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GEODETIC ECCENTRIC DOMINATION IN GRAPHS

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Abstract

A subset D of the vertex set $V(G)$ of a graph G is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is said to be an eccentric dominating set if for every $v \in V - D$, there exists at least one eccentric vertex of v in D. The minimum cardinality of an eccentric dominating set is called the eccentric domination number and is denoted by $\gamma_{ed}(G)$. A set S of vertices in a graph G is a geodetic dominating set if S is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set is the geodetic domination number of G and is denoted by $\gamma_g(G)$. A set S of vertices in a graph G is a geodetic eccentric dominating set if S is both a geodetic set and an eccentric dominating set. The minimum cardinality of a geodetic eccentric dominating set is the geodetic eccentric domination number of G and is denoted by $\gamma_{ged}(G)$. In this paper, we obtain some bounds for $\gamma_{ged}(G)$. Exact values of $\gamma_{ged}(G)$ for some particular classes of graphs are obtained. Also, we characterize graphs for which $\gamma_{ged}(G) = 2,\ p - 1$ and p.

1. Introduction

Let G be a finite, simple, connected and undirected (p, q) graph with vertex set $V(G)$ and edge set $E(G)$. For graph theoretic terminology refer to Harary [17], Buckley and Harary [11].

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The concept of domination in graphs is originated from the chess games theory and that paved the way to the development of the study of various domination parameters and its relation to various other graph parameters. For details on domination theory, refer to Haynes, Hedetniemi, and Slater [18].

Definition 1.1. The *distance* $d(u, v)$ between two vertices u and v in G is the minimum length of a $u - v$ path.

Definition 1.2. Let G be a connected graph and v be a vertex of G. The *eccentricity* $e(v)$ of v is the distance to a vertex farthest from v. Thus, $e(v) = \max\{d(u, v) : u \in V\}$. The radius $r(G)$ is the minimum eccentricity of the vertices, whereas the diameter $diam(G) = d(G)$ is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. The vertex v is a central vertex if $e(v) = r(G)$. The center $C(G)$ is the set of all central vertices. For a vertex v, each vertex at a distance $e(v)$ from v is an eccentric vertex of v. Eccentric vertex set of a vertex v is defined as $E(v) = \{u \in V(G)/d(u, v) = e(v)\}$. The set E_k denotes the set of vertices of G with eccentricity k.

The concept of domination in graphs was introduced by Ore in [21]. In 1977, Cockayne and Hedetniemi explained importance and properties of domination in [15].

Definition 1.3 [15, 18]**.** A set $D \subseteq V$ is said to be a *dominating set* in G if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number and is denoted by $\gamma(G)$.

Chartrand et al. studied the concept of geodetic sets in graphs and on the geodetic number of a graph [12, 13, 14]. They also studied the concept of geodomination in graphs. Escuadro et al. [16] studied the concept of geodetic domination in graphs.

Definition 1.4 [13]**.** An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. The closed interval $I[x, y]$ consists of x, y and all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V(G)$, $I[S] = \bigcup_{x, y \in S} I[x, y]$.

Definition 1.5 [13]. A set S of vertices in a graph G is a *geodetic set* if $I[S] = V(G)$. The minimum cardinality of a geodetic set is the geodetic number of G and is denoted by $g(G)$.

Definition 1.6 [16]. A set S of vertices in a graph G is a *geodetic dominating set* if S is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set is the geodetic domination number of G and is denoted by $\gamma_g(G)$.

Janakiraman, Bhanumathi and Muthammai [19] introduced Eccentric domination in Graphs. Bhanumathi and Muthammai studied some bounds for $\gamma_{ed}(G)$ and $\gamma_{ed}(T)$ in [1, 2, 3]. Bhanumathi, John Flavia and Kavitha [4] studied the concept of Restrained Eccentric domination in graphs. Bhanumathi and John Flavia studied the concept of Total Eccentric domination in graphs and some more bounds for $\gamma_{ed}(G)$ in [5, 7]. Bhanumathi and Sudhasenthil [6, 8, 9] studied the concept of the split and Nonsplit Eccentric domination, Distance closed eccentric domination, Eccentric domination and chromatic number in graphs. Bhanumathi and Meenal Abirami [10] studied the concept of Upper Eccentric Domination in Graphs. Geodetic eccentric dominating set was defined by Nishanthi in [20].

Definition 1.7 [19]. A set $D \subseteq V(G)$ is an *eccentric dominating set* if D is a *dominating set* of G and for every $v \in V - D$, there exists at least one eccentric vertex of v in D. The minimum cardinality of an eccentric dominating set is called the *eccentric domination number* and is denoted by $\gamma_{ed}(G)$.

Let S be a vertex set of G. Then S is known as an eccentric vertex set of G if for every v is belongs to $V - S$, S has at least one vertex u such that vertex u belongs to eccentric vertex set $E(v)$. An eccentric vertex set S of G is a minimal eccentric vertex set if no proper subset S' of S is an eccentric vertex set of G. S is known as a minimum eccentric vertex set if S is an eccentric vertex set with minimum cardinality. Let $e(G)$ be the cardinality of a minimum eccentric vertex set of G, $e(G)$ is known as eccentric number of G.

Theorem 1.1 [15]. *For any graph* G, $\lceil p/(1 + \Delta(G)) \rceil \leq \gamma(G) \leq p - \Delta(G)$.

Theorem 1.2 [16]. (i) $\gamma_g(K_{1,n}) = n$.

(ii) $\gamma_g(K_{m,n}) = \min\{m, n, 4\}$ *for* $m, n \geq 2$.

Theorem 1.3 [16]. *If G is a connected graph of order* $p \geq 2$, *then* $2 \leq \max\{g(G), r(G)\} \leq \gamma_g(G) \leq p$.

Theorem 1.4 [16]. *Let G be a connected graph of order* $p \geq 2$. *Then.*

(a) $\gamma_g(G) = 2$ *if and only if there exists a geodetic set* $S = \{u, v\}$ *of G such that* $d(u, v) \leq 3$.

(b) $\gamma_g(G) = p$ *if and only if G is the complete graph on p vertices.*

(c) $\gamma_g(G) = p - 1$ *if and only if there is a vertex v in G such that v is adjacent to every other vertex of G and* $G - v$ *is the union of at least two complete graphs.*

Theorem 1.5 [16]. *If G is a connected graph with* $\gamma_g(G) = 1$, *then* $\gamma_g(G) = g(G)$.

Theorem 1.6 [19]. (i) $\gamma_{ed}(G) = 1$ *if and only if* $G = K_p$.

(ii) $\gamma_{ed}(K_{1,n}) = 2$, $n \geq 2$.

(iii) $\gamma_{ed}(K_{m,n}) = 2$.

(iv) $\gamma_{ed}(W_3) = 1$, $\gamma_{ed}(W_4) = 2$, $\gamma_{ed}(W_n) = 3$ *for* $n = 5$, $\gamma_{ed}(W_6) = 2$, $\gamma_{ed}(W_n) = 3$ *for* $n \geq 7$.

(v) $\gamma_{ed}(P_p) = \lceil p/3 \rceil$ *if* $p = 3k + 1$

$\gamma_{ed}(P_p) = \lceil p/3 \rceil + 1$ *if* $p = 3k$ *or* $3k + 2$

(vi) $\gamma_{ed}(C_p) = p/2$ *if p is even.*

$\gamma_{ed}(C_p) = \lceil p/3 \rceil$ or $\lceil p/3 \rceil + 1$ *if p is odd.*

In [22] Sudhasenthil proved the following theorem.

Theorem 1.7 [22]. *Let G be a connected graph with* $r(G) = rad(G) = 1$,

$diam(G) = 2$ with t central vertices. Then $\gamma_{ed}(G) = 2$ if and only if any one of the following is true.

(i) *G has at least one vertex of degree t.*

(ii) *There exist $u, v \in V(G)$ such that $D = \{u, v\}$ is a maximal independent set and $d_{\langle E_2 \rangle}(u, v) \geq 3$, that is $e(u) = 2 = e(v)$ and a vertex of eccentricity two is adjacent to exactly one of u and v.*

Theorem 1.8 [19]. *If G is a two self-centered graph, then $\gamma_{ed}(G) = 2$ if and only if G has a dominating edge which is not in a triangle.*

Theorem 1.9 [19]. *If G is a graph with radius two and diameter three then $\gamma_{ed}(G) = 2$ if and only if G has a γ-set $D = \{u, v\}$ of cardinality two with $d\{u, v\} = 3$ and for any $u - v$ path in G, $e(u) = e(v) = 3$ and $e(x) = e(y) = 2$.*

2. Geodetic Eccentric Domination in Graphs

In [20], Nishanthi defined Geodetic eccentric domination number. She did not study any properties or bounds about this domination number. In this paper, we study some bounds for Geodetic eccentric domination number and characterize graphs for which Geodetic eccentric domination number $\gamma_{ged}(G) = 2,\ p - 1$ and p.

A set S of vertices in a graph G is a geodetic eccentric dominating set if S is both a geodetic set and an eccentric dominating set. The minimum cardinality of a geodetic eccentric dominating set is the geodetic eccentric domination number of G and is denoted by $\gamma_{ged}(G)$. We have, $\gamma(G) \leq \gamma_{ed}(G) \leq \gamma_{ged}(G)$. Also, $\gamma_g(G) \leq \gamma_{ged}(G)$. $\gamma_{ged}(G)$ exists for all graphs, since $V(G)$ is always a geodetic eccentric dominating set. By Theorem 1.3, $2 \leq \gamma_g(G) \leq \gamma_{ged}(G)$. Any geodetic eccentric dominating set must contains all pendant vertices of G.

For any graph G, $\gamma_{ged}(G) \leq \gamma_g(G) + e(G)$ and $\gamma_{ged}(G) \leq \gamma_{ed}(G) + g(G)$, where $g(G)$ is geodetic number of G and $e(G)$ is eccentric number of G.

Example 2.1.

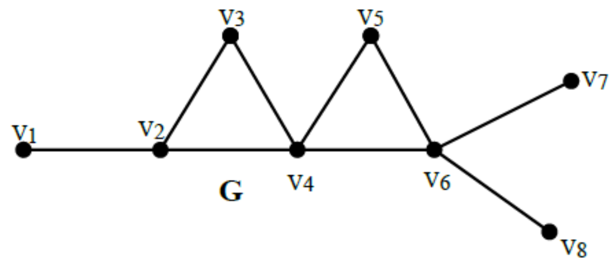

Figure 2.1

In Figure 2.1, $S_1 = \{v_2, v_6\}$ is a minimum dominating set of G, $\gamma(G) = 2$.

$S_2 = \{v_1, v_4, v_7, v_8\}$ is a minimum eccentric dominating set of G, $\gamma_{ed}(G) = 4$.

$S_3 = \{v_1, v_3, v_5, v_7, v_8\}$ is a minimum geodetic dominating set of G. It is also a minimum geodetic eccentric dominating set of G, $\gamma_g(G) = \gamma_{ged}(G) = 5$.

Example 2.2.

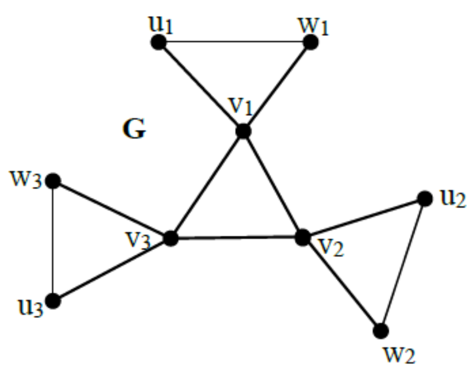

Figure 2.2

In Figure 2.2, $S_1 = \{v_1, v_2, v_3\}$ is a minimum dominating set of G, $\gamma(G) = 3$.

$S_2 = \{u_1, u_2, u_3\}$ is a minimum eccentric dominating set of G, $\gamma_{ed}(G) = 3$.

$S_3 = \{u_1, u_2, u_3, w_1, w_2, w_3\}$ is a minimum geodetic dominating set of G and is a minimum geodetic eccentric dominating set of $G, \gamma_g(G) = \gamma_{ged}(G) = 6$.

In the following theorems, we discuss about the geodetic eccentric dominating set and find out some more bounds for $\gamma_{ged}(G)$.

Theorem 2.1. *Let G be a graph of radius one and diameter two. Then* $D \subseteq V(G)$ *is a geodetic eccentric dominating set if and only if* $D \subseteq E_2$ *and is a geodetic eccentric dominating set of the subgraph* $\langle E_2 \rangle$.

Proof. G is a graph with radius one and diameter two. Hence, $\gamma_{ged}(G) \geq 2$. Assume that $D \subseteq E_2$ is a geodetic eccentric dominating set of the subgraph $\langle E_2 \rangle$. This implies that $I[D] = E_2$ and D has at least two vertices of eccentricity two at distance two. Hence, $I[D]$ in G contains all central vertices also. Thus, $D \subseteq E_2$ and D is a geodetic eccentric dominating set of G. Converse is obvious.

Theorem 2.2. *If a connected G has pendant vertices, then any* $\gamma_{ged}(G)$*-set of G contains all its pendant vertices.*

Proof. Every geodetic eccentric dominating set is a geodetic set. Therefore, $\gamma_{ged}(G)$-set of G must contain all pendant vertices of G.

Theorem 2.3. *Let G be a graph of radius one and diameter two and let* $|E_1| = t$ *and* $|E_2| = s$. *Then* $\gamma_{ged}(G) \leq s$.

Proof. E_2 contains at least two non-adjacent vertices. Hence, $I[E_2] = V(G)$. Therefore, E_2 is a geodetic eccentric dominating set of G. Hence, $\gamma_{ged}(G) \leq s$.

Theorem 2.4. *If G is a self-centered graph of diameter two, then* $\gamma_{ged}(G) \leq p - \Delta(G)/2$.

Proof. Let $u \in V(G)$ such that $\deg u = \Delta(G)$. The vertex u and $N_2(u)$ dominate all other vertices of G. Each vertex of $N_2(u)$ is an eccentric vertex of u in G. $N_2(u) \cup \{u\}$ is a geodetic set of G. Hence, $N_2(u) \cup \{u\}$ is a geodetic dominating set of G. Vertices in $N(u)$ may have eccentric vertices in $N(u)$. In this case, let S be a subset of $N(u)$ such that vertices in $N(u) - S$ have eccentric vertices in S. Thus, $|S| \leq \Delta(G)/2$ and $S \cup N_2(u) \cup \{u\}$ is a geodetic eccentric dominating set. Hence, $\gamma_{ged}(G) \leq \Delta(G)/2 + (p - \Delta(G)) = p - \Delta(G)/2$.

Theorem 2.5. *If G is a graph of radius greater than two, then* $\gamma_{ged}(G) \leq p - \delta(G)$.

Proof. Let $u \in V(G)$ such that u is not a support vertex in G. $V(G) - N(u)$ dominate all other vertices of G. Also, since radius > 2, each vertex in $N(u)$ has eccentric vertices in $V - N(v)$ only. $D = V(G) - N(u)$ is a geodetic set, since u is not a support. D is also a geodetic eccentric dominating set. Thus, $\gamma_{ged}(G) \leq |V - N(u)| \leq p - \delta(G)$.

Corollary 2.1. *If there exists u such that* $\deg u = \Delta(G)$ *and u is not a support then* $\gamma_{ged}(G) \leq p - \Delta(G)$.

Theorem 2.6. *If G is a tree, then* $\gamma_{ged}(G) \leq \gamma(G) + t$, *$t$-number of pendant vertices.*

Proof. Let d be the diameter of G. Let t be the number of pendant vertices of G. Let D be any dominating set of G and S be the set of all pendant vertices of G. Then $D \cup S$ is a geodetic eccentric dominating set of G. Hence, $\gamma_{ged}(G) \leq \gamma(G) + t$.

Theorem 2.7. *If G is a tree of order* $p \geq 3$, *then the following conditions are equivalent.*

(i) $\gamma_{ged}(G) = \gamma_g(G) = \gamma(G) = g(G)$.

(ii) $L(G)$ *is a minimum dominating set of G, where* $L(G)$ *is the number of pendant vertices.*

Proof. $L(G)$ is a minimum geodetic set of a tree G. Also, since eccentric vertices are in $L(G)$, (i) and (ii) are equivalent.

Following theorems characterize graphs for which $\gamma_{ged}(G) = 2$, $p - 1$ and p.

Theorem 2.8. *Let G be a graph with radius one and diameter two. Then* $\gamma_{ged}(G) = 2$ *if and only if* $G = \overline{K_2} + K_{p-2}$.

Proof. Assume $\gamma_{ged}(G) = 2$. This implies that $\gamma_{ed}(G) = 2$. We know by Theorem 1.7, $\gamma_{ed}(G) = 2$ if and only if G is any one of the following.

(i) G has a vertex of degree m, where m is the number of central vertices.

(ii) G has vertices u, v with $e(u) = e(v) = 2$ such that each vertex of eccentricity two is adjacent to either u or v.

If G has a vertex of degree m then $D = \{x, y\}, e(y) = 1, e(x) = 2$ and $\deg x = m$ is a γ_{ed}-set but in this case D is not geodetic, since $d(x, y) = 1$.

Let $D = \{x, y\}$ be a γ_{ged}-set with $e(x) = e(y) = 2$. Vertices x and y are not adjacent, since D is a geodetic set. If E_2 has more than two vertices by (ii) any vertex of E_2 is not adjacent to both x and y in G. Hence, D is not geodetic if E_2 has more than two vertices.

Hence, $E_2 = \{x, y\} = D$ and $E_1 = V(G) - \{x, y\}$. That is, $G = \overline{K_2} + K_{p-2}$.

Conversely, when $G = \overline{K_2} + K_{p-2}$, it is clear that $\gamma_{ged}(G) = 2$.

Theorem 2.9. *Let G be a two self-centered graph. Then* $\gamma_{ged}(G) \neq 2$.

Proof. Suppose $\gamma_{ged}(G) = 2$. Then $\gamma(G) = \gamma_{ed}(G) = \gamma_{ged}(G) = 2$, since $\gamma_{ed}(G) \neq 1$ and $\gamma(G) \neq 1$ for G. $\gamma_{ed}(G) = 2$ implies that there is a γ_{ed}-set $D = \{x, y\}$ such that $e = xy$ is a dominating edge of G which is not in a triangle by Theorem 1.8. But D is a geodetic eccentric dominating set implies that x and y are not adjacent in G. Hence, $\gamma_{ged}(G)$ cannot be two.

Theorem 2.10. *Let G be a connected graph of radius two and diameter three. Then* $\gamma_{ged}(G) = 2$ *if and only if G has only two peripheral vertices such that all other vertices lie on a diametral path from x to y.*

Proof. We know that $\gamma_{ged}(G) = 2$ implies that $\gamma_{ed}(G) = 2$. But by Theorem 1.9, $\gamma_{ed}(G) = 2$ if and only if G has a γ-set $D = \{x, y\}$ such that

$d(x, y) = 3, e(x) = e(y) = 3$ and for any shortest path $xuvy$ in G, $e(u) = e(v) = 2$.

But, D is a geodetic set in G if and only if all other vertices lie on a diametral path from x to y.

Suppose there exists $z \in V(G)$ such that $e(z) = 3$, then z is adjacent to x (or y) and $d(z, y) = 2$ (or $d(z, x) = 2$). Thus, z has no eccentric vertex in D. Hence, except x and y, all other vertices are of eccentricity two. This proves the theorem.

Example 2.3.

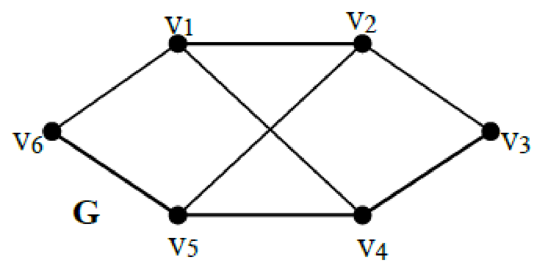

Figure 2.3.

In Figure 2.3, $S = \{v_3, v_6\}$ is a minimum eccentric dominating set of G and is also a minimum geodetic eccentric dominating set of G, $\gamma_{ed}(G) = \gamma_{ged}(G) = 2$.

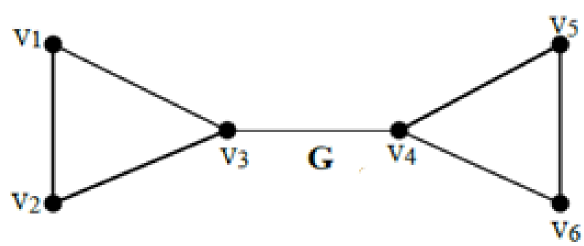

Figure 2.4.

In Figure 2.4, $S_1 = \{v_1, v_5\}$ is a minimum eccentric dominating set of G, $\gamma_{ed}(G) = 2$. $S_2 = \{v_1, v_2, v_5, v_6\}$ is a minimum geodetic eccentric dominating set of G, $\gamma_{ged}(G) = 4$.

Theorem 2.11. *Let G be a graph of radius one and diameter two with $p \geq 4$. Then $\gamma_{ged}(G) = p - 1$ if and only if G is unicentral with $\langle E_2 \rangle$ is disconnected whose components are complete graphs.*

Proof. Let D be γ_{ged}-set with $|D| = p - 1$. Let $v \in V(G)$ such that $D = V - \{v\}$.

Case (i). v is not a central vertex.

Suppose v is not a central vertex. Then D contains a central vertex u and $e(v) = 2$. Vertex u dominates all the vertices, u dominates v and v is eccentric to at least one vertex of E_2. Hence, it is not adjacent to some $w \in E_2$, $w \in D$. Since D is a γ_{ged}-set there exists $x, y \in E_2$ such that xvy is a path P_3 in $\langle E_2 \rangle$. In this case, $D - \{u\}$ is also geodetic eccentric dominating set, which is a contradiction to D is a γ_{ged}-set.

Hence, this case is not possible.

Case (ii). v is a central vertex.

Suppose there exists another central vertex z then $V(G) - \{v, z\}$ is also geodetic eccentric dominating set, which is a contradiction to $\gamma_{ged}(G) = p - 1$. Hence, G is unicentral with centre v and $D = V(G) - \{v\}$ is a geodetic eccentric dominating set. Now, if there exists an induced $P_3 = u_1u_2u_3$ in $\langle E_2 \rangle$, (In this case $p \geq 4$) then $D - \{u_2\}$ is also a geodetic eccentric dominating set, which is a contradiction. Hence, $\langle E_2 \rangle$ has no induced P_3. This implies that $\langle E_2 \rangle$ is disconnected whose components are complete graphs. ($\langle E_2 \rangle$ cannot be complete. If the components are not complete, there exists induced P_3).

Theorem 2.12. *Let G be a self-centered graph of radius two. Then $\gamma_{ged}(G) = p - 1$ if and only if $G = C_4$.*

Proof. Let G be a two self-centered graph. Suppose $\gamma_{ged}(G) = p - 1$, there exists $D = V - \{x\}$ such that $x \in I[D]$. But G is two self-centered implies that G has more than three vertices. Also, $e(x) = 2$ in G implies that $\deg(x) \geq 2$. Since, D is a dominating set and $|D| = p - 1$, there exists $u, v \in D$ such that u and v are adjacent to x. Since G is two self-centered, G is two connected and hence there exists another path from u to v of length two or three.

If $V = \{u, v, x, y\}$. Then x is eccentric to y. Hence $x \notin D$ implies that y must be in D. Hence $G = C_4$, $\gamma_{ged}(G) = 3 = p - 1$. Suppose there exists another path from u to v of length three. Let $uyzv$ be such a path. Then either y or z must be in D. We can get a geodetic eccentric dominating set $V(G) - \{v, z\}$ or $V(G) - \{x, z\}$. Hence, $\gamma_{ged}(G) \neq p - 1$.

Suppose there exists more than two paths from u to v or G has more than four elements. In this case, D is not a minimum geodetic eccentric dominating set. Hence, $\gamma_{ged}(G) = p - 1$ implies that $G = C_4$.

Remark 2.1. If G is a connected graph with radius two and diameter three, then $\gamma_{ged}(G) \neq p - 1$.

Remark 2.2. If G is a connected graph with radius two and diameter four, then $\gamma_{ged}(G) \neq p - 1$.

Remark 2.3. If G is a connected graph with radius greater than two, then $\gamma_{ged}(G) \neq 2$ and $\gamma_{ged}(G) \neq p - 1$.

Theorem 2.13. *Let G be a connected graph. Then $\gamma_{ged}(G) = p - 1$ if and only if* (i) *G is unicentral with centre v and $G - v$ is disconnected whose components are complete graphs.* (ii) $G = C_4$.

Proof. Proof follows from Theorem 2.11 and 2.12 and Remarks 2.1, 2.2, and 2.3.

Theorem 2.14. *Let G be a connected graph. Then $\gamma_{ged}(G) = p$ if and only if $G = K_p$.*

Proof. $\gamma_{ged}(G) = p$ implies that $D = V(G)$ is the only γ_{ged}-set. If there exists non-adjacent vertices then we can find a γ_{ged}-set D such that $|D| < p$.

Hence, any two vertices of G are adjacent to each other. Hence, $G = K_p$ only.

Conversely, suppose $G = K_p$, we know that $\gamma_g(G) = p$. Therefore, $\gamma_{ged}(G) = p$.

3. Geodetic Eccentric Domination in some particular classes of Graphs

The geodetic eccentric domination number of some classes of graphs is given in the following theorems.

Theorem 3.1. *If K_p is a complete graph on p vertices, then $\gamma_{ged}(K_p) = p$.*

Proof. Let $v_1, v_2, v_3, \ldots, v_p$ be the vertices of the complete graph K_p. $S = \{v_1, v_2, v_3, \ldots, v_p\}$ is the minimum geodetic eccentric dominating set of K_p, where $I[S] = V(K_p)$.

Therefore, $\gamma_{ged}(K_p) = p$.

Theorem 3.2. *If W_p is a wheel graph, then*

(i) $\gamma_{ged}(W_p) = \lceil p/2 \rceil$ *if p is odd.*

(ii) $\gamma_{ged}(W_p) = p/2$ *if p is even,* $p > 4$.

Proof. Let $v, v_1, v_2, v_3, \ldots, v_p$ be the vertices of the wheel graph W_p, where v is the central vertex.

Case (i). p is odd.

$S = \{v_1, v_3, v_5, \ldots, v_{p-2}, v_p\}$ is the minimum geodetic eccentric dominating set of W_p. Hence, $\gamma_{ged}(W_p) = \lceil p/2 \rceil$.

Case (ii). p is even.

$S = \{v_1, v_3, v_5, \ldots, v_{p-3}, v_{p-1}\}$ is the minimum geodetic eccentric dominating set of W_p. Therefore, $\gamma_{ged}(W_p) = p/2$.

Remark 3.1. $\gamma_{ged}(W_4) = 3$.

Theorem 3.3. *If* F_p *is a fan graph, then*

(i) $\gamma_{ged}(F_p) = \lceil p/2 \rceil$ *if* p *is odd.*

(ii) $\gamma_{ged}(F_p) = p/2 + 1$ *if* p *is even.*

Proof. Let $w, w_1, w_2, w_3, \ldots, w_p$ be the vertices of the fan graph F_p.

Case (i). p is odd.

$S = \{w_1, w_3, w_5, \ldots, w_{p-2}, w_p\}$ is the minimum geodetic eccentric dominating set of F_p. Hence, $\gamma_{ged}(F_p) = \lceil p/2 \rceil$.

Case (ii). p is even.

$S = \{w_1, w_3, w_5, \ldots, w_{p-1}, w_p\}$ is the minimum geodetic eccentric dominating set of F_p. Therefore, $\gamma_{ged}(F_p) = p/2 + 1$.

Theorem 3.4. *If* $K_{m,n}$ *is a complete bi-partite graph with* $m, n > 2$, *then* $\gamma_{ged}(K_{m,n}) = 4$, *where* $m + n = p$.

Proof. Let $A = \{v_1, v_2, v_3, \ldots, v_m\}$ and $B = \{w_1, w_2, w_3, \ldots, w_n\}$ be the set of vertices of $K_{m,n}$. $S = \{v_1, v_2, w_1, w_2\}$ is the minimum geodetic eccentric dominating set of $K_{m,n}$.

Therefore, $\gamma_{ged}(K_{m,n}) = 4$.

Remark 3.2. $\gamma_{ged}(K_{1,n}) = n, n > 1$ and $\gamma_{ged}(K_{2,n}) = 3, n > 1$.

Theorem 3.5. *If* $K_{1,n}$ *is a star graph, then* $\gamma_{ged}(K_{1,n}) = n, n > 1$, *where* $p = n + 1$.

Proof. Let $v, v_1, v_2, v_3, \ldots, v_n$ be the vertices of the star graph $K_{1,n}$, where v is the central vertex of $K_{1,n}$. $S = \{v_1, v_2, v_3, \ldots, v_n\}$ is the minimum geodetic eccentric dominating set of $K_{1,n}$, where $I[S] = V(K_{1,n})$. Therefore, $\gamma_{ged}(K_{1,n}) = n, n > 1$, where $p = n + 1$.

Remark 3.3. $\gamma_{ged}(K_{1,n}) = 2.$

Theorem 3.6. *If* P_p *is a path graph,* $p \geq 3.$ *Then*

(i) $\gamma_{ged}(P_p) = \lceil p/3 \rceil$ *if* $p = 3k + 1.$

(ii) $\gamma_{ged}(P_p) = \lceil p/3 \rceil + 1$ *if* $p = 3k$ *or* $3k + 2.$

Proof. Let $v_1, v_2, v_3, \ldots, v_p$ represent the path P_p.

Case (i). $p = 3k.$

$S = \{v_2, v_5, v_8, \ldots, v_{3k-1}\}$ is the only minimum dominating set in P_p.

$S' = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k}\}$ is a geodetic dominating set of P_p. S' is also a geodetic eccentric dominating set of P_p.

Thus, $\gamma_{ged}(P_p) = \gamma_g(P_p) = \lceil p/3 \rceil + 1.$

Case (ii). $p = 3k + 1.$

$S = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k+1}\}$ is the minimum geodetic dominating set of P_p. S is also a geodetic eccentric dominating set of P_p. Thus, $\gamma_{ged}(P_p) = \gamma_g(P_p) = \lceil p/3 \rceil.$

Case (iii). $p = 3k + 2.$

$S = \{v_1, v_2, v_5, v_8, \ldots, v_{3k+2}\}$ is the minimum geodetic dominating set of P_p. S is also a geodetic eccentric dominating set of P_p. Thus, $\gamma_{ged}(P_p) = \gamma_g(P_p) = \lceil p/3 \rceil + 1.$

Theorem 3.7. *If* C_p *is a cycle graph,* $p \geq 6,$ *then*

(i) $\gamma_{ged}(C_p) = p/2$ *if p is even.*

(ii) $\gamma_{ged}(C_p) = \lceil p/3 \rceil$ *(or)* $\lceil p/3 \rceil + 1$ *if p is odd.*

Proof of (i). Let $p = 2k$ and $k > 2.$

Let the cycle C_p be $v_1 v_2 v_3 \ldots v_{2k} v_1$. Each vertex of C_p has exactly one eccentric vertex (that is C_p unique eccentric vertex graph).

Hence, $\gamma_{ged}(C_p) \geq \gamma_g(C_p) \geq \gamma_{ed}(C_p) \geq p/2.$ (1)

Case (i). k-odd.

$S = \{v_1, v_3, \ldots, v_k, v_{k+2}, \ldots, v_{2k-1}\}$ is the minimum geodetic dominating set of C_p. S is also a geodetic eccentric dominating set of C_p. Therefore, $\gamma_{ged}(C_p) \leq p/2.$ (2)

From (1) and (2), $\gamma_{ged}(C_p) = p/2.$

Case (ii). k-even.

$S = \{v_1, v_3, \ldots, v_{k-1}, v_{k+2}, \ldots, v_{2k}\}$ is the minimum geodetic dominating set of C_p. Vertex V_i is an eccentric vertices of v_{i+k}. S is also a geodetic eccentric dominating set of C_p. Therefore, $\gamma_{ged}(C_p) \leq p/2.$ (3)

From (1) and (3), $\gamma_{ged}(C_p) = p/2.$

Proof of (ii). When p is odd, each vertex of C_p has exactly two eccentric vertices. If $p = 2k+1$, $v_i \in V(G)$ has v_{i+1}, v_{i+k+1} as eccentric vertices.

Case (i). $p = 3m, m \geq 3.$

Also $p = 3m$, p is odd $\Rightarrow m$ is odd.

$S = \{v_1, v_4, \ldots, v_k, v_{k+3}, \ldots, v_{2k-1}\}$ is the minimum geodetic dominating set of C_p. Vertex V_i is an eccentric vertex of v_{i+k} and v_{i+k+1}. S is also a geodetic eccentric dominating set of C_p. Therefore, $\gamma_{ged}(C_p) \leq \lceil p/3 \rceil.$ (4)

By Theorem 1.6, $\gamma_{ged}(C_p) = \lceil p/3 \rceil \leq \gamma_{ged}(C_p).$ (5)

From (4) and (5), $\gamma_{ged}(C_p) = \lceil p/3 \rceil.$

Case (ii). $p = 3m+1, m \geq 2.$

Also $p = 3m+1$, p is odd $\Rightarrow m$ is even.

$S = \{v_1, v_4, \ldots, v_{k+1}, v_{k+3}, v_{k+6}, \ldots, v_{2k-1}\}$ is the minimum geodetic dominating set of C_p. Vertex V_i is an eccentric vertex of v_{i+k} and v_{i+k+1}. S is also a geodetic eccentric dominating set of C_p. Therefore, $\gamma_{ged}(C_p) \leq \lceil p/3 \rceil$. (6)

By Theorem 1.6, $\gamma_{ed}(C_p) = \lceil p/3 \rceil \leq \gamma_{ged}(C_p)$. (7)

From (6) and (7), $\gamma_{ged}(C_p) = \lceil p/3 \rceil$.

Case (iii). $p = 3m + 2$, $m \geq 1$.

Also $p = 3m + 2$, p is odd $\Rightarrow m$ is odd.

$S = \{v_1, v_4, \ldots, v_{k-1}, v_k, v_{k+3}, \ldots, v_{2k+1}\}$ is the minimum geodetic dominating set of C_p. Vertex V_i is an eccentric vertex of v_{i+k} and v_{i+k+1}. S is also a geodetic eccentric dominating set of C_p. Therefore, $\gamma_{ged}(C_p) \leq \lceil p/3 \rceil + 1$. (8)

By Theorem 1.6, $\gamma_{ed}(C_p) = \lceil p/3 \rceil + 1 \leq \gamma_{ged}(C_p)$. (9)

From (8) and (9), $\gamma_{ged}(C_p) = \lceil p/3 \rceil + 1$.

Remark 3.4. $\gamma_{ged}(C_3) = \gamma_{ged}(C_4) = \gamma_{ged}(C_5) = 3$.

Theorem 3.8. *If $P_n \circ K_1$ is a path corona, then $\gamma_{ged}(P_n \circ K_1) = n$, where $2n = p$.*

Proof. Let $A = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices of P_n and $B = \{w_1, w_2, w_3, \ldots, w_n\}$ be the set of pendant vertices attached at $v_1, v_2, v_3, \ldots, v_n$ respectively. $S = \{w_1, w_2, w_3, \ldots, w_n\}$ is the minimum eccentric dominating set of $P_n \circ K_1$. $I[S] = V(P_n \circ K_1)$. Hence, S is also a geodetic eccentric dominating set. Therefore, $\gamma_{ged}(P_n \circ K_1) = n$, where $2n = p$.

Theorem 3.9. *If $C_n \circ K_1$ is a cycle corona, then $\gamma_{ged}(C_n \circ K_1) = n$, where $2n = p$.*

Proof. Let $A = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices of C_n and

$B = \{w_1, w_2, w_3, \ldots, w_n\}$ be the set of pendant vertices attached at $v_1, v_2, v_3, \ldots, v_n$ respectively. $S = \{w_1, w_2, w_3, \ldots, w_n\}$ is the minimum eccentric dominating set of $C_n \circ K_1$. $I[S] = V(C_n \circ K_1)$. Hence, S is also a geodetic eccentric dominating set. Therefore, $\gamma_{ged}(C_n \circ K_1) = n$, where $2n = p$.

Theorem 3.10. *If* $K_{1,n} \circ K_1$ *is a star corona, then* $\gamma_{ged}(K_{1,n} \circ K_1) = n + 1$, *where* $2n + 2 = p$.

Proof. Let $A = \{v, v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices of $K_{1,n}$ and $B = \{w, w_1, w_2, w_3, \ldots, w_n\}$ be the set of pendant vertices attached at $v, v_1, v_2, v_3, \ldots, v_n$ respectively. $S = \{w, w_1, w_2, w_3, \ldots, w_n\}$ is the minimum eccentric dominating set of $K_{1,n} \circ K_1$. $I[S] = V(K_{1,n} \circ K_1)$. Hence, S is also a geodetic eccentric dominating set.

Therefore, $\gamma_{ged}(K_{1,n} \circ K_1) = n + 1$, where $2n + 2 = p$.

Theorem 3.11. *If* $K_{1,n,n}$ *is a spider, then* $\gamma_{ged}(K_{1,n,n}) = n + 1$ *where* $2n + 1 = p$.

Proof. Let $K_{1,n,n}$ be a spider. Let w be a vertex of maximum degree $\Delta(K_{1,n}, n)$ and S be the set of pendant vertices. The set $S \cup \{w\}$ form a minimum geodetic eccentric dominating set, where $I[S \cup \{w\}] = V(K_{1,n}, n)$. Therefore, $\gamma_{ged}(K_{1,n}, n) = n + 1$.

Theorem 3.12. *If* G *is a wounded spider (not a path), then* $\gamma_{ged}(G) = r + h$, *where* r *is the number of non-wounded legs,* h *is the number of wounded legs.*

Proof. Let G be a wounded spider. Let w be a vertex of maximum degree $\Delta(G)$ and R be the set of pendant vertices which are adjacent to vertices of degree two, H be the set of pendant vertices which are adjacent to w. The set $R \cup H$ form a minimum geodetic eccentric dominating set, where $I[R \cup H] = V(G), |R| = r, |H| = h$. Therefore, $\gamma_{ged}(G) = r + h$.

Conclusion

Here, we have studied geodetic eccentric domination in some families of graphs and also found out some bounds for geodetic eccentric domination number of a graph. Also, we have characterized graphs for which $\gamma_{ged}(G) = 2,\ p-1,$ and p.

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