations (20) and (21) become respectively,

$$\frac{d\lambda_1}{dt} = \lambda_1\left(\frac{rx}{k} + 2\gamma_1 x^2\right) + \lambda_2(-\beta y) - \left[\frac{p_1 q_1}{b_1 + E} Ee^{-\delta t}\right] \quad (23)$$

and

$$\frac{d\lambda_2}{dt} = \lambda_1(\alpha x) + \lambda_2(-\gamma_2 y) - \left[\frac{p_2 q_2}{b_2 + E} Ee^{-\delta t}\right] \quad (24)$$

Eliminating λ_2 from the equations (23) and (24), we have

$$\frac{d^2\lambda_1}{dt^2} - \left(\frac{rx}{k} + 2\gamma_1 x^2 + \gamma_2 y\right)\frac{d\lambda_1}{dt}$$

$$+\left[\left(\frac{rx}{k} + 2\gamma_1 x^2\right)\gamma_2 y + \alpha\beta xy\right]\lambda_1 = M_1 e^{-\delta t} \quad (25)$$

where $D = \frac{d}{dt}$ and $M_1 = \frac{p_1 q_1 E\delta}{b_1 + E} + \frac{p_2 q_2 \beta y E}{b_2 + E} + \frac{p_1 q_1 E \gamma_2 y}{b_1 + E}$

The auxiliary equation for (25) is

$$\mu^2 - \left(\frac{rx}{k} + 2\gamma_1 x^2 + \gamma_2 y\right)\mu + \left[\left(\frac{rx}{k} + 2\gamma_1 x^2\right)\gamma_2 y + \alpha\beta xy\right] = 0 \tag{26}$$

This is a quadratic equation in μ where sum of the roots $= \left(\frac{rx}{k} + 2\gamma_1 x^2 + \gamma_{2y}\right) > 0$ and product of the roots $\left[\left(\frac{rx}{k} + 2\gamma_1 x^2\right)\gamma_2 y + \alpha\beta xy\right] > 0.$

Therefore the roots m_1 and m_2 of the above equation are either both real and positive or complex conjugates with positive parts. The complete solution for equation (26) is of the form

$$\lambda_1(t) = A_1 e^{\mu_1(t)} + A_2 e^{\mu_2(t)} + \left(\frac{M_1}{N}\right)e^{-\delta t} \tag{27}$$

where $N = \left[\delta^2 - \delta\left(\frac{rx}{k} + 2\gamma_1 x^2 + \gamma_2 y\right) + \left(\frac{rx}{k} + 2\gamma_1 x^2\right)\gamma_2 y + \alpha\beta xy\right] \neq 0.$ It is true that λ_1 is bounded if $A_1 = A_2 = 0$. Then we have

$$\lambda_1(t) = \left(\frac{M_1}{N}\right)e^{-\delta t} = \text{constant} \tag{28}$$

Similarly, we get

$$\lambda_2(t) = \left(\frac{M_2}{N}\right)e^{-\delta t} = \text{constant} \tag{29}$$

where $M_2 = \frac{p_2 q_2 E\delta}{b_2 + E} + \left(\frac{rx}{k} + 2\gamma_1 x^2\right)\frac{p_2 q_2 E}{b_2 + E} - \alpha x \frac{p_1 q_1 E}{b_1 + E}$.

Hence the shadow prices $\lambda_i(t)e^{\delta t}$, $i = 1, 2$ remain constant over time in optimal equilibrium when they satisfy the transversality condition at ∞, i.e., when they remain bounded as $t \to \infty$. Again the condition that the Hamiltonian H must be a maximum gives the condition

$$\frac{\partial H}{\partial E} = e^{-\delta t}\left[\frac{p_1 q_1 b_1 x}{(b_1 + E)^2} + \frac{p_2 q_2 b_2 y}{(b_2 + E)^2} - C\right] + \lambda_1\left[\frac{-q_1 b_1 x}{(b_1 + E)^2}\right] + \lambda_2\left[\frac{-q_2 b_2 y}{(b_2 + E)^2}\right]$$

$$= 0. \tag{30}$$

The right hand side represents the discounted value of the future profit per unit effort at the steady state effort level.

Putting the values of $\lambda_1(t)$ and $\lambda_2(t)$ in equation (31), we get

$$\left(p_1 - \frac{M_1}{N}\right)\frac{q_1 b_1 x}{(b_1 + E)^2} + \left(p_2 - \frac{M_2}{N}\right)\frac{q_2 b_2 x}{(b_2 + E)^2} = C \tag{31}$$

Equation (31) together with equation (22) gives the optimal equilibrium populations

$x = x_\delta,\ y = y_\delta$

When $\delta \to \infty$, equation (32) leads to the result $\frac{p_1 q_1 b_1 x}{(b_1 + E)^2} + \frac{p_2 q_2 b_2 y}{(b_2 + E)^2} = C$

which implies $\frac{\partial \pi}{\partial E}(x_\infty, y_\infty, E) = 0$

Thus the economic rent is completely dissipated and hence the fishery remains unexploited. When the discount rate is infinite, using Equation (32) we get

$$\frac{\partial \pi}{\partial E} = \frac{p_1 q_1 b_1 x}{(b_1 + E)^2} + \frac{p_2 q_2 b_2 y}{(b_2 + E)^2} - C$$

$$= \frac{M_1}{N}\frac{q_1 b_1 x}{(b_1 + E)^2} + \frac{M_2}{N}\frac{q_2 b_2 x}{(b_2 + E)^2} \tag{32}$$

Since M_1 and M_2 is of $o(\delta)$ where N is of $o(\delta^2)$, we see that $\frac{\partial \pi}{\partial E}$ is of $o(\delta^{-1})$. Thus $\frac{\partial \pi}{\partial E}$ is a decreasing function of $\delta(\geq 0)$.

We, therefore, conclude that $\delta = 0$ leads to maximization of $\frac{\partial \pi}{\partial E}$.

8. Numerical Results

Numerical simulations have been carried out by making use of MATLAB-2016a and Maple-18. It is very difficult to validate the model results with realistic data so far toxic effect and harvesting are considered in the natural field. These results are all verified by means of numerical illustrations of which some chosen ones are shown in the figures. So we take a set of hypothetical parameter values to illustrate the results, we have established in the present model: $r = 15$, $k = 1000$, $\alpha = 0.02$, $q_1 = 1.6$, $E = 11$, $b_1 = 0.3$, $\gamma_1 = 0.00005$, $s = 0.6$, $\beta = 0.005$, $\gamma_2 = 0.000008$, $q_2 = 1.01$, $b_2 = 0.2$.

Example 1. We take the parameter values as $r = 15$, $k = 1000$, $\alpha = 0.02$, $q_1 = 1.6$, $E = 11$, $b_1 = 0.3$, $\gamma_1 = 0.000005$, $s = 0.6$, $\beta = 0.005$, $\gamma_2 = 0.000008$, $q_2 = 1.01$, $b_2 = 0.2$. in appropriate units.

For the above values, we find that

(i) $S_0(0, 0)$ is unstable.

(ii) $S_1(722.27, 0)$ is unstable.

(iii) $S_1(320.15, 408.26)$ is both locally and globally asymtotically stable node.

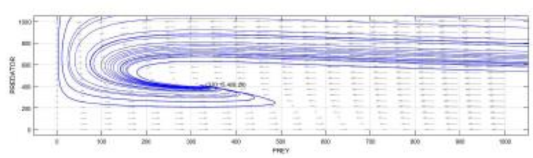

Figure 1. Phase plane trajectories of the prey–predator fishery with different initial values.

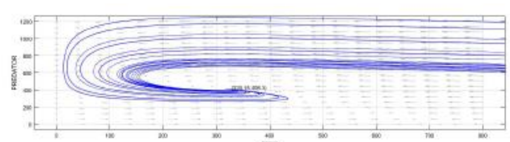

Figure 2. Phase plane trajectories of the prey –predator fishery with different initial values.

From Figure 1 and Figure 2 it is clear that for the above set of parameter values, the system possesses an interior equilibrium point $S_1(320.15, 408.26)$. It is also observed that the system (2) is globally asymptotically stable around the coexistence equilibrium S_*.

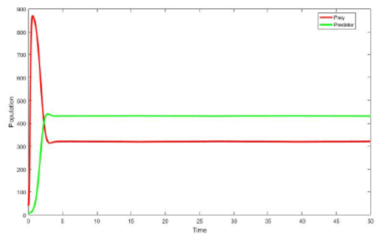

Figure 3. Solution curves of the prey-predator fishery for a period $t = 0$ to 50 weeks.

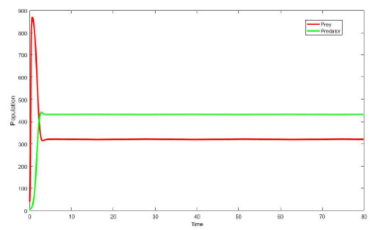

Figure 4. Solution curves of the prey-predator fishery for a period $t = 0$ to 80 weeks.

From Figure 3 and Figure 4, it is clear that the biomass density of prey species increases sharply with respect to time and then decreases and settles down at its equilibrium level. The biomass density of predator species increases with respect to time and then decreases slightly and settles down at its equilibrium level.

Example 2. Taking the same values of the parameters together with $p_1 = 5,\ p_2 = 6,\ C = 50$ and $E = \dfrac{\alpha y b_1 + \gamma_1 x^2 b_1 - r\left(1 - \dfrac{x}{k}\right) b_1}{r\left(1 - \dfrac{x}{k}\right) - \alpha y - \gamma_1 x^2 - q_1}$ from Figure 5 we find that the bionomic equilibrium exists $(x_\infty,\ y_\infty) = (201.51, 1044.59)$.

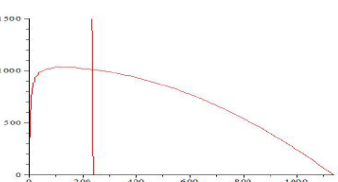

Figure 5. Bionomic equilibrium.

In absence of toxicity $(\gamma_1 = \gamma_2 = 0)$, we found $(x_\infty, y_\infty) = (198.21, 1078.42)$. If we take $\delta = 0.1$ together with the same parameter values, we find that the optimal equilibrium $(x_\delta, y_\delta) = (196.82, 539.14)$.

In absence of toxicity $(\gamma_1 = \gamma_2 = 0)$, we found $(x_\delta, y_\delta) = (1435.26, 236.06)$.

From numerical example we may draw the following notes:

(i) The optimal equilibrium level for the prey species $x_\delta = 196.82$ is lower than that of the corresponding steady state level $x^* = 320.15$. But the optimal equilibrium population level for the predator fish species $y_\delta = 539.14$ is much higher than that of the corresponding steady state level $y^* = 408.26$. In absence of toxicity, the optimal equilibrium level $(x_\delta, y_\delta) = (1435.26, 236.06)$ exists at a higher population level for the prey species and lower population level at the predator species compared to $(x_\delta, y_\delta) = (196.82, 539.14)$ in presence of toxicity.

(ii) Bionomic equilibrium exists in absence of toxicity at a higher population level $(x_\infty, y_\infty) = (318.21, 1078.42)$ compared to $(x_\infty, y_\infty) = (201.51, 1044.59)$ in presence of toxicity) for the first population but at slightly lower population for the second species.

(iii) The bionomic equilibrium and optimal equilibrium both are critically depended upon the parameter values of the parameters $p_1, p_2, C, q_1, q_2, r, s, b_1, b_2, k$.

(iv) As the effects of toxicity increase gradually the population density of both the species will decline and finally will tend to extinction.

9. Existence of Limit Cycle

We use Bendixon-Du Lac test to determine the existence of limit cycle. Consider the dynamical system of equations

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \alpha xy - \gamma_1 x^3 - \frac{q_1 Ex}{b_1 + E} = F(x, y)$$

$$\frac{dy}{dt} = -sy + \beta xy - \gamma_2 y^2 - \frac{q_2 Ey}{b_2 + E} = G(x, y) \tag{33}$$

where the functions $F(x, y)$ and $G(x, y)$ are smooth in a given simply connected region D in the 1sr quadrant of the (x, y) phase plane. Now, consider the function $B(x, y) = \frac{1}{xy}$ which is also smooth in the region D. Consider the function

$$\frac{\partial(BF)}{\partial x} + \frac{\partial(BG)}{\partial y} = -\frac{rx}{k} - \frac{2\gamma_1 x}{y} - \frac{\gamma_2}{x} \tag{34}$$

We thus observed that whatever may be the values of the parameters (as they are assumed to be positive) the above expression is always negative i.e. it keeps the same sign throughout the region D and hence there are no closed orbits lying entirely in the region D.

10. Conclusions

In this paper, we have discussed the effects of toxicants released by some other resources in the aquatic environment in a prey-predator fishery model where both the species are harvested with a modified catch rate function. We modify the Lotka-Volterra system by taking into consideration the environmental factors to restrict the growth of the prey species to a finite level in absence of predator. Here we use modified catch rate function for harvesting. The local and global stabilities are observed here.

The existence of bionomic equilibria is examined. These are the zero profit line and the biological equilibrium line. The bionomic equilibrium is computed for a set of values of the parameters. Finally, the optimal harvesting policy is discussed. The present value of revenue is maximised by using Pontryagin’s maximum principle [14] subject to the state equations and the control constraints. The various cases of optimal equilibrium are shown here. It is found that the shadow prices remain constraint over time in optimal equilibrium when they satisfy the transversality condition. It is also shown that the zero discounting leads to the maximization of economic revenue and that an infinite discount rate lead to complete dissipation of economic rent. Optimal steady state solution is found out for a set of data.

From numerical simulation, we have observed that gradual increasing of γ_1 causes decrease of prey species and increase of the predator species. Similarly, the gradual increasing of γ_2 causes increasing of the prey species and decreasing of predator species. When both the toxicants are increasing then both the species are decreasing. So we can conclude that gradual increase of toxicants released by the other resources have detrimental effects on each other and finally will go into extinction simultaneously, i.e., the system will tend to extinction. Solution curves corresponding to the steady state and bionomic equilibrium are shown by using Matlab and Maple package. Growth curves and phase plane trajectories are discussed. Lastly, existence of limit cycle is discussed here.

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