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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH THREE-POINT FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

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Abstract

In this paper, we study the existence and uniqueness of solutions for a three-point fractional integral boundary value problem. New existence and uniqueness results are established using Banach contraction principle. Other existence results are obtained using Krasnoselskii and Scheafer's fixed point theorem. At the end, some illustrative examples are presented.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, and fitting of experimental data. Multipoint boundary value problems arise in different areas of physics and mathematics. The most

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commonly quoted example in this respect is modeling the vibration of a guy-wire with n parts of different densities, but having uniform cross-section. Bitsadze and Samarski [1] initiated the study of multi-point boundary value problems for integer order differential equations. Later II'in and Moiseev [2, 3] played the leading role for the development of the existence theory of such problems. Since then, the multi-point boundary value problems have been investigated by several researchers.

In 2011, Ahmed et al. [4] studied the following boundary value problem of fractional order differential equations with three-point integral boundary conditions:

$$^{c}D^{q}x(t) = f(t, x(t)),\ 0 < t < 1,\ 1 < q \leq 2,$$

$$x(0) = 0,\ x(1) = \alpha\int_{0}^{\eta} x(s)ds,\ 0 < \eta < 1,$$

where $^{c}D^{q}$ denotes the Caputo fractional derivative of order q, $f : [0, 1] \times X \to X$ is continuous, and $\alpha \in \mathbb{R}$ is such that $\alpha \neq \frac{2}{\eta^2}$. Existence and uniqueness results are proved via Banach's contraction principle, Krasnoselskii's fixed point theorem and Leray-Schauder's degree theory.

In 2012, Sudsutad et al. [5] discussed the existence and uniqueness of the following boundary value problem of fractional order differential equations with three-point fractional integral boundary conditions:

$$^{c}D^{q}x(t) = f(t, x(t)),\ t \in [0, 1],\ q \in (1, 2],$$

$$x(0) = 0,\ \alpha[I^{p}x](\eta) = x(1),\ 0 < \eta < 1,$$

where $^{c}D^{q}$ denotes the Caputo fractional derivative of order q, I^{p} is the Riemann-Liouville fractional integral of order p, $f \in C([0, 1] \times \mathbb{R})$ and $\alpha \in \mathbb{R}$, $\alpha \neq \Gamma(p+2)/\eta^{p+1}$. Existence and uniqueness results are proved via Banach's contraction principle and Schaefer's fixed point theorem. The results of [5] are completed in [6], by existence results via Krasnoselskii's fixed point theorem and Leray-Schauders degree theory, and extended to cover the multivalued case.

Lakoud and Khaldi [7] discussed the existence and uniqueness results via Banach's contraction principle and Leray-Schauder's nonlinear alternative of fractional differential equations with a fractional integral condition. Ntouyas [8] proved existence and uniqueness results for a single and multivalued case for fractional order differential equations with nonlocal and fractional integral boundary conditions. Ahmad et al. [9] investigated the existence and uniqueness results for fractional order differential equations with four-point nonlocal Riemann-Liouville fractional integral boundary conditions. For some recent development on the topic, see [10]-[14] and the references therein.

Motivated by above mentioned works, in this article, we have presented a new type of three-point nonlinear fractional boundary value problem

$$\begin{cases} {}^cD^q z(\xi) = h(\xi, z(\xi)), \ 2 < q \leq 3, \ \xi \in [0, 1] \\ z(\eta) = 0, \ z'(0) = 0, \ I^p z(1) = 0, \ 0 < \eta < 1, \end{cases} \tag{1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, I^p is the Riemann-Liouville fractional integral of order p, $h : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\eta^2 \neq \frac{2}{p^2 + 3p + 2}$. By $C([0, 1], \mathbb{R})$ we denote the Banach space of all continuous functions from [0; 1] into R with the norm

$$\| z \| = \sup \{| z(\xi) | : \xi \in [0, 1]\}.$$

In this paper, first we prove the uniqueness of solution of (1) by using Banach fixed point theorem under suitable conditions. Secondly, we discuss the existence of solution by means of Krasnoselskii's fixed point theorem and Schaefer's fixed point theorem on the interval [0, 1]. Finally, we illustrate some examples for validation of our results.

2. Preliminaries

In this section, we introduce notations, definitions of fractional calculus and prove a lemma before stating our main results.

Definition 1. For a continuous function $f : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^qf(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}f^{(n)}(s)ds,\ n-1 < q < n,\ n = [q]+1$$

provided that $f^n(t)$ exists, where $[q]$ denotes the integer part of the real number q.

Definition 2. The Riemann-Liouville fractional integral of order q for a continuous function $f(t)$ is defined as

$$I^qf(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds,\ q > 0,$$

provided that such integral exists.

Lemma 3 ([15]). *Let* $q > 0$, *then*

$$I^q\ {}^cD^qu(t) == u(t) + k_0 + k_1t + +k_2t^2 + \ldots + k_{n-1}t^{n-1},$$

for some $k_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1$ *where n is the smallest integer greater than or equal to* q.

Lemma 4. *Let* $\eta^2 \neq \frac{2}{p^2+3p+2}$, $2 < q \leq 3,\ 0 < \eta < 1.$ *Then for* $w \in C([0, 1], \mathbb{R})$, *the problem*

$$^cD^qz(t) = w(\xi),\ 0 < \xi < \xi < 1, \tag{2}$$

$$z(\eta) = 0,\ z'(0) = 0,\ I^pz(1) = 0 \tag{3}$$

has a unique solution

$$z(\xi) = \frac{1}{\Gamma(q)}\int_0^\xi (\xi-s)^{q-1}w(s)ds - \frac{1}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}w(s)ds$$

$$+ \frac{(\eta^2-\xi^2)\Gamma(p+3)}{\Gamma(p+q)Q}\int_0^1 (1-s)^{p+q-1}w(s)ds$$

$$- \frac{(p+2)(p+1)(\eta^2-\xi^2)}{\Gamma(p)Q}\int_0^\eta (1-s)^{q-1}w(s)ds, \tag{4}$$

where

$$Q = 2 - \eta^2(p^2+3p+2).$$

Proof. In view of Lemma 3, we may reduce (2) to an equivalent integral equation

$$z(\xi) = \frac{1}{\Gamma(q)}\int_0^{\xi}(\xi - s)^{q-1}w(s)ds - c_0 - c_1\xi - c_2\xi^2 \tag{5}$$

for some $c_0, c_1, c_2 \in \mathbb{R}$.

From $z'(0) = 0$, it follows $c_1 = 0$.

$$z(\eta) = 0 \Rightarrow \frac{1}{\Gamma(q)}\int_0^{\eta}(\eta - s)^{q-1}w(s)ds - c_0 - c_2\eta^2 = 0 \tag{6}$$

$$I^p[z(\xi)] = \frac{1}{\Gamma(p)}\int_0^{\xi}(\xi - s)^{p-1}[I^q w(s) - c_0 - c_2 s^2]ds$$

$$= I^p I^q w(\xi) - \frac{c_0\xi^p}{\Gamma(p+1)} - \frac{c_2\xi^{p+2}\Gamma(3)}{\Gamma(p+3)}$$

$$= \frac{1}{\Gamma(p+q)}\int_0^{\xi}(\xi - s)^{p+q-1}w(s)ds - \frac{c_0\xi^p}{\Gamma(p+1)} - \frac{2c_2\xi^{p+2}}{\Gamma(p+3)} \tag{7}$$

$$I^p[z(1)] = 0 \Rightarrow \frac{1}{\Gamma(p+q)}\int_0^{1}(1 - s)^{p+q-1}w(s)ds - \frac{c_0}{\Gamma(p+1)} - \frac{2c_2}{\Gamma(p+3)} = 0$$

solving (6) and (7) for $c_0, c_2,$ we have:

$$c_2 = \frac{\Gamma(p+3)}{\Gamma(p+q)Q}\int_0^{1}(1 - s)^{p+q-1}w(s)ds$$

$$- \frac{(p+2)(p+1)}{\Gamma(q)Q}\int_0^{\eta}(\eta - s)^{q-1}w(s)ds$$

and

$$c_0 = \frac{1}{\Gamma(q)}\int_0^{\eta}(\eta - s)^{q-1}w(s)ds - \frac{\eta^2\Gamma(p+3)}{\Gamma(p+q)Q}\int_0^{1}(1 - s)^{p+q-1}w(s)$$

$$+ \frac{\eta^2(p+2)(p+1)}{\Gamma(q)Q}\int_0^{\eta}(\eta - s)^{q-1}w(s)ds$$

putting the values of c_0, c_1, c_2 in (5), we obtain the solution (4). □

In view of Lemma 4, the BVP (1) can be written in the fixed point problem. For this, we consider the operator $P : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ defined by

$$P(z)(\xi) = \frac{1}{\Gamma(q)} \int_0^{\xi} (\xi - s)^{q-1} h(s, z(s)) ds - \frac{1}{\Gamma(q)} \int_0^{\eta} (\eta - s)^{q-1} h(s, z(s)) ds$$

$$+ \frac{(\eta^2 + \xi^2)\Gamma(p+3)}{\Gamma(p+q)Q} \int_0^1 (1-s)^{p+q-1} h(s, z(s)) ds$$

$$- \frac{(p+2)(p+1)(\eta^2 - \xi^2)}{\Gamma(q)Q} \int_0^{\eta} (\eta - s)^{q-1} h(s, z(s)) ds. \tag{8}$$

To prove the main results, we need the following assumptions:

(A) Assume that there exists a constant $L > 0$ such that $| h(\xi, z_1) - f(\xi, z_2) | \leq L | z_1 - z_2 |$ for each $\xi \in [0, 1]$, and all $z_1, z_2 \in \mathbb{R}$

(B) $| h(\xi, z) | \leq \phi(\xi)$ for all $(\xi, z) \in [0, 1] \times \mathbb{R}$ and $\phi \in C([0, 1], \mathbb{R}^+)$

(C) There exists a constant $\mu > 0$ such that $| h(\xi, z) | \leq \mu$ for each $\xi \in [0, 1]$ and $z \in \mathbb{R}$.

To simplify and our convenience, we put

$$\Lambda = \frac{2}{\Gamma(q+1)} + \frac{(p+2)(p+1)}{\Gamma(q+1)|Q|} + \frac{\Gamma(p+3)}{\Gamma(p+q+1)|Q|}. \tag{9}$$

3. Main Results

Theorem 5. *Assume that* (A) *holds and if* $L < \frac{1}{\Lambda}$, *where* Λ *is defined by* (8), *then the BVP* (1) *has a unique solution on* [0, 1].

Proof. Obviously, the fixed points of the operator P defined by (8) are solution of the problem (1). We shall use the Banach Contraction Principle to prove that P has a fixed point. For that, we shall prove P is a contraction.

Let $z_1, z_2 \in C([0, 1], \mathbb{R})$, then for each $\xi \in [0, 1]$, we have

$$
\begin{aligned}
| P(z_1)(\xi) - P(z_2)(\xi) | &\leq \frac{1}{\Gamma(q)} \int_0^{\xi} (\xi - s)^{q-1} | h(s, z_1(s)) - h(s, z_2(s)) | ds \\
&+ \frac{1}{\Gamma(q)} \int_0^{\eta} (\eta - s)^{q-1} | h(s, z_1(s)) - h(s, z_2(s)) | ds \\
&+ \frac{\Gamma(p+3)}{\Gamma(p+q)|Q|} \int_0^1 (1-s)^{p+q-1} | h(s, z_1(s)) - h(s, z_2(s)) | ds \\
&+ \frac{(p+2)(p+1)}{\Gamma(q)|Q|} \int_0^{\eta} (\eta - s)^{q-1} | h(s, z_1(s)) - h(s, z_2(s)) | ds \\
&\leq \frac{L\| z_1 - z_2 \|}{\Gamma(q)} \int_0^{\xi} (\xi - s)^{q-1} ds + \frac{L\| z_1 - z_2 \|}{\Gamma(q)} \int_0^{\eta} (\eta - s)^{q-1} ds \\
&+ \frac{(p+2)(p+1)L\| z_1 - z_2 \|}{\Gamma(q)|Q|} \int_0^{\eta} (\eta - s)^{q-1} ds \\
&+ \frac{\Gamma(p+3)L\| z_1 - z_2 \|}{\Gamma(p+q)|Q|} \int_0^1 (1-s)^{p+q-1} ds \\
&\leq \frac{L\| z_1 - z_2 \|}{\Gamma(q+1)} + \frac{L\| z_1 - z_2 \|}{\Gamma(q+1)} + \frac{(p+2)(p+1)L\| z_1 - z_2 \|}{\Gamma(q+1)|Q|} \\
&+ \frac{\Gamma(p+3)L\| z_1 - z_2 \|}{\Gamma(p+q+1)|Q|}.
\end{aligned}
$$

Thus

$$\| P(z_1) - P(z_2) \| \leq L\Lambda \| z_1 - z_2 \|.$$

As $L < \frac{1}{\Lambda}$, therefore, P is a contraction, which satisfies all the conditions of Banach Contraction Principle. Hence, P has a unique fixed point which is a solution of the problem (1). □

We prove the following result by using Krasnoselskii's fixed point theorem.

Theorem 6. *Let* $h : [0, 1] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function and the assumption* (A), (B) *holds with*:

$$L \left\{ \frac{(p+2)(p+1)}{\Gamma(q+1)|Q|} + \frac{\Gamma(p+3)}{\Gamma(p+q+1)|Q|} \right\} < 1. \tag{10}$$

Then the boundary value problem (1) *has at least one solution defined on* $[0, 1]$.

Proof.

Suppose $\sup_{\xi\in[0,1]} |\phi(\xi)| = \|\phi\|$, we fix

$$\epsilon \geq \|\phi\|\left\{\frac{2}{\Gamma(q+1)} + \frac{(p+1)(p+2)}{\Gamma(q+1)|Q|} + \frac{\Gamma(p+3)}{\Gamma(p+q+1)|Q|}\right\}$$

and consider $B_\epsilon = \{z \in C([0, 1], \mathbb{R}) : \|z\| \leq \epsilon\}$, We define the operators P_1 and P_2 on B_ϵ as:

$$(P_1 z)(\xi) = \frac{1}{\Gamma(q)}\int_0^\xi (\xi - s)^{q-1} h(s, z(s))ds$$

$$-\frac{1}{\Gamma(q)}\int_0^\eta (\eta - s)^{q-1} h(s, z(s))ds,\ \xi \in [0, 1]$$

$$(P_2 z)(\xi) = \frac{(\eta^2 - \xi^2)\Gamma(p+3)}{\Gamma(p+q)Q}\int_0^1 (1-s)^{p+q-1} h(s, z(s))ds$$

$$-\frac{(p+2)(p+1)(\eta^2 - \xi^2)}{\Gamma(q)Q}\int_0^\eta (\eta - s)^{q-1} h(s, z(s))ds,\ \xi \in [0, 1].$$

For $z_1, z_2 \in B_\epsilon$, we have:

$$|P_1 z_1 + P_2 z_2\| \leq \frac{\|\phi\|}{\Gamma(q)}\int_0^\xi (\xi - s)^{q-1} ds + \frac{\|\phi\|}{\Gamma(q)}\int_0^\eta (\eta - s)^{q-1} ds$$

$$+ \frac{\|\phi\||\eta^2 - \xi^2|\Gamma(p+3)}{\Gamma(p+q)|Q|}\int_0^1 (1-s)^{p+q-1} ds$$

$$+ \frac{\|\phi\|(p+2)(p+1)|\eta^2 - \xi^2|}{\Gamma(q)|Q|}\int_0^\eta (\eta - s)^{q-1} ds$$

$$\leq \frac{2\|\phi\|}{\Gamma(q+1)} + \frac{\|\phi\|(p+1)(p+2)}{\Gamma(q+1)|Q|}$$

$$+ \frac{\|\phi\|\Gamma(p+3)}{\Gamma(p+q+1)|Q|}$$

$$\leq \epsilon.$$

Thus $z_1, z_2 \in B_\epsilon \Rightarrow P_1 z_1 + P_2 z_2 \in B_\epsilon$.

From (A) and (10), P_2 is a contraction mapping.

From the continuity of h, one obtains that the operator P_1 is continuous. Also P_1 is uniformly bounded on B_ϵ as

$$\| P_1 z \| \leq \frac{2\| \phi \|}{\Gamma(q+1)}.$$

Now we prove that the operator P_1 is compact.

Define $\sup_{(\xi, z)\in[0, \;]\times B_\epsilon} | h(\xi, z) | = h_s$.

Then, we have

$$| (P_1 z)(\xi_1) - (P_1 z)(\xi_2) | = | \frac{1}{\Gamma(q)} \int_0^{\xi_1} [(\xi_2 - s)^{q-1} - (\xi_1 - s)^{q-1}] h(s, z(s)) ds$$

$$+ \int_{\xi_1}^{\xi_2} (\xi_2 - s)^{q-1} h(s, z(s)) ds |$$

$$\leq \frac{h_s}{\Gamma(q+1)} | 2(\xi_2 - \xi_1)^q + \xi_1^q - \xi_2^q |$$

which is independent of z and tends to zero as $\xi_1 \to \xi_2$. Thus P_1 is equicontinuous.

Hence by Arzela-Ascoli theorem P_1 is compact on B_ϵ. Thus all the assumption of theorem 3.2 are satisfied. So the conclusion of theorem 3.2 implies that the boundary value problem (1) has at least one solution defined on [0, 1]. □

Our next result is based on Schaefer's fixed point theorem.

Theorem 7. *Assume that the function* $h : [0, 1] \times \mathbb{R} \to \mathbb{R}$ *is continuous and the assumption* (*C*) *holds. Then the* BVP (1) *has at least one solution on* [0, 1].

Proof. Step I. *P* is continuous

Let $\{z_n\}$ be a sequence such that $z_n \to z$ in $C([0, 1], \mathbb{R})$. Then for each $\xi \in [0, 1]$

$$| P(z_n)(\xi) - P(z)(\xi) | \le \frac{1}{\Gamma(q)} \int_0^{\xi} (\xi - s)^{q-1} | h(s, z_n(s)) - h(s, z(s)) | ds$$

$$+ \frac{1}{\Gamma(q)} \int_0^{\eta} (\eta - s)^{q-1} | h(s, z_n(s)) - h(s, z(s)) | ds$$

$$+ \frac{\Gamma(p+3)}{\Gamma(p+q) | Q |} \int_0^1 (1-s)^{p+q-1} | h(s, z_n(s)) - h(s, z(s)) | ds$$

$$+ \frac{(p+2)(p+1)}{\Gamma(q) | Q |} \int_0^{\eta} (\eta - s)^{q-1} | h(s, z_n(s)) - h(s, z(s)) | ds$$

$$+ \frac{1}{\Gamma(q)} \int_0^{\xi} (\xi - s)^{q-1} \sup_{s \in [0, 1]} | h(s, z_n(s)) - h(s, z(s)) | ds$$

$$+ \frac{1}{\Gamma(q)} \int_0^{\eta} (\eta - s)^{q-1} \sup_{s \in [0, 1]} | h(s, z_n(s)) - h(s, z(s)) | ds$$

$$+ \frac{\Gamma(p+3)}{\Gamma(p+q) | Q |} \int_0^1 (1-s)^{p+q-1} \sup_{s \in [0,1]} | h(s, z_n(s)) - h(s, z(s)) | ds.$$

$$+ \frac{(p+2)(p+1)}{\Gamma(q) | Q |} \int_0^{\eta} (\eta - s)^{q-1} \sup_{s \in [0,1]} | h(s, z_n(s)) - h(s, z(s)) | ds.$$

Since h is a continuous function, then $\| P(z_n) - P(z) \| \to 0$ as $n \to \infty$. This means that P is continuous.

Step II. P maps bounded sets into bounded sets in $C([0,1], \mathbb{R})$. So, let us prove that for any $\epsilon > 0$, there exists a positive constant M such that for each $z \in B_{\epsilon} = \{z \in C([0, 1], \mathbb{R}) : \| z \| \le \epsilon\}$, we have $\| P(z) \| \le M$.

Now, for any $z \in B_{\epsilon}$, by using (2.7) and (C), ones obtain:

$$| P(z)(\xi) | \le \frac{1}{\Gamma(q)} \int_0^{\xi} (\xi - s)^{q-1} | h(s, z(s)) | ds + \frac{1}{\Gamma(q)} \int_0^{\eta} (\eta - s)^{q-1} | h(s, z(s)) | ds$$

$$+ \frac{\Gamma(p + 3)}{\Gamma(p + q) | Q |} \int_0^1 (1 - s)^{p+q-1} | h(s, z(s)) | ds$$

$$+\frac{(p+2)(p+1)}{\Gamma(q)|Q|}\int_0^\eta(\eta-s)^{q-1}|h(s,z(s))|ds$$

$$\leq\frac{\mu}{\Gamma(q)}\int_0^\xi(\xi-s)^{q-1}ds+\frac{\mu}{\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}ds$$

$$+\frac{\mu\Gamma(p+3)}{(p+q)|Q|}\int_0^1(1-s)^{p+q-1}ds$$

$$+\frac{\mu(p+2)(p+1)}{\Gamma(q)|Q|}\int_0^\eta(\eta-s)^{q-1}ds$$

$$\leq\frac{\mu}{\Gamma(q+1)}+\frac{\mu}{\Gamma(q+1)}+\frac{\mu\Gamma(p+3)}{\Gamma(p+q+1)|Q|}+\frac{\mu(p+2)(p+1)}{\Gamma(q+1)|Q|}$$

$$=\Lambda\mu.$$

Thus,

$$\|P(z)\|\leq\Lambda\mu:=M$$

which implies P maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.

Step III. P is completely continuous operator

Let $0\leq\xi_1<\xi_2\leq 1$ and $z\in B_\epsilon$, using (2.7) and (C), ones obtain:

$$|P(z)(\xi_2)-P(z)(\xi_1)|\leq\frac{\mu}{\Gamma(q)}\int_0^{\xi_1}[(\xi_2-s)^{q-1}-(\xi_1-s)^{q-1}]ds+\frac{\mu}{\Gamma(q)}\int_{\xi_1}^{\xi_2}(\xi_2-s)^{q-1}ds$$

$$+\frac{\mu(p+1)(p+2)|\xi_1^2-\xi_2^2|}{\Gamma(q)|Q|}\int_0^\eta(\eta-s)^{q-1}ds$$

$$+\frac{\mu|\xi_1^2-\xi_2^2|\Gamma(p+3)}{\Gamma(p+q)|Q|}\int_0^1(1-s)^{p+q-1}ds$$

$$\leq\frac{\mu}{\Gamma(q+1)}[(\xi_2-\xi_1)^q+(\xi_2^q-\xi_1^q)]+\frac{\mu(\xi_2-\xi_1)^q}{\Gamma(q+1)}$$

$$+\frac{\mu\Gamma(p+3)|\xi_1^2-\xi_2^2|}{\Gamma(p+q+1)|Q|}+\frac{\mu(p+1)(p+2)}{\Gamma(q+1)|Q|}.$$

As $\xi_1 \to \xi_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps I to III together with the Arzela-Ascoli theorem, we get that $P : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ is completely continuous.

Step IV. We show that the set $\Theta = \{z \in C([0, 1], \mathbb{R}) : z = \theta P(z)$ for some $0 < \theta < 1\}$ is bounded.

Let $z \in \Theta$, then $z(\xi) = \theta P(z)(\xi)$ for some $0 < \theta < 1$.

Now, for each $\xi \in [0, 1]$, using (2.7) and (C), we have

$$|z(\xi)| = |\theta P(z)(\xi)| \leq \frac{\theta\mu}{\Gamma(q)}\int_0^{\xi}(\xi - s)^{q-1}ds + \frac{\theta\mu}{\Gamma(q)}\int_0^{\eta}(\eta - s)^{q-1}ds$$

$$+ \frac{\theta\mu\Gamma(p+3)|\eta^2 - \xi^2|}{\Gamma(p+q)|Q|}\int_0^1(\eta - s)^{p+q-1}ds$$

$$+ \frac{\theta\mu(p+1)(p+2)|\eta^2 - \xi^2|}{\Gamma(q)|Q|}\int_0^{\eta}(\eta - s)^{q-1}ds$$

$$\leq \frac{\mu}{\Gamma(q+1)} + \frac{\mu}{\Gamma(q+1)} + \frac{\mu\Gamma(p+3)}{\Gamma(p+q+1)|Q|}$$

$$+ \frac{\mu(p+1)(p+2)}{\Gamma(q+1)|Q|}.$$

Hence, we get $\|z\| \leq \Lambda\mu := N$.

This implies that the set Θ is bounded. By Schaefer's fixed point theorem, P has a fixed point which is a solution of the problem (1). □

4. Examples

In this section, we have discussed some examples to illustrate our results.

Example 1. Consider the following three-point fractional integral boundary value problem

$${}^cD^{\frac{5}{2}}z(\xi) = \frac{1}{(\xi + 10)^2}\frac{|z(\xi)|}{1 + |z(\xi)|},\ \xi \in [0, 1] \tag{11}$$

$$z\left(\frac{1}{2}\right) = 0,\ z'(0) = 0,\ I^{\frac{3}{2}}z(1) = 0. \tag{12}$$

Here $q = \frac{5}{2}, \eta = \frac{1}{2}, p = \frac{3}{2}, \eta^2 = \frac{1}{4} \neq \frac{2}{p^2 + 3p + 2} = \frac{8}{35}$ and $h(\xi, z(\xi)) = \frac{1}{(\xi + 10)^2}$ $\frac{|z(\xi)|}{1 + |z(\xi)|}$. As $|h(\xi, z_1) - h(\xi, z_2)| \leq \frac{1}{100} |z_1 - z_2|$, therefore (A) is satisfied with $L = \frac{1}{100}$. Further,

$$L\Lambda = L\left[\frac{2}{\Gamma(q+1)} + \frac{(P+2)(p+1)}{\Gamma(q+1)|Q|} + \frac{\Gamma(p+3)}{\Gamma(p+q+1)|Q|}\right]$$

$$= \frac{1}{100}\left[\frac{16}{15\sqrt{\pi}} + \frac{70}{15\sqrt{\pi} \times 0.1875} + \frac{105\sqrt{\pi}}{16 \times 24 \times 0.1875}\right]$$

$$\approx 0.28615186 < 1.$$

Hence all the conditions of Theorem 5 are satisfied, therefore the boundary value problem (11)-(12) has a unique solution on [0, 1].

Example 2. Consider the following three-point fractional integral boundary value problem

$$^{c}D^{\frac{5}{2}} z(\xi) = \xi^2 + \frac{1}{(\xi + 4)^2} \sin z(\xi), \; \xi \in [0, 1] \tag{13}$$

$$z\left(\frac{1}{4}\right) = 0, \; z'(0) = 0, \; I^{\frac{1}{2}} z(1) = 0. \tag{14}$$

Here $\eta = \frac{1}{4}, q = \frac{5}{2}, p = \frac{1}{2}, \eta^2 = \frac{1}{16} \neq \frac{2}{p^2 + 3p + 2} = \frac{8}{15}, h(\xi, z) = \xi^2 + \frac{1}{(\xi + 4)^2} \sin z.$

Clearly $|h(\xi, z_1) - h(\xi, z_1)| \leq \frac{1}{16} |\sin z_1 - \sin z_2| \leq \frac{1}{16} |z_1 - z_2|$. Thus (A) is satisfied with $L = \frac{1}{16} > 0$. Also $|h(\xi, z)| \leq \frac{17}{16} = \phi(\xi)$ i.e. (B) is satisfied.

$$L\left\{\frac{(p+2)(p+1)}{\Gamma(q+1)|Q|} + \frac{\Gamma(p+3)}{\Gamma(p+q+1)|Q|}\right\}$$

$$= \frac{1}{16}[0.849192 + 2.887994]$$

$$= 0.233574 < 1.$$

Hence, all the conditions of Theorem 6 are satisfied and consequently the boundary value problem (13)-(14) has at least one solution on [0, 1].

Example 3. Consider the following three-point fractional integral boundary value problem

$$ {}^{c}D^{\frac{9}{4}} z(\xi) = \frac{e^{-2t} \cos \xi}{5 + \sin z(\xi)}, \ \xi \in [0, 1] \tag{15}$$

$$ z\left(\frac{1}{3}\right) = 0, \ z'(0) = 0, \ I^{\frac{4}{5}} z(1) = 0 \tag{16}$$

Here $\eta = \frac{1}{3}, q = \frac{9}{4}, p = \frac{4}{5}, \eta^2 = \frac{1}{9} \neq \frac{2}{p^2 + 3p + 2} = \frac{25}{43}, h(\xi, z) = \frac{e^{-2t} \cos t}{5 + \sin z}.$

Clearly $|h(\xi, z)| \leq \frac{1}{4} = \mu.$

Hence, all the conditions of Theorem 7 are satisfied and consequently the boundary value problem (15)-(16) has at least one solution on [0, 1].

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