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SOLVING LINEAR SYSTEM OF EQUATIONS WITH FUZZY REVISED DECOMPOSITION METHOD

D. STEPHEN DINAGAR and A. HETCIYAL SANGEETHA PRIYAM

PG and Research Department of Mathematics
T.B.M.L College (Affiliated to Bharathidasan
University) Porayars, Tamil Nadu, India
E-mail: asangeethapriyam@gmail.com
dsdina@rediffmail.com

Abstract

In this paper, we have studied fuzzy linear systems with the aid of generalized triangular fuzzy number. A new procedure is proposed namely 'Revised LU Decomposition method' for solving fuzzy linear systems of equations. A new notion namely close interval approximation of fuzzy number is also reviewed and their arithmetic operations are also generated. Finally, the proposed method is illustrated by solving relevant numerical examples.

1. Introduction

The concept of fuzzy logic was introduced by Lotfi Zadeh in 1965. Equations linearized for iterative solutions arise in many fields. Alan Turing was developed the concept of LU-decomposition, $A = LU$ [8]. Rao and Chen [5] proposed the solutions of simultaneous linear equations.

Dehghan et al. [2] extended the adomain decomposition method for fully fuzzy linear system of equations. Mosleh et al. [3] discussed a new decomposition of a non-singular fuzzy matrix.

This paper organized as follows. In section 2, we give some basic preliminaries. In section 3 arithmetic operations are given. In section 4, Revised LU decomposition method is proposed. In section 5 deals with a numerical illustrations to illustrate the above proposed method. The concept

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of close interval approximation is reviewed and their arithmetic operations are proposed in section 6. Section 7 ends this paper with conclusion and followed by references.

2. Preliminaries

In this section, some basic definitions and results are recalled.

Definition 2.1. A membership function is a generalization of a characteristic function or an indicator function of a subset defined.

Definition 2.2. A membership function $\mu_{\widetilde{A}}(x)$ is associated with a fuzzy sets $\widetilde{A}$ such that the function maps every element of universe of discourse X written as $\mu_{\widetilde{A}}(x) : x \to [0, 1]$.

Definition 2.3. Let $\widetilde{A} = (\widetilde{a}_{ij})$ is called a fuzzy matrix, if every element of $\widetilde{A}$ is a fuzzy number. If $\widetilde{A}$ is called as positive (negative) and represented by $\widetilde{A} > 0, (\widetilde{A} < 0)$ if each element of $\widetilde{A}$ be positive (negative).

3. Arithmetic Operations on Generalized Triangular Fuzzy Numbers (GTFNs)

In this section, two types of arithmetic operations between generalized triangular fuzzy numbers (GTFNs) are given below. Let us consider $\widetilde{A} = (a_1, a_2, a_3; w_{\widetilde{A}})$ and $\widetilde{B} = (b_1, b_2, b_3; w_{\widetilde{B}})$ be two generalized triangular fuzzy numbers, then,

(i) Addition:

$$\widetilde{A} \oplus \widetilde{B} = (a_1, a_2, a_3; w_{\widetilde{A}}) \oplus (b_1, b_2, b_3; w_{\widetilde{B}})$$

$$= (a_1 + b_1, a_2 + b_2, a_3 + b_3; \min (w_{\widetilde{A}}, w_{\widetilde{B}})).$$

(ii) Subtraction:

$$\widetilde{A} \ominus \widetilde{B} = (a_1, a_2, a_3; w_{\widetilde{A}}) \ominus (b_1, b_2, b_3; w_{\widetilde{B}})$$

$$= (a_1 - b_3, a_2 - b_2, a_3 - b_1; \min (w_{\widetilde{A}}, w_{\widetilde{B}})).$$

Type -1

Multiplication:

$$\widetilde{A} \otimes \widetilde{B} = (a_1, a_2, a_3; w_{\widetilde{A}}) \otimes (b_1, b_2, b_3; w_{\widetilde{B}})$$

$$= (a, a_1 \times b_2, c; \min(w_{\widetilde{A}}, w_{\widetilde{B}})).$$

Where, $a = \min(a_1 \times b_1, a_1 \times b_3, a_3 \times b_1, a_3 \times b_3)$ and

$$c = \max = (a_1 \times b_1, a_1 \times b_3, a_3 \times b_1, a_3 \times b_3).$$

Division:

$$\widetilde{A} \oslash \widetilde{B} = (a_1, a_2, a_3; w_{\widetilde{A}}) \oslash (b_1, b_2, b_3; w_{\widetilde{B}})$$

$$= (a, a_1 \div b_2, c; \min(w_{\widetilde{A}}, w_{\widetilde{B}})).$$

Where, $a = \min(a_1 \div b_1, a_1 \div b_3, a_3 \div b_1, a_3 \div b_3)$ and

$$c = \max = (a_1 \div b_1, a_1 \div b_3, a_3 \div b_1, a_3 \div b_3)., b_1, b_2, b_3 \neq 0.$$

Type 2:

Multiplication:

$$\widetilde{A} \otimes \widetilde{B} = (a_1, a_2, a_3; w_{\widetilde{A}}) \otimes (b_1, b_2, b_3; w_{\widetilde{B}})$$

$$= (a_1 R(B), a_2 R(B), a_3 R(B); \min(w_{\widetilde{A}}, w_{\widetilde{B}})).$$

Where, $R(B) = \left(\frac{b_1 + b_2 + b_3}{3}\right)$, if $R(B) > 0$

$$\widetilde{A} \otimes \widetilde{B} = (a_3 R(B), a_2 R(B), a_1 R(B); \min(w_{\widetilde{A}}, w_{\widetilde{B}})), \text{ if } R(B) < 0.$$

Division:

$$\widetilde{A} \oslash \widetilde{B} = (a_1, a_2, a_3; w_{\widetilde{A}}) \oslash (b_1, b_2, b_3; w_{\widetilde{B}})$$

$$= (a_1 / R(B), a_2 / R(B), a_3 / R(B); \min(w_{\widetilde{A}}, w_{\widetilde{B}})) \text{ if } R(B) > 0.$$

$$\widetilde{A} \oslash \widetilde{B} = (a_3 / R(B), a_2 / R(B), a_1 / R(B); \min(w_{\widetilde{A}}, w_{\widetilde{B}})) \text{ if } R(B) < 0.$$

4. Revised *LU* Decomposition Method for Fuzzy Linear System

Consider a generalized triangular fuzzy linear system of n equations in n unknowns.

$$(a_{11}^{(1)}, a_{11}^{(2)}, a_{11}^{(3)}; w_1)(x_1', x_2', x_3'; w_1) + (a_{12}^{(1)}, a_{12}^{(2)}, a_{12}^{(3)}; w_2)(x"_1, x"_2, x"_3; w_2) + \ldots$$

$$+ (a_{1n}^{(1)}, a_{1n}^{(2)}, a_{1n}^{(3)}; w_n)(x_1^{n'}, x_2^{n'}, x_3^{n'}; w_n) = (r_1^{(1)}, r_1^{(2)}, r_1^{(3)}; w_r^{(1)})$$

$$(a_{21}^{(1)}, a_{21}^{(2)}, a_{21}^{(3)}; w_1)(x_1', x_2', x_3'; w_1) + (a_{22}^{(1)}, a_{22}^{(2)}, a_{22}^{(3)}; w_2)(x"_1, x"_2, x"_3; w_2) + \ldots$$

$$+ (a_{2n}^{(1)}, a_{2n}^{(2)}, a_{2n}^{(3)}; w_n)(x_1^{n'}, x_2^{n'}, x_3^{n'}; w_n) = (r_2^{(1)}, r_2^{(2)}, r_2^{(3)}; w_r^{(2)})$$

$$(a_{n1}^{(1)}, a_{n1}^{(2)}, a_{n1}^{(3)}; w_1)(x_1', x_2', x_3'; w_1) + (a_{n2}^{(1)}, a_{n2}^{(2)}, a_{n2}^{(3)}; w_2)(x"_1, x"_2, x"_3; w_2) + \ldots$$

$$+ (a_{nn}^{(1)}, a_{nn}^{(2)}, a_{nn}^{(3)}; w_n)(x_1^{n'}, x_2^{n'}, x_3^{n'}; w_n) = (r_n^{(1)}, r_n^{(2)}, r_n^{(3)}; w_r^{(2)}).$$

The matrix representation of the system of equations will be

$$\tilde{A}\tilde{X} = \tilde{B}.$$

The proposed method establishes a solution that transforms both the co-efficient matrix A and the right hand side column vector B are transformed through the equations,

$$\tilde{L}\tilde{U}\tilde{X} = \tilde{B};\ \tilde{U}\tilde{X} = \tilde{Y};\ \tilde{L}\tilde{Y} = \tilde{B}.$$

Let the matrix $\tilde{L}\tilde{U}$ be,

$$\tilde{L} = \begin{bmatrix} (1, 1, 1; 1) & (0, 0, 0; 1) & (0, 0, 0; 1) & \ldots & (0, 0, 0; 1) \\ (l_{21}^{(1)}, l_{21}^{(2)}, l_{21}^{(3)}; w) & (1, 1, 1; 1) & (0, 0, 0; 1) & \ldots & (0, 0, 0, ; 1) \\ (l_{31}^{(1)}, l_{31}^{(2)}, l_{31}^{(3)}; w) & (l_{32}^{(1)}, l_{32}^{(2)}, l_{32}^{(3)}; w) & (1, 1, 1; 1) & \ldots & (0, 0, 0, ; 1) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ (l_{n1}^{(1)}, l_{n1}^{(2)}, l_{n1}^{(3)}; w) & (l_{n2}^{(1)}, l_{n2}^{(2)}, l_{n2}^{(3)}; w) & (l_{n3}^{(1)}, l_{n3}^{(2)}, l_{n3}^{(3)}; w) & \ldots & (1, 1, 1; 1) \end{bmatrix}$$

$$\widetilde{U} = \begin{bmatrix} (u_{11}^{(1)}, u_{11}^{(2)}, u_{11}^{(3)}; w) & (u_{12}^{(1)}, u_{12}^{(2)}, u_{12}^{(3)}; w) & (u_{13}^{(1)}, u_{13}^{(2)}, u_{13}^{(3)}; w) & \dots & (u_{1n}^{(1)}, u_{1n}^{(2)}, u_{1n}^{(3)}; w) \\ (0, 0, 0; 1) & (u_{22}^{(1)}, u_{22}^{(2)}, u_{22}^{(3)}; w) & (u_{23}^{(1)}, u_{23}^{(2)}, u_{23}^{(3)}; w) & \dots & (u_{2n}^{(1)}, u_{2n}^{(2)}, u_{2n}^{(3)}; w) \\ (0, 0, 0; 1) & \vdots & (u_{33}^{(1)}, u_{33}^{(2)}, u_{33}^{(3)}; w) & \dots & (u_{3n}^{(1)}, u_{3n}^{(2)}, u_{3n}^{(3)}; w) \\ \vdots & (0, 0, 0; 1) & \vdots & \dots & \vdots \\ (0, 0, 0; 1) & (0, 0, 0; 1) & (0, 0, 0; 1) & \dots & (u_{nn}^{(1)}, u_{nn}^{(2)}, u_{nn}^{(3)}; w) \end{bmatrix}$$

Where,

$$(u_{ij}^{(1)}, u_{ij}^{(2)}, u_{ij}^{(3)}; w) = (a_{ij}^{(1)}, u_{ij}^{(2)}, u_{ij}^{(3)}; w)$$

$$(l_{i1}^{(1)}, l_{i1}^{(2)}, l_{i1}^{(3)}; w) = \frac{(a_{i1}^{(1)}, a_{i1}^{(2)}, a_{i1}^{(3)}; w)}{(u_{11}^{(1)}, u_{11}^{(2)}, u_{11}^{(3)}; w)}.$$

And,

$$(u_{rj}^{(1)}, u_{rj}^{(2)}, u_{rj}^{(3)}; w) = (a_{rj}^{(1)}, a_{rj}^{(2)}, a_{rj}^{(3)}; w) - \sum_{k=1}^{r-1} (l_{rk}^{(1)}, l_{rk}^{(2)}, k_{rk}^{(3)}; w)(u_{kj}^{(1)}, u_{kj}^{(2)}, u_{kj}^{(3)}; w)$$

$$(l_{ir}^{(1)}, l_{ir}^{(2)}, l_{ir}^{(3)}; w) = \frac{(a_{ir}^{(1)}, a_{ir}^{(2)}, a_{ir}^{(3)}; w) - \sum_{k=1}^{r-1} (l_{ik}^{(1)}, l_{ik}^{(2)}, l_{ik}^{(3)}; w)(u_{kr}^{(1)}, u_{kr}^{(2)}, u_{kr}^{(3)}; w)}{(u_{rr}^{(1)}, u_{rr}^{(2)}, u_{rr}^{(3)}; w)}.$$

Where, $r = 2, 3, \dots, n$, $j = r, r+1, \dots, n$, $i = r, r+1, \dots, n$.

Note:

In selection of $\widetilde{L}$ and $\widetilde{U}$ we can also take as,

$$\widetilde{L} = \begin{bmatrix} (l_{11}^{(1)}, l_{11}^{(2)}, l_{11}^{(3)}; w) & (0, 0, 0; 1) & (0, 0, 0; 1) & \dots & (0, 0, 0; 1) \\ (l_{21}^{(1)}, l_{21}^{(2)}, l_{21}^{(3)}; w) & (l_{22}^{(1)}, l_{22}^{(2)}, l_{22}^{(3)}; w) & (0, 0, 0; 1) & \dots & (0, 0, 0, ; 1) \\ (l_{31}^{(1)}, l_{31}^{(2)}, l_{31}^{(3)}; w) & (l_{32}^{(1)}, l_{32}^{(2)}, l_{32}^{(3)}; w) & (l_{33}^{(1)}, l_{33}^{(2)}, l_{33}^{(3)}; w) & \dots & (0, 0, 0, ; 1) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ (l_{n1}^{(1)}, l_{n1}^{(2)}, l_{n1}^{(3)}; w) & (l_{n2}^{(1)}, l_{n2}^{(2)}, l_{n2}^{(3)}; w) & (l_{n3}^{(1)}, l_{n3}^{(2)}, l_{n3}^{(3)}; w) & \dots & (l_{nn}^{(1)}, l_{nn}^{(2)}, l_{nn}^{(3)}; w) \end{bmatrix}$$

$$\widetilde{U}=\begin{bmatrix}(1,1,1;1) & (u_{12}^{(1)},u_{12}^{(2)},u_{12}^{(3)};w) & (u_{13}^{(1)},u_{13}^{(2)},u_{13}^{(3)};w) & \dots & (u_{1n}^{(1)},u_{1n}^{(2)},u_{1n}^{(3)};w)\\ (0,0,0;1) & (1,1,1;1) & (u_{23}^{(1)},u_{23}^{(2)},u_{23}^{(3)};w) & \dots & (u_{2n}^{(1)},u_{2n}^{(2)},u_{2n}^{(3)};w)\\ (0,0,0;1) & \vdots & (1,1,1;1) & \dots & (u_{3n}^{(1)},u_{3n}^{(2)},u_{3n}^{(3)};w)\\ \vdots & (0,0,0;1) & \vdots & \dots & \vdots\\ (0,0,0;1) & (0,0,0;1) & (0,0,0;1) & \dots & (1,1,1;1)\end{bmatrix}.$$

5. Numerical illustration

In this section a simple example is illustrated to justify the discussed notion.

Illustration 5.1. Consider the following fuzzy linear system and solve by revised LU decomposition method.

$$(-1,\ 1,\ 3;\ 0.2)(x_1',\ x_2',\ x_3';\ w_1)+(-1,\ 1,\ 3;\ 0.3)(x''_1,\ x''_2,\ x''_3;\ w_2)$$

$$+\ (-1,\ 1,\ 3\ 0;\ 2)(x_1''',\ x_2''',\ x_3''';\ w_3)=(-1,\ 1,\ 3;\ 0.5)$$

$$(2,\ 4,\ 6;\ 0.5)(x_1',\ x_2',\ x_3';\ w_1)+(2,\ 3,\ 4;\ 0.6)(x''_1,\ x''_2,\ x''_3;\ w_2)$$

$$-\ (-1,\ 1,\ 3;\ 0.7)(x_1''',\ x_2''',\ x_3''';\ w_3)=(3,\ 6,\ 9;\ 0.8)$$

$$(2,\ 3,\ 4;\ 0.6)(x_1',\ x_2',\ x_3';\ w_1)+(3,\ 5,\ 7;\ 0.8)(x''_1,\ x''_2,\ x''_3;\ w_2)$$

$$+\ (2,\ 3,\ 4,\ 0.9)(x_1''',\ x_2''',\ x_3''';\ w_3)=(2,\ 4,\ 8;\ 0.9).$$

Solution: 1

The given fuzzy linear system, can be solved with the aid of type -1 operations of fuzzy arithmetic operations (sec 3)

$$\widetilde{A}=\begin{pmatrix}(-1,\ 1,\ 3;\ 0.5) & (-1,\ 1,\ 3;\ 0.3) & (-1,\ 1,\ 3;\ 0.3)\\ (2,\ 4,\ 6;\ 0.5) & (2,\ 3,\ 4;\ 0.6) & -(-1,\ 1,\ 3;\ 0.7)\\ (2,\ 3,\ 4;\ 0.6) & (3,\ 5,\ 7;\ 0.8) & (2,\ 3,\ 4;\ 0.9)\end{pmatrix},$$

Where,

$$L\widetilde{U}=\widetilde{A}$$

$$\widetilde{L} = \begin{pmatrix} (1, 1, 1; 1) & (0, 0, 0; 1) & (0, 0, 0; 1) \\ (l_{21}^{(1)}, l_{21}^{(2)}, l_{21}^{(3)}; w) & (1, 1, 1; 1) & (0, 0, 0; 1) \\ (l_{31}^{(1)}, l_{31}^{(2)}, l_{31}^{(3)}; w) & (l_{32}^{(1)}, l_{32}^{(2)}, l_{32}^{(3)}; w) & (1, 1, 1; 1) \end{pmatrix}$$

$$\widetilde{U} = \begin{pmatrix} (l_{11}^{(1)}, l_{11}^{(2)}, l_{11}^{(3)}; w) & (u_{12}^{(1)}, u_{12}^{(2)}, u_{12}^{(3)}; w) & (u_{13}^{(1)}, u_{13}^{(2)}, u_{13}^{(3)}; w) \\ (0, 0, 0; 1) & (u_{22}^{(1)}, u_{22}^{(2)}, u_{2}^{(3)}; w) & (u_{23}^{(1)}, u_{23}^{(2)}, u_{23}^{(3)}; w) \\ (0, 0, 0; 1) & (0, 0, 0; 1) & (u_{33}^{(1)}, u_{33}^{(2)}, u_{33}^{(3)}; w) \end{pmatrix}.$$

Solving the above we get,

$$(u_{11}^{(1)}, u_{11}^{(2)}, u_{11}^{(3)}; w) = (-1, 1, 3; 0.2)$$

$$(u_{12}^{(1)}, u_{12}^{(2)}, u_{12}^{(3)}; w) = (-1, 1, 3; 0.3)$$

$$(u_{13}^{(1)}, u_{13}^{(2)}, u_{13}^{(3)}; w) = (-1, 1, 3; 0.3)$$

$$(l_{21}^{(1)}, l_{21}^{(2)}, l_{21}^{(3)}; w) = (-6, 4, 2; 0.2)$$

$$(l_{31}^{(1)}, l_{31}^{(2)}, l_{31}^{(3)}; w) = (-4, 3, 1.33; 0.2)$$

$$(l_{32}^{(1)}, l_{32}^{(2)}, l_{32}^{(3)}; w) = (-4, 3, 1.33; 0.2)$$

$$(u_{22}^{(1)}, u_{22}^{(2)}, u_{22}^{(3)}; w) = (-4, -1, 22; 0.2)$$

$$(u_{23}^{(1)}, u_{23}^{(2)}, l_{23}^{(3)}; w) = (-19, 5, 9; 0.2)$$

$$(u_{33}^{(1)}, u_{33}^{(2)}, u_{33}^{(3)}; w) = (-44.75, -10, 106.25; 0.2).$$

$$\widetilde{L}\widetilde{Y} = \widetilde{B}$$

$$\begin{pmatrix} (1, 1, 1; 1) & (0, 0, 0; 1) & (0, 0, 0; 1) \\ (-6, 4, 2; 0.2) & (1, 1, 1; 1) & (0, 0, 0; 1) \\ (-4, 3, 1.33; 0.2) & (-4, 3, 1.33; 0.2) & (1, 1, 1; 1) \end{pmatrix}$$

$$\begin{pmatrix} (y'_1,\ y'_2,\ y'_3;\ w_1) \\ (y''_1,\ y''_2,\ y''_3;\ w_2) \\ (y'''_1,\ y'''_2,\ y'''_3;\ w_3) \end{pmatrix} = \begin{pmatrix} (-1,\ 1,\ 3;\ 0.5) \\ (3,\ 6,\ 9;\ 0.8) \\ (2,\ 4,\ 6;\ 0.9) \end{pmatrix}.$$

Solve the above fuzzy system we get,

$$(y'_1,\ y'_2,\ y'_3;\ w_1) = (-1,\ 1,\ 3;\ 0.5)$$

$$(y''_1,\ y''_2,\ y''_3;\ w_2) = (-3,\ 2,\ 27;\ 0.2)$$

$$(y'''_1,\ y'''_2,\ y'''_3;\ w_3) = (-0.42,\ -4,\ 0.22,\ 0.2).$$

Now

$$\widetilde{U}\widetilde{X} = \widetilde{Y}$$

$$\begin{pmatrix} (-1,\ 1,\ 3;\ 0.2) & (-1,\ 1,\ 3;\ 0.2) & (-1,\ 1,\ 3;\ 0.3) \\ (0,\ 0,\ 0;\ 1) & (-4,\ -1,\ 22;\ 0.2) & -(-19,\ 5,\ 9;\ 0.2) \\ (0,\ 0,\ 0;\ 1) & (0,\ 0,\ 0;\ 1) & (-44.75,\ -10,\ 106.2;\ 0.2) \end{pmatrix}$$

$$\begin{pmatrix} (x'_1,\ x'_2,\ x'_3;\ w_1) \\ (x''_1,\ x''_2,\ x''_3;\ w_2) \\ (x'''_1,\ x'''_2,\ x'''_3;\ w_3) \end{pmatrix} = \begin{pmatrix} (1,\ 2,\ 3;\ 0.8) \\ (3,\ 5,\ 7;\ 0.8) \\ (4,\ 6,\ 8;\ 0.9) \end{pmatrix}.$$

The solution is,

$$(x'_1,\ x'_2,\ x'_3;\ w_1) = (-23.2,\ 4.6,\ 7.73;\ 0.2)$$

$$(x''_1,\ x''_2,\ x''_3;\ w_2) = (-6.76,\ -4,\ 1.23;\ 0.2)$$

$$(x'''_1,\ x'''_2,\ x'''_3;\ w_3) = (-0.0049,\ 0.4,\ 0.002;\ 0.2). \text{ (I)}$$

Solution: 2

The given fuzzy linear system, can be solved with the aid of type-2 operations of fuzzy arithmetic operations (sec. 3)

$$\widetilde{A} = \begin{pmatrix} (-1, 1, 3; 0.5) & (-1, 1, 3; 0.3) & (-1, 1, 3; 0.3) \\ (2, 4, 6; 0.5) & (2, 3, 4; 0.6) & -(-1, 1, 3; 0.7) \\ (2, 3, 4; 0.6) & (3, 5, 7; 0.8) & (2, 3, 4; 0.9) \end{pmatrix}.$$

Where,

$$L\widetilde{U} = \widetilde{A}$$

$$\widetilde{L} = \begin{pmatrix} (1, 1, 1; 1) & (0, 0, 0; 1) & (0, 0, 0; 1) \\ (l_{21}^{(1)}, l_{21}^{(2)}, l_{21}^{(3)}; w) & (1, 1, 1; 1) & (0, 0, 0; 1) \\ (l_{31}^{(1)}, l_{31}^{(2)}, l_{31}^{(3)}; w) & (l_{32}^{(1)}, l_{32}^{(2)}, l_{32}^{(3)}; w) & (1, 1, 1; 1) \end{pmatrix}$$

$$\widetilde{U} = \begin{pmatrix} (u_{11}^{(1)}, u_{11}^{(2)}, u_{11}^{(3)}; w) & (u_{12}^{(1)}, u_{12}^{(2)}, u_{12}^{(3)}; w) & (u_{13}^{(1)}, u_{13}^{(2)}, u_{13}^{(3)}; w) \\ (0, 0, 0; 1) & (u_{22}^{(1)}, u_{22}^{(2)}, u_{2}^{(3)}; w) & (u_{23}^{(1)}, u_{23}^{(2)}, u_{23}^{(3)}; w) \\ (0, 0, 0; 1) & (0, 0, 0; 1) & (u_{33}^{(1)}, u_{33}^{(2)}, u_{33}^{(3)}; w) \end{pmatrix}.$$

Solving by method 2 the above we get,

$$(u_{11}^{(1)}, u_{11}^{(2)}, u_{11}^{(3)}; w) = (-1, 1, 3; 0.2)$$

$$(u_{12}^{(1)}, u_{12}^{(2)}, u_{12}^{(3)}; w) = (-1, 1, 3; 0.3)$$

$$(u_{13}^{(1)}, u_{13}^{(2)}, u_{13}^{(3)}; w) = (-1, 1, 3; 0.3)$$

$$(l_{21}^{(1)}, l_{21}^{(2)}, l_{21}^{(3)}; w) = (2, 4, 6; 0.2)$$

$$(l_{31}^{(1)}, l_{31}^{(2)}, l_{31}^{(3)}; w) = (2, 3, 4; 0.2)$$

$$(l_{32}^{(1)}, l_{32}^{(2)}, l_{32}^{(3)}; w) = (-5, -2, 1; 0.2)$$

$$(u_{22}^{(1)}, u_{22}^{(2)}, u_{22}^{(3)}; w) = (-4, -1, 2; 0.2)$$

$$(u_{23}^{(1)}, u_{23}^{(2)}, l_{23}^{(3)}; w) = -(1, 5, 9; 0.2)$$

$$(u_{33}^{(1)}, u_{33}^{(2)}, u_{33}^{(3)}; w) = (-27, -10, 7; 0.2).$$

$$\widetilde{L}\widetilde{Y} = \widetilde{B}$$

$$\begin{pmatrix} (1, 1, 1; 1) & (0, 0, 0; 1) & (0, 0, 0; 1) \\ (2, 4, 6; 0.2) & (1, 1, 1; 1) & (0, 0, 0; 1) \\ (2, 3, 4; 0.2) & (-5, -2, 1; 0.2) & (1, 1, 1; 1) \end{pmatrix}$$

$$\begin{pmatrix} (y_1', y_2', y_3'; w_1) \\ (y''_1, y''_2, y''_3; w_2) \\ (y_1''', y_2''', y_3'''; w_3) \end{pmatrix} = \begin{pmatrix} (-1, 1, 3; 0.5) \\ (3, 6, 9; 0.8) \\ (2, 4, 6; 0.9) \end{pmatrix}.$$

Solve the above fuzzy system we get,

$$(y_1', y_2', y_3'; w_1) = (-1, 1, 3; 0.5)$$

$$(y''_1, y''_2, y''_3; w_1) = (-3, 2, 7; 0.2)$$

$$(y_1''', y_2''', y_3'''; w_1) = (-4, 5, 14, 0.2).$$

Now,

$$\widetilde{U}\widetilde{X} = \widetilde{Y}$$

$$\begin{pmatrix} (-1, 1, 3; 0.2) & (-1, 1, 3; 0.2) & (-1, 1, 3; 0.3) \\ (0, 0, 0; 1) & (-4, -1, 2; 0.2) & -(1, 5, 9; 0.2) \\ (0, 0, 0; 1) & (0, 0, 0; 1) & (-27, -10, 7; 0.2) \end{pmatrix}$$

$$\begin{pmatrix} (x_1', x_2', x_3'; w_1) \\ (x''_1, x''_2, x''_3; w_2) \\ (x_1''', x_2''', x_3'''; w_3) \end{pmatrix} = \begin{pmatrix} (-1, 1, 3; 0.5) \\ (-3, 2, 7; 0.2) \\ (-4, 5, 14; 0.2) \end{pmatrix}.$$

The solution is,

$$(x_1', x_2', x_3'; w_1) = (-3, 1, 5; 0.2)$$

$$(x_1'', x_2'', x_3''; w_2) = (-6.5, 0.5, 7.5; 0.2)$$

$$(x_1''', x_2''', x_3'''; w_3) = (-1.4, -0.5, 0.4; 0.2). \quad \text{(II)}$$

Comparing the two solutions I and II, it is clear that solution II given the final result will be in generalized triangular fuzzy number form than the other solution type.

6. Close Interval Approximation

In this section, some definitions and notions of fuzzy numbers [7] close interval approximation of fuzzy numbers have been reviewed.

Definition 6.1. If $\widetilde{A} = (a_1, a_2, a_3)$ is a triangular fuzzy number (TFN), then its associated (crisp) number is given by $\widetilde{A}_i = \frac{(a_1 + a_2 + a_3)}{3}$. We can find the same ordinary number by $\widetilde{A}_i = \frac{(a_1 + a_2 + a_3)}{4}$, also.

Definition 6.2. An interval approximation $[a_\alpha^L, a_\alpha^U]$ of a triangular fuzzy number $\widetilde{A}$ is said to be closed interval approximation if,

$$a_\alpha^L = \inf \{x \in R / \mu_{\widetilde{A}} \geq 0.5\} \text{ and}$$

$$a_\alpha^U = \sup \{x \in R / \mu_{\widetilde{A}} \geq 0.5\}.$$

Which is denoted by $[\widetilde{A}]$. Mathematically, if $\widetilde{A} = (a_1, a_2, a_3)$, then the closed interval approximation of TFN is, $[\widetilde{A}] = \left[\frac{(a_1 + a_2)}{2}, \frac{(a_2 + a_3)}{2}\right] = [a_\alpha^L, a_\alpha^U]$.

Definition 6.3. If $[\widetilde{A}] = [a_\alpha^L, a_\alpha^U]$ is a closed interval approximation of TFN, then associated real number of $[\widetilde{A}]$ is denoted by Re $[\widetilde{A}]$ and Re $[\widetilde{A}] = \left[\frac{[a_\alpha^L, a_\alpha^U]}{2}\right]$,

If Re$[\widetilde{A}]$ then by $[\widetilde{A}]$ is positive, and if Re $[\widetilde{A}] < 0$, then by $[\widetilde{A}]$ is negative.

Arithmetic operations on close interval approximation:

The arithmetic operations of close interval approximation of fuzzy numbers are given below,

Addition:

$$[\widetilde{A}] + [\widetilde{B}] = [a_\alpha^L + b_\alpha^U,\ a_\alpha^U + b_\alpha^U].$$

Subtraction:

$$[\widetilde{A}] - [\widetilde{B}] = [a_\alpha^L - b_\alpha^U,\ a_\alpha^U - b_\alpha^U].$$

Scalar Multiplication:

$$\alpha[\widetilde{A}] = \begin{cases} [\alpha a_\alpha^L,\ \alpha a_\alpha^U];\ \alpha > 0 \\ [\alpha a_\alpha^U,\ \alpha a_\alpha^L];\ \alpha < 0 \end{cases}.$$

Multiplication:

$$[\widetilde{A}] \cdot [\widetilde{B}] = [\frac{1}{2}[\alpha a_\alpha^U b_\alpha^L + a_\alpha^L b_\alpha^U],\ \frac{1}{2}[\alpha_\alpha^L b_\alpha^L + a_\alpha^U b_\alpha^U].$$

Division:

$$[\widetilde{A}]/[\widetilde{B}] = \begin{cases} \left[2\left(\frac{a_\alpha^L}{b_\alpha^L + b_\alpha^U}\right),\ 2\left(\frac{a_\alpha^U}{b_\alpha^L + b_\alpha^U}\right)\right],\ \text{if } [B] > 0 \\ \left[2\left(\frac{a_\alpha^U}{b_\alpha^L + b_\alpha^U}\right),\ 2\left(\frac{a_\alpha^L}{b_\alpha^L + b_\alpha^U}\right)\right],\ \text{if } [B] < 0. \end{cases}$$

Numerical Example 6.1. Let us take the same illustration as in 5 and it can be solved by using the notion of 'Close Interval Approximation' as follows.

Let $\widetilde{A}$ be the co-efficient fuzzy matrix,

$$\widetilde{A} = \begin{pmatrix} (-1,\ 1,\ 3;\ 0.5) & (-1,\ 1,\ 3;\ 0.3) & (-1,\ 1,\ 3;\ 0.3) \\ (2,\ 4,\ 6;\ 0.5) & (2,\ 3,\ 4;\ 0.6) & -(-1,\ 1,\ 3;\ 0.7) \\ (2,\ 3,\ 4;\ 0.6) & (3,\ 5,\ 7;\ 0.8) & (2,\ 3,\ 4;\ 0.9) \end{pmatrix}.$$

The close interval approximation representation as follows,

$$\widetilde{A} = \begin{pmatrix} [0,\ 2] & [0,\ 2] & [0,\ 2] \\ [3,\ 5] & [2.5,\ 3.5] & -[0,\ 2] \\ [2.5,\ 3.5] & [4,\ 6] & [2.5,\ 3.5] \end{pmatrix}$$

$$\widetilde{L}\widetilde{U} = \widetilde{A}$$

$$\begin{pmatrix} [1, 1] & [0, 0] & [0, 0] \\ [l_{21}^L, l_{21}^U] & [1, 1] & [0, 0] \\ [l_{31}^L, l_{31}^U] & [l_{32}^L, l_{32}^U] & [1, 1] \end{pmatrix} \begin{pmatrix} [u_{11}^L, u_{11}^U] & [u_{12}^L, u_{12}^U] & [u_{13}^L, u_{13}^U] \\ [0, 0] & [u_{22}^L, l_2^U] & [u_{23}^L, u_{23}^U] \\ [0, 0] & [0, 0] & [u_{33}^L, u_{33}^U] \end{pmatrix}$$

$$= \begin{pmatrix} [0, 2] & [0, 2] & [0, 2] \\ [3, 5] & [2.5, 3.5] & -[0, 2] \\ [2.5, 3.5] & [4, 6] & [2.5, 3.5] \end{pmatrix}.$$

Solving the above, we have obtained the close interval approximation matrices $\widetilde{L}$ and $\widetilde{U}$

$$\widetilde{L} = \begin{pmatrix} [1, 1] & [0, 0] & [0, 0] \\ [3, 5] & [1, 1] & [0, 0] \\ [2.5, 3.5] & [-3.5, -0.5] & [1, 1] \end{pmatrix}$$

$$\widetilde{U} = \begin{pmatrix} [0, 2] & [0, 0] & [0, 2] \\ [0, 2] & [-2.5, 0.5] & -[4.5, 9.5] \\ [0, 0] & [0, 0] & [-0.5, 4.5] \end{pmatrix}.$$

And the close-interval approximation solution of the given fuzzy linear system follows as,

$$[x_1^L, x_1^U] = [5.3125, 30.6875]$$

$$[x_2^L, x_2^U] = [-28.5625, -10.4375]$$

$$[x_3^L, x_3^U] = [-0.125, 5.125]s.$$

7. Conclusion

In this work, a system of fuzzy linear equations has been studied with the aid of generalized triangular fuzzy numbers together with the proposed new arithmetic operations. A new 'Revised LU method' is used to solve the fuzzy linear system and a new notion namely Close Interval Approximation of fuzzy number is reviewed and attempt to solve the fuzzy linear system by employing the Close Interval Approximation of the fuzzy numbers. Relevant numerical examples are also included to justify the proposed notions.

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