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A SHORT REVIEW ON IDEAL CONVERGENCE AND ITS GENERALIZATIONS

MEENAKSHI and SURAJ SHARMA

Department of Mathematics
Chandigarh University
Gharuan, Punjab, India
E-mail: chawlameenakshi7@gmail.com
bhardwajsuraj070@gmail.com

Abstract

In this article, our aim is to review the generalized convergence named as Ideal convergence. In section 1, the basic terminologies or notation related to usual convergence are given. The basic definitions and examples related to ideal convergence and its generalized forms are given in section 2. In section 3, the main purpose is to review the ideal convergence in different spaces.

1. Introduction

The usual convergence has generalized in many approaches. One of the most recent generalization named as statistical convergence of idea of usual convergence was introduced in 1951 by Fast [3] as a generalized summability method by using the notion of natural density. The another generalized concept named as Ï- convergence of sequence was introduced by Kostyrko, Macaj and Salat [7] that depends on the belief of ideals in $\mathbb{N}$. The aim is to explore elementary view and results in the field of ideal convergence. First of all, the basic terminologies are given as follows:

Definition 1.1. A sequence on a set S is a function having domain as the set $\mathbb{N}$ ("set of all natural numbers") and range can be any subset of set S, i.e. a function $f : \mathbb{N} \to K$ is called a sequence. The range set $\{f_1, f_2, f_3, \ldots\}$ represents a sequence. The name of the sequences is decided by the range of

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the sequence. If the range is $\mathbb{R}$ ("set of real numbers"), then the sequence is called "real sequence".

Remark 1. Notation: Usually, the sequence is represented by the notation $x = \{x_r\}; r \in N$. For every value of r, the term x_r is known as r^{th} term of x.

Example 1.1. The function $f : \mathbb{N} \to \mathbb{R}$ illustrate by $f(r) = \frac{1}{r}; r \in \mathbb{N}$ is a Sequence, i.e., $\left\{1, \frac{1}{2}, \frac{1}{3} \dots, \frac{1}{r}, \dots\right\}$ is a sequence.

Definition 1.2. A real sequence $x = \{x_r\}; r \in \mathbb{N}$ will converge to a number c if for any $\varepsilon > 0$, we are able to find positive integer $k(\varepsilon) \in \mathbb{N}$ depending on ε such that $|x_r - c| < \varepsilon, \forall r \geq k(\varepsilon)$. In this case, we can write it as : $x_r \to c$ as $r \to \infty$. Then c is termed as the limit of sequence x_r.

Example 1.2. The sequence $x = \{x_r\}; r \in N$ where $x_r = 1 + \frac{1}{r}$ is a sequence $\left\{2, \frac{3}{2}, \frac{4}{3}, \dots, 1 + \frac{1}{r}, \dots\right\}$ converging to 1.

Definition 1.3. The sequence $x = \{x_r\} : r \in \mathbb{N}$ is bounded if we are able to find a real constant $B > 0$, which satisfies the inequality $|x_r| \leq B : \forall r \in \mathbb{N}$.

Example 1.3. The sequence given by $x = \{x_r\} : r \in \mathbb{N}$ where $x_r = (-1)^r$ is a bounded sequence.

Next, we are defining the concept of generalized convergence named as statistical convergence that is depend on natural density, which is one of the possibilities to measure how large a subset of natural numbers is.

Definition 1.4. Any subset B of natural numbers has natural density a if the proportion of elements of B among all natural numbers from 1 to n converges to a as n tends to infinity.

The notation of natural density for a set B is given by $\delta(B)$.

2. Ideal Convergence

The convergence of sequence has always been the centre of attraction for the researchers. Many different types of convergence of Sequences were studied such as usual convergence, pointwise convergence and uniform convergence.

In 1951, Fast [3] generalized the notion of usual convergence and introduced a new convergence named as statistical convergence as a method of generalized summability by using the idea of natural density of the subset of $\mathbb{N}$.

Definition 2.1 [3]**.** The sequence of numbers $x = \{x_r\}$; $r \in \mathbb{N}$ will converge statistically to a number A, for each $\epsilon > 0$, the following condition holds

$$\delta(\{r \leq n : | x_r - A | > \epsilon\}) = 0.$$

In that case, we can write $S - \lim_{k \to \infty} x_r = A$.

Definition 2.2 [3]**.** A sequence $\{x_r\}$; $r \in \mathbb{N}$ is called statistically bounded if we are able to find a number S_B such that

$$\delta(\{r \leq n : |x_r| > S_B\}) = 0.$$

Our main focus is to review ideal convergence and its basic properties.

In 2000, Kostyrko et al. [7] introduced a new type of convergence called $\ddot{I}$-convergence of real sequences that is depend on the belief of ideals of subset of $\mathbb{N}$ (“set of all natural numbers”). Different types of convergence relateded to the statistical convergence are linked by $\ddot{I}$- Convergence. $\ddot{I}$- convergence gives a common idea to examine properties of many different types of convergence of a sequence. The authors introduce some examples and several results for $\ddot{I}$- Convergence by using the axioms of usual Convergence which are given as follows:

Definition 2.3. A non-void class $\ddot{I} \subset 2^{\mathbb{N}}$ will be an ideal in $\mathbb{N}$ if,

1. $\emptyset \in \ddot{I}$.

2. $S, R \in \ddot{I}$ implies $S \cup R \in \ddot{I}$ and

3. $S \in \ddot{I}$ and $R \subset S$ implies $R \in \ddot{I}$.

if $\ddot{I} \neq 2^{\mathbb{N}}$, then it is termed as non-trivial ideal.

Definition 2.4. A non-null class $F \subset 2^{\mathbb{N}}$ is called a Filter in $\mathbb{N}$ if,

1. $\emptyset \notin F$.

2. $S, R, \in F$ implies $S \cap R \in F$ and

3. $S \in F$ and $R \supset S$ implies $R \in F$.

Definition 2.5. A non-trivial ideal $\ddot{I} \subset 2^{\mathbb{N}}$ having all the singletons *i. e.* $\{\{y\} : y \in \mathbb{N}\}$ is known as admissible ideal in $\mathbb{N}$.

Definition 2.6 [7]**.** A real sequence $x = \{x_r\}; r \in \mathbb{N}$ of real numbers will $\ddot{I}$- converge to a number R provided that, for any $\epsilon > 0$,

$$\{r \in \mathbb{N} : | x_r - R | \geq \epsilon\} \in \ddot{I}.$$

Example 2.1. Every constant sequence $x = \{r, r, r, r \ldots\}$ is converge ideally to r.

In 2001, Kostyrko and Salat [7] further introduced and extended the belief of $\ddot{I}$- convergence of sequence of numbers defined on a metric space to the $\ddot{I}$- convergence on sequence of functions defined on a metric space, where $\ddot{I}$ is an Ideal.

In 2004, Nabiev et al. [2] defined and studied some axioms of $\ddot{I}$- Cauchy sequence and $\ddot{I}^*$- Cauchy sequence and the decomposition theorem is proved for $\ddot{I}$ -convergent sequences, which was introduced by Kostyrko [7] in 2000.

Definition 2.7 [2]**.** Let (Z, σ) denotes a metric space and for the admissible ideal $\ddot{I} \subset 2^{\mathbb{N}}$, the sequence $\{x_r\} : r \in \mathbb{N}$ in Z is called an

$\ddot{\text{I}}$- Cauchy sequence in Z if for any $\varepsilon > 0$ we are able to find a positive integer $S = S(\varepsilon)$ such that

$$A(\varepsilon) = \{r \in \mathbb{N} : \sigma(x_r, x_S) > \epsilon\} \in \ddot{\text{I}}.$$

In 2004, Salat et al. [13] build up some ideas like $\ddot{\text{I}}$- monotonic increasing (decreasing) sequence outcomes from the statistical convergence to the ideal convergence, introduced in [7] as the generalization of the statistical convergence.

Definition 2.8 [13]**.** Let $x = \{x_r\} : r \in \mathbb{N}$ denotes a real sequence. Then x will be $\ddot{\text{I}}$- monotonically increasing (decreasing), if we are able to find a set $\{s_1 < s_2 < \ldots\} \in \ddot{F}$ such that $x_{s_i} \leq x_{s_{i+1}} (x_{s_i} \geq x_{s_{i+1}})$ for each $i \in \mathbb{N}$.

In 2005, Salat et al. [6] further bring up the conception $\ddot{\text{I}}$- limit points and derive its various elementary results in the field of $\ddot{\text{I}}$- convergence.

In 2007, Filipow et al. [4] has generalized Bolzano-Weierstrass theorem (BWT) on ideal convergence and give some examples of the ideals having Bolzano-Weierstrass property, and examples of Ideals not having Bolzano-Weierstrass property.

In 2010, Albayrak and Pehlivan [1] examine and give two important results on the ideal convergence of subsequences of a real sequence by using the concept of statistical convergence of subsequences of a real sequence, which is given in 2000 by Zeager [15].

In 2012, Pal et al [11] investigated the belief of rough ideal convergence by extending the idea of rough convergence with the ideals which naturally develops the previous concept of rough convergence and rough stat-convergence.

In 2014, Filipow and Staniszewski [5] introduce ideal equal convergence of a sequence of functions as a generalization of equal convergence introduced by Csaszar and Laczkovich in 1975.

3. Ideal Convergence in Different Spaces

The main purpose is to review the ideal convergence in different spaces.

In 2005, Lahiri and Das [8] broad the notion of Ï- convergence and $\ddot{I}^*$- convergence of sequences in the topological space and deduce various elementary characteristics of this generality in the topological space.

In 2007, Gunawan et al. [12] proposed and examined Ï- Convergence in 2-normed spaces and explains some different spaces for convergent sequences by the belief of 2-normed space.

In 2009, Mursaleen et al. [10] studied the concept of Ï- convergence in intuitionistic fuzzy normed spaces for the convergence of double sequences and introduced the concept of Ï- convergence and Ï- Cauchy for double sequences in intuitionistic fuzzy normed spaces.

In 2010, Mursaleen and Mohiudden [10] studied and proposed a relation between $\ddot{I}_2$- convergence and $\ddot{I}_2^*$- convergence in prob-normed space.

i.e. show that,

$$\ddot{I}_2^* \text{-convergence} \Rightarrow \ddot{I}_2\text{-convergence},$$

and also give example to prove that, in general,

$$\ddot{I}_2\text{-convergence} \not\Rightarrow \ddot{I}_2^*\text{-convergence}$$

in the prob-normed space.

In 2012, Mohiuddine et al. [9] investigated and characterized the belief of Ï- convergence and $\ddot{I}^*$- convergence in random “2-normed space” for the double sequences and proposed a relation in between the two different types of convergence.

In 2012, Sarabadan and Talebi [14] originated and examined the belief of Ï -convergence in 2-normed space for double sequences and extended this generality to Ï- limit points. The authors also prove various elementary properties.

4. Observations

Ideal Convergence is the generalized case of usual as well as statistical convergence because for the consideration of the collection of finite subsets of $\mathbb{N}$ ("the set of natural numbers"), this class will become an admissible ideal. The convergence will correspond to usual convergence of real numbers.

Also if we consider the collection of those subsets of natural numbers whose natural density is zero, then that collection will be an admissible ideal and this convergence will correspond to with stat-convergence.

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