Advances and Applications in Mathematical Sciences
Volume 19, Issue 8, June 2020, Pages 811-823

FIXED POINT THEOREM FOR FOUR SELF MAPPINGS IN DISLOCATED QUASI *B*-METRIC SPACE

QAZI AFTAB KABIR[1,*], R. VERMA[2], M. S. CHAUHAN[3] and R. SHRIVASTAVA[4]

[1]Department of Mathematics
Saifia Science College
Bhopal-462001, M.P, India

[2,3]Department of Mathematics
Institute for Excellence in Higher Education
Bhopal-462007, India
E-mail: rupali.varma1989@gmail.com
dr.msc@rediffmail.com

[4]Department of Mathematics
Govt. Science and Commerce College Benazir
Bhopal-462008, India
E-mail: rajeshraju0101@rediffmail.com

Abstract

In this paper we have proved a fixed point theorem for Banach contraction mapping in dislocated quasi b-metric space. We have also obtained a common fixed point theorem for a pair of four self mapping in dislocated quasi b-metric spaces. We have concluded examples in order to validate our establish theorem, main results and corollaries. Our result unifies, extends and improvises many known results from the current literature.

1. Introduction

In 1906, Frechet presented the idea of metric space, which is one of the foundations of mathematics as well as in a few quantitative sciences. Because of its significance and application potential, this idea has been broadened,

---

2010 Mathematics Subject Classification: 47H10, 54H25.
Keywords: common fixed point; dislocated quasi b-metric space; self-mapping; generalized contraction condition.
*Corresponding author, E-mail: qaziaftabkabir@gmail.com
Received November 2, 2019; Accepted November 22, 2019

improved and summed up from various perspectives. An inadequate rundown of such endeavors are as following: quasi metric space, symmetric space, partial metric space, cone metric space, G-metric space, b-metric space, dislocated metric space, dislocated quasi metric space, partial b-metric space and so on.

Zeyada [15] initiated the concept of dislocated quasi-metric space and generalized the result of Hitzler and Seda in dislocated quasi-metric spaces. Recently Rahman and Sarwar [7] presented the idea of dislocated quasi b-metric space and proved Banach's contraction principle, Kannan and Chatterjea type fixed point results for self mapping in such space. Further work about dislocated quasi b-metric space can be found in ([1], [6], [7], [10], [14]). The Banach contraction theorem [2] is a very papular tool in solving existence problems in many branches of mathematical analysis. In the current work the concept of dislocated quasi b-metric space which generalized the notation of b-metric, partial b-metric and b-metric like spaces, has been studied. The famous Banach's contraction principle and many other well known results in so called dislocated quasi b-metric space have been proved.

In this paper, we proved a common fixed point theorem satisfying Banach contraction conditions in the context of four self-mapping in dislocated quasi b-metric space.

2. Preliminaries

Definition 2.1 [15]**.** Let X be a nonempty and let $d : X \times X \to [o, \infty)$ be a function, called a distance function, satisfies:

$d1 : d(x, x) = 0,$

$d2 : d(x, y) = d(y, x) = 0$ then $x = y,$

$d3 : d(x, y) = d(y, x),$

$d4 : d(x, y) \leq d(x, z) + d(z, y).$

For all $x, y, z \in X.$ If d satisfies the condition $d1 - d4,$ then d is called a metric on $X.$ if it satisfies the condition $d1, d2$ and $d4$ it is called a quasi metric space. If d satisfies conditions $d2, d3, d4$ it is called a dislocated metric and if d satisfies only $d2$ and $d4$ then d is called a dislocated quasi-metric on X Non empty set X together with dq-metric d, i.e. (X, d) is called a dislocated quasi-metric space.

Definition 2.2 [15]. Let X be a non-empty and let $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called b-metric if;

$d1 : d(x, x) = 0,$

$d2 : d(x, y) = d(y, x) = 0$ then $x = y,$

$d3 : d(x, y) = d(y, x),$

$d4 : d(x, y) \leq kd(x, z) + d(z, y).$

For all $x, y, z \in X$ the pair (X, d) is called b-metric space.

It is clear that b-metric is more generalization of usual metric.

Definition 2.3 [7]. Let X be a non-empty and let $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called dislocated quasi b-metric if;

$d1 : d(x, y) = d(y, x) = 0$ then $x = y,$

$d2 : d(x, y) \leq kd(x, z) + d(z, y).$

For all $x, y, z \in X$ the pair (X, d) is called dislocates quasi b-metric or shortly (dq b-metric) space.

Definition 2.4 [7]. A sequence $\{x_n\}$ is called dq b-convergent in X if for $n \geq N$ we have $d(x_n, x) < \epsilon$ where $\epsilon > 0,$ then X is called the dq b-limit of the sequence $\{x_n\}$.

Proposition 2.5 [7]. *Let X be a non-empty set such that* d^* *is dq-metric and* d^{**} *is a b-metric with* $k \geq 1$ *on X. Then the function* $d : X \times X \to [0, \infty)$ *defined by* $d(x, y) = d^*(x, y) + d^{**}(x, y)$ *for all* $x, y \in X$ *is a dq b-metric on X.*

Definition 2.6. A sequence $\{x_n\}$ is called dq b-metric space X is called Cauchy sequence if for $\epsilon > 0, \exists n_0 \in N$ such that for all $m, n \geq n_0,$ $d(x_m, x_n) < \epsilon.$

Definition 2.7 [8]**.** A dislocated quasi b-metric space (X, d) is complete if every Cauchy sequence in it is dislocated quasi b-convergent.

Lemma 2.8 [6]**.** *Limit of convergent sequence in a dislocated quasi b-metric space is unique.*

Lemma 2.9 [2]**.** *Let A be a Banach algebra with a unit e, and* $x \in A$. *If the spectral radius* $\rho(x) < 1$, *i.e.*,

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \| x^n \|^{\frac{1}{n}} < 1,$$

then $e - x$ *is invertible. Actually*,

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

If $\rho(x) < |\lambda|$, then $\lambda e - x$ is invertible in A moreover,

$$(\lambda e - x)^{-1} = \sum_{i=0}^{\infty} \frac{x^i}{\lambda^{i-1}},$$

where λ is a complex constant.

Example 2.10. Let $A = \ell^1 = \{a = (a_n)_{n\geq 0} : \sum_{i=0}^{\infty} | a_n | < \infty\}$ with convolution as multiplication:

$$ab = (a_n)_{n\geq 0}(b_n)_{n\geq 0} = \left(\sum_{i+j=n} a_i b_j\right)_{n\geq 0}.$$

Thus A is a Banach contraction. The unit e is $(1, 0, 0, 0 \ldots)$.

Let $P = \{a = (a_n)_{n\geq 0} \in A : a_n \geq 0, \forall_n \geq 0\}$, which is a normal in A.

And let $X = \ell^1$ with the metric $d : X \times X \to A$ defined by

$$d(x, y) = d(x_n)_{n\geq 0}, (y_n)_{n\geq 0}) = (| x_n - y_n |)_{n\geq 0}.$$

Then (x, d) is a dislocated quasi b-metric space with A.

Lemma 2.11 [6]. *Let* (X, d) *be a dislocated quasi b-metric space and* $\{x_n\}$ *be a sequence in* X *such that* $d(x_n, x_{n+1}) \le \alpha d(x_{n-1}, x_n)$ *for* $n = 1, 2, 3, 4, \ldots$ *and* $0 \le \alpha k < 1$, $\alpha \in [0, 1)$, *and* k *is defined in dislocated quasi b-metric space. Then* $\{x_n\}$ *is a Cauchy sequence.*

Theorem 2.12 [8]. *Let* (X, d) *be a complete dislocated quasi b-metric space and let* $T : X \to X$ *be a continuous self mapping with* $\alpha \in \left[0, \frac{1}{2}\right)$ *and* $k \ge 1$ *satisfying the condition* $d(Tx, Ty) \le \alpha[d(x, Tx) + d(y, Ty)]$ *for all* $x, y \in X$. *Then T has a unique fixed point in X.*

Theorem 2.13 [1]. *Let* (X, d) *be a complete dislocated quasi b-metric space and let* $T : X \to X$ *be a continuous self mapping with* $\alpha \in \left[0, \frac{1}{4}\right)$ *and* $k \ge 1$ *satisfying the condition* $d(Tx, Ty) \le \alpha[d(x, Ty) + d(y, Tx)]$ *for all* $x, y \in X$. *Then T has a common fixed point in X.*

Lemma 2.14. *Let* (X, d) *be a dislocated quasi b-metric space. Let* $\{x_n\}$ *be a sequence in X. Then* $\{x_n\}$ *converges to x if and only if* $d(x_n, x) \to 0$ *as* $n \to \infty$.

Lemma 2.15. *Let* (X, d) *be a dislocated quasi b-metric space. Let* $\{x_n\}$ *be a sequence in X. Then* $\{x_n\}$ *is a Cauchy sequence if and only if* $d(x_n, x_m) \to 0$ *as* $n, m \to \infty$.

3. Main Results

In this section, we shall prove common fixed point theorem of generalized Banach Contraction mappings in the setting of dislocated quasi b-metric spaces.

Theorem 3.1. *Let* (X, d) *be a complete dislocated quasi b-metric space. Suppose that F, G, S and T are four self-maps on X such that* $T(X) \subseteq F(X)$ *and* $S(X) \subseteq G(X)$ *and suppose that at least one of these four subsets of X is complete. Let*

$$d(Sx, Ty) \le \lambda_1 d(Fx, Sx) + \lambda_2 d(Gy, Sx) + \lambda_3 d(Fx, Ty) + \lambda_4 d(Gy, Ty)$$

$$+ \lambda_5 d(Fx, Gy) + \lambda_6 \frac{d(Gy, Sy) + d(Sx, Ty)}{1 + d(Fx, Ty)} + \lambda_7 d(Ty, Gy)$$

$$+ \lambda_8 \frac{1}{2} d(Fx, Ty) + d(Gy, Sx). \tag{3.1}$$

For all $x, y \in X$. *If* $\mu\left(\frac{\lambda_1 + \lambda_2 + \lambda_8}{2}\right) + \mu\left(\frac{\lambda_1 + \lambda_3 + \lambda_5 + \lambda_7 + \lambda_8}{2}\right) < 1$ *and* $\mu\left(\frac{\lambda_3 + \lambda_4 + \lambda_8}{2}\right) + \mu\left(\frac{\lambda_2 + \lambda_4 + \lambda_6 + \lambda_7 + \lambda_8}{2}\right) < 1, \lambda_i$ *where* $1 \le i \le 8$ *are generalized constants. Then the pairs* (F, S) *and* (G, T) *has a unique point of coincidence. Moreover, F, G, S and T have common fixed point provided that the pair in* (F, S) *and* (G, T) *self mappings.*

Proof. Choose $x_0 \in X$, define a sequence $\{x_n\}$ in X as:

$$x_{2n} = Sx_{2n} = Gx_{2n+1},\ x_{2n+1} = Tx_{2n+1} = Fx_{2n+2} = \forall n \ge 0$$

$$d(x_{2n}, x_{2n+1}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\le \lambda_1 d(Fx_{2n}, Sx_{2n}) + \lambda_2 d(Gx_{2n+1}, Sx_{2n})$$

$$+\lambda_3 d(Fx_{2n}, Tx_{2n+1}) + \lambda_4 d(Gx_{2n+1}, Tx_{2n+1})$$

$$+ \lambda_5 d(Fx_{2n}, Gx_{2n+1}) + \lambda_6 \frac{d(Gx_{2n+1}, Sx_{2n+1}) + d(Sx_{2n}, Tx_{2n+1})}{1 + d(Fx_{2n}, Tx_{2n+1})}$$

$$+ \lambda_7 d(Tx_{2n+1}, Gx_{2n+1}) + \lambda_8 \frac{1}{2} d(Fx_{2n}, Tx_{2n+1}) + d(Gx_{2n+1}, Sx_{2n})$$

$$= \lambda_1 d(x_{2n-1}, x_{2n}) + \lambda_2 d(x_{2n}, x_{2n}) + \lambda_3 d(x_{2n-1}, x_{2n+1}) + \lambda_4 d(x_{2n}, x_{2n+1})$$

$$+ \lambda_5 d(x_{2n-1}, x_{2n}) + \lambda_6 \frac{d(x_{2n}, x_{2n}) + d(x_{2n}, x_{2n+1})}{1 + d(x_{2n-1}, x_{2n+1})} + \lambda_7 d(x_{2n}, x_{2n})$$

$$+ \lambda_8 \frac{1}{2} d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n})$$

$$\le \lambda_1 d(x_{2n-1}, x_{2n}) + \lambda_3 [d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})] + \lambda_4 d(x_{2n}, x_{2n+1})$$

$$+ \lambda_5 d(x_{2n-1}, x_{2n}) + \lambda_6 \frac{[d(x_{2n}, x_{2n-1}) + d(x_{2n}, x_{2n+1})}{1 + [d(x_{2n-1}, x_{2n+1})]}$$

$$+\lambda_8[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})]$$

$$d(x_{2n}, x_{2n+1}) \leq (\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 + \lambda_8)d(x_{2n-1}, x_{2n})$$

$$+(\lambda_3 + \lambda_4 + \lambda_6 + \lambda_8)d(x_{2n}, x_{2n+1})$$

$$(e - \lambda_3 - \lambda_4 - \lambda_6 - \lambda_8)d(x_{2n}, x_{2n+1}) \leq (\lambda_1 + \lambda_3 - \lambda_5 - \lambda_6 - \lambda_8)d(x_{2n-1}, x_{2n})$$

As $\mu(\lambda_3 + \lambda_4 + \lambda_6 + \lambda_8) + \mu(\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8) < 1$ implies $\mu(\lambda_3 + \lambda_4) < 1$. Hence by lemma 2.9 $(e - \lambda_3 - \lambda_4 - \lambda_6 - \lambda_8)$ is invertible, so

$$d(x_{2n}, x_{2n+1}) \leq (e - \lambda_3 - \lambda_4 - \lambda_6 - \lambda_8)^{-1}(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 + \lambda_8)d(x_{2n-1}, x_{2n})$$

$$\Rightarrow d(x_{2n}, x_{2n+1}) \leq \psi d(x_{2n-1}, x_{2n}), \tag{3.2}$$

Where $\psi = (e - \lambda_3 - \lambda_4 - \lambda_6 - \lambda_8)^{-1}(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 + \lambda_8)$.

Again by (3.1) we have.

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n+2}, Tx_{2n+1})$$

$$\leq \lambda_1 d(F_{2n+2}, Sx_{2n+2}) + \lambda_2 d(Gx_{2n+1}, Sx_{2n+2})$$

$$+\lambda_3 d(Fx_{2n+2}, Tx_{2n+1})$$

$$+ \lambda_4 d(Gx_{2n+1}, Tx_{2n+1}) + \lambda_5 d(Fx_{2n+2}, Gx_{2n+1})$$

$$+ \lambda_6 \frac{d(Gx_{2n+1}, Sx_{2n+1}) + (Sx_{2n+2}, Tx_{2n+1})}{1 + d(Fx_{2n+2}, Tx_{2n+1})}$$

$$+\lambda_7 d(Tx_{2n+1}, Gx_{2n+1})$$

$$+ \lambda_8 \frac{1}{2} d(Fx_{2n+2}, Tx_{2n+1}) + d(Gx_{2n+1}, Sx_{2n+2})$$

$$= \lambda_1 d(x_{2n+1}, x_{2n+2}) + \lambda_2 d(x_{2n}, x_{2n+2}) + \lambda_3 d(x_{2n+1}, x_{2n+1})$$

$$+\lambda_4 d(x_{2n}, x_{2n+1}) + \lambda_5 d(x_{2n+1}, x_{2n})$$

$$+ \lambda_6 \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+2}, x_{2n+1})}{1 + d(x_{2n+1}, x_{2n+1})} + \lambda_7 d(x_{2n+1}, x_{2n})$$

$$+ \lambda_8 \frac{1}{2} d(x_{2n+1}, x_{2n+1}) + d(x_{2n}, x_{2n+2})$$

$$\leq \lambda_1 d(x_{2n+1}, x_{2n+2}) + \lambda_2[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$
$$+\lambda_4 d(x_{2n}, x_{2n+1}) + \lambda_5 d(x_{2n+1}, x_{2n}) + \lambda_6[d(x_{2n}, x_{2n+1})$$
$$+d(x_{2n+2}, x_{2n+1})] + \lambda_8[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$
$$= (\lambda_1 + \lambda_2 + \lambda_6 + \lambda_8) d(x_{2n+1}, x_{2n+2}) + (\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6$$
$$+\lambda_8) d(x_{2n}, x_{2n+1})(e - \lambda_1 - \lambda_2 - \lambda_6 - \lambda_8) d(x_{2n+1}, x_{2n+2})$$
$$\leq (\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8) d(x_{2n}, x_{2n+1}).$$

As $\rho(\lambda_1 + \lambda_2 + \lambda_6 + \lambda_8) + \rho(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 + \lambda_8) < 1$ implies $\rho(\lambda_1 + \lambda_2) < 1$. Hence by lemma 2.9 $(e - \lambda_1 - \lambda_2 - \lambda_7 - \lambda_8)$ is invertible, so

$$d(x_{2n+1}, x_{2n+2}) \leq (e - \lambda_1 - \lambda_2 - \lambda_6 - \lambda_8)^{-1}(\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8) d(x_{2n}, x_{2n+1})$$

$$d(x_{2n+1}, x_{2n+2}) \leq \gamma d(x_{2n}, x_{2n+1}), \tag{3.3}$$

Where $\gamma = (e - \lambda_1 - \lambda_2 - \lambda_6 - \lambda_8)^{-1}(\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8)$.

From (3.2) and (3.3) we get

$$d(x_{2n+1}, x_{2n+2}) \leq \gamma\psi d(x_{2n-1}, x_{2n})$$

and

$$d(x_{2n}, x_{2n+1}) \leq \psi\gamma d(x_{2n-2}, x_{2n-1}).$$

Since, $\gamma\psi = \psi\gamma$, as $k_i k_j = k_j k_i$.

Thus, we have

$$d(x_{2k+1}, x_{2k+2}) \leq Ld(x_{2k-1}, x_{2k}) \leq \ldots \leq L^k Ld(x_1, x_2) \tag{3.4}$$

and

$$d(x_{2k}, x_{2k+1}) \leq Ld(x_{2k-2}, x_{2k-1}) \leq \ldots \leq L^k Ld(x_0, x_1). \tag{3.5}$$

By (3.4) and by (3.5) for any k we have

$$d(x_n, x_{n+1}) \leq L^{\frac{n-1}{2}} d(x_1, x_2)(where\, n = 2k + 1) \tag{3.6}$$

and

$$d(x_n, x_{n+1}) \leq L^{\frac{n}{2}d} (x_0, x_1) \text{ (where } n = 2k). \tag{3.7}$$

By Lemma 2.11 and Lemma 2.14 we have

$$\rho(L) = \rho(\gamma\psi) = \rho[(e - \lambda_1 - \lambda_2 - \lambda_6 - \lambda_8)^{-1}(\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8)$$

$$(e - \lambda_3 - \lambda_4 - \lambda_6 - \lambda_8)^{-1}(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 + \lambda_8)]$$

$$\leq \rho((e - \lambda_1 - \lambda_2 - \lambda_6 - \lambda_8)^{-1}(\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8))\rho((e - \lambda_3 - \lambda_4 - \lambda_6 - \lambda_8)^{-1}$$

$$(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 - \lambda_8))$$

$$= \frac{\rho(\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8)}{1 - \rho(\lambda_1 + \lambda_2 + \lambda_6 + \lambda_8)} \cdot \frac{\rho(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 + \lambda_8)}{1 - \rho(\lambda_3 + \lambda_4 + \lambda_6 + \lambda_8)} < 1.$$

Which implies $(e - L)^{-1} = \sum_{i=0}^{\infty} L^i$ and $\| L^n \| \to 0,\ n \to \infty.$

Without loss of generality, for each $m > n$, let m be even and n be odd. Thus by (3.6) and (3.7)

$$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m)$$

$$\leq \left(L^{\frac{n-1}{2}} + L^{\frac{n+1}{2}} + L^{\frac{n-2}{2}} + L^{\frac{n+2}{2}} \ldots + L^{\frac{m-2}{2}} \right) d(x_1, x_2)$$

$$+ \left(L^{\frac{n+1}{2}} + L^{\frac{n+1}{2}} + L^{\frac{n+2}{2}} + L^{\frac{n+2}{2}} \ldots + L^{\frac{m-2}{2}} \right) d(x_0, x_1)$$

$$\leq L^{\frac{n-1}{2}}(e + L + L^2 \ldots) d(x_1, x_2) + L^{\frac{n+1}{2}}(e + L + L^2 + \ldots)$$

$$d(x_0, x_1)(e - L)^{-1}[L^{\frac{n-1}{2}} d(x_1, x_2) + L^{\frac{n+1}{2}} d(x_0, x_1)].$$

Now,

$$\| L^{\frac{n-1}{2}} d(x_1, x_2) + L^{\frac{n+1}{2}} d(x_0, x_1) \| \leq \| L^{\frac{n-1}{2}} \| \| d(x_1, x_2) \| + \| L^{\frac{n+1}{2}} \| \| d(x_0, x_1) \| \to 0,$$ as $n \to \infty$.

Hence, by lemma $\left\{ L^{\frac{n-1}{2}} d(x_1, x_2) + L^{\frac{n+1}{2}} d(x_0, x_1) \right\}$ is a Cauchy sequence.

By lemma 2.14 and lemma 2.15 we conclude that $\{x_n\}$ is a Cauchy sequence in X.

Suppose that $f(X)$ is complete subset of X. Then there exists $x \in X$ such that $x_n \to x = f\omega^*$, as $n \to \infty$ for $\omega^* \in X$. Implies $\{x_n\}$ and $\{x_{n+1}\}$ also converge to x.

Now we will prove that $x = S\omega^*$.

By using (3.1) we have

$$d(S\omega^*, x) \leq d(S\omega^*, Ty_{n+1}) + d(Ty_{n+1}, x)$$

$$\leq \lambda_1 d(F\omega^*, S\omega^*) + \lambda_2 d(Gy_{n+1}, S\omega^*) + \lambda_3 d(F\omega^*, Ty_{n+1})$$

$$+ \lambda_4 d(Gy_{n+1}, Ty_{n+1}) + \lambda_5 d(F\omega^*, Gy_{n+1})$$

$$+ \lambda_6 \frac{d(Gy_{n+1}, Sy_{n+1}) + d(S\omega^*, Ty_{n+1})}{1 + d(F\omega^*, Ty_{n+1})} + \lambda_7 d(Ty_{n+1}, Gy_{n+1})$$

$$+ \lambda_8 \frac{1}{2} d(F\omega^*, Ty_{n+1}) + d(Gy_{n+1}, S\omega^*) + d(Ty_{n+1}, x)$$

$$= \lambda_1 d(x, S\omega^*) + \lambda_2 d(Gy_{n+1}, S\omega^*) + \lambda_3 d(x, Ty_{n+1}) + \lambda_4 d(Gy_{n+1}, Ty_{n+1})$$

$$+ \lambda_5 d(x, Gy_{n+1}) + \lambda_6 \frac{d(Gy_{n+1}, Sy_{n+1}) + d(S\omega^*, Ty_{n+1})}{1 + d(x, Ty_{n+1})}$$

$$+ \lambda_7 d(Ty_{n+1}, Gy_{n+1}) + \lambda_8 \frac{1}{2} d(x, Ty_{n+1}) + d(Gy_{n+1}, S\omega^*) + d(Ty_{n+1}, x)$$

$$\leq \lambda_1 d(x, S\omega^*) + \lambda_2 d(Gy_{n+1}, S\omega^*) + \lambda_3 d(x, Ty_{n+1}) + \lambda_4 d(Gy_{n+1}, Ty_{n+1})$$

$$+ \lambda_5 d(x, Gy_{n+1}) + \lambda_6 \frac{d(Gy_{n+1}, Sy_{n+1}) + d(S\omega^*, Ty_{n+1})}{1 + d(x, Ty_{n+1})} + \lambda_7 d(Ty_{n+1}, Gy_{n+1})$$

$$+ \lambda_8 [d(x, Ty_{n+1}) + d(Gy_{n+1}, S\omega^*)] + d(Ty_{n+1}, x)$$

$$\leq \lambda_1 d(x, S\omega^*) + \lambda_2 d(Gy_{n+1}, S\omega^*) + \lambda_3 d(x, Ty_{n+1}) + \lambda_4 d(Gy_{n+1}, Ty_{n+1})$$

$$+ \lambda_5 d(x, Gy_{n+1}) + \lambda_6 d(Gy_{n+1}, Sy_{n+1}) + d(S\omega^*, Ty_{n+1}) + \lambda_7 d(Ty_{n+1}, Gy_{n+1})$$

$$+ \lambda_8 [d(x, Ty_{n+1}) + d(Gy_{n+1}, S\omega^*)] + d(Ty_{n+1}, x)$$

$$\leq \lambda_1 d(x, S\omega^*) + \lambda_2[d(Gy_{n+1}, x) + d(x, S\omega^*)] + \lambda_3 d(x, Ty_{n+1}) + \lambda_4 d(Gy_{n+1}, x)$$

$$+[d(x, Ty_{n+1})] + \lambda_5 d(x, Gy_{n+1}) + \lambda_6[d(Gy_{n+1}, x) + d(x, Ty_{n+1})]$$

$$+\lambda_7 d(Ty_{n+1}, Gy_{n+1}) + \lambda_8[d(x, Ty_{n+1}) + d(Gy_{n+1}, x)] + d(Ty_{n+1}, x) \tag{3.8}$$

$$\Rightarrow d(S\omega^*, x) \leq \frac{1}{e - \lambda_1 - \lambda_4 - \lambda_8}[(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8)]d(Gy_{n+1}, x)$$

$+(e + \lambda_5 + \lambda_6 + \lambda_8)d(Tx_{n+1}, x)]$. Because $(e - \lambda_1 - \lambda_4 - \lambda_8)$ is invertible.

Since $\{d(Gy_{n+1}, x)\}$ and $\{(Tx_{n+1}, x)\}$ are Cauchy sequences, therefore, by Lemma 2.11 and Lemma 2.14 it follows that $x = S\omega^*$. Hence $x = F\omega^* = S\omega^*$. Since $x = S\omega^* \in S(X) \subseteq G(X)$, then there exists $\omega^{**} \in X$ such that $x = G\omega^{**}$.

Now we will prove that $x = T\omega^*$. By (3.1) we have

$$d(x, T\omega^{**}) = d(x, Sy_{2n}) + d(Sy_{2n}, Tz^{**})$$

$$d(x, Sy_{2n}) \leq \lambda_1 d(Fy_{2n}, Sy_{2n}) + \lambda_2 d(G\omega^{**}, Sy_{2n}) + \lambda_3 d(Fy_{2n}, T\omega^{**})$$

$$+\lambda_4 d(G\omega^{**}, T\omega^{**})$$

$$+\lambda_5 d(Fy_{2n}, G\omega^{**}) + \lambda_6 \frac{d(G\omega^{**}, S\omega^{**}) + d(Sy_{2n}, T\omega^{**})}{1 + d(Fy_{2n}, T\omega^{**})}$$

$$+\lambda_7 d(T\omega^{**}, G\omega^{**}) + \lambda_8 \frac{1}{2} d(Fy_{2n}, T\omega^{**}) + d(G\omega^{**} Sy_{2n})$$

$$= d(x, Sy_{2n}) + \lambda_1 d(Fy_{2n}, Sy_{2n}) + \lambda_2 d(x, Sy_{2n})$$

$$+\lambda_3 d(Fy_{2n}, T\omega^{**}) + \lambda_4 d(x, T\omega^{**})$$

$$+\lambda_5 d(Fy_{2n}, x) + \lambda_6 \frac{d(x, S\omega^{**}) + d(Sy_{2n}, T\omega^{**})}{1 + d(Fy_{2n}, T\omega^{**})}$$

$$+\lambda_7 d(T\omega^{**}, x) + \lambda_8 \frac{1}{2} d(Fy_{2n}, T\omega^{**}) + d(x, Sy_{2n})$$

$$\leq d(x, Sy_{2n}) + \lambda_1[d(Fy_{2n}, x) + d(x, Sy_{2n})] + \lambda_2 d(x, Sy_{2n})$$

$$+ \lambda_3[d(Fy_{2n}, x) + d(x, T\omega^{**}) + \lambda_4 d(x, T\omega^{**})$$

$$+ \lambda_5 d(Fy_{2n}, x) + \lambda_6[d(Fy_{2n}, x)] + d(x, T\omega^{**})$$

$$+ \lambda_7[d(Fy_{2n}, T\omega^{**}) + d(x, Sy_{2n})]$$

$$\Rightarrow d(x, T\omega^{**}) \leq \frac{1}{e - \lambda_3 - \lambda_4 - \lambda_8}[(e + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8)]d(x, Sy_{2n})$$

$$+ (\lambda_5 + \lambda_6 + \lambda_8) d(Fy_{2n}, x).$$

Since $\{d(x, Sy_{2n})\}$ and $\{(Fx_{2n}, x)\}$ are Cauchy sequences, then by Lemma 2.11 and Lemma 2.14 it follows that $x = T\omega^{**}$. Hence $x = G\omega^{**} = T\omega^{**}$. Thus we have proved that x is a common point of coincidence for pairs (F, S) and (G, T).

Conclusion

In this paper, we have proved a fixed point theorem for new banach contraction condition with pair of four self mappings in dislocated quasi b-metric space the presented result generalize some existing results due to Aage [1], Kineam and Suanoom [6], and Sharma [14].

References

[1] C. T. Aage and P. G. Golhare, On fixed point theorems in dislocated quasi-b-metric spaces, International Journal of Advances in Mathametics (1) (2016), 55-70.

[2] I. A. Bakhtin, The contraction mapping principle in quasi metric spaces, Funct. Anal. 2(30) (1989), 26-37.

[3] S. Czerwik, Contraction mappings in b-metric spaces, Acta. Math. Inf. Univ. Ostraviensis 1(23) (1993), 5-11.

[4] P. Hitzler and A. K. Seda, Dislocated topologies, J. Electr. Engin. 51(12) (2000), 3-7.

[5] Wu Hao and Wu. Dingping, Some fixed point theorems in complete dislocated quasi-b-metric space, Journal of Mathematics Research 8(4) (2016), 68-73.

[6] C. Kineam and C. Suanoom, Dislocated quasi-b-metric spaces and fixed point theorems for cyclic contractions, fixed point theory and applications 74 (2015), 12 pages.

[7] M. Rahman and M. Sarwar, Dislocated quasi b-metric space and fixed point theorems, Electronic Journal of Mathematical Analysis and Applications 4(2) (2016), 16-24.

[8] M. U. Rahmana and M. Sarwar, Coupled fixed point theorem in dislocated quasi b-metric spaces, Communications in Nonlinear Analysis 2 (2016), 113-118.

[9] M. U. Rahman and M. Sarwar, Coupled fixed point theorem for rational type contraction conditions in dislocated quasi-metric space, Palestine J. Math. 6(11) (2016), 17-27.

[10] M. U. Rahaman, New fixed point theorems in dislocated quasi-b-metric space, Appl. Math. SciLett 5(1) (2017), 7-11.

[11] B. E. A Rhoades, Comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226 (1977), 257-290.

[12] W. Rudin, Functional Analysis, 2nd ed., McGraw-Hill, New York, 1991.

[13] Suanoom, Cholatis, C. Klin-eam and Suantai, Suthep, Dislocated quasi-b-metric spaces and fixed point theorems for cyclic weakly contractions, J. Nonlinear Sci. Appl. 9 (2016), 2779-2788.

[14] M. A Sharma, Fixed point result in dislocated quasi b-metric space, Int. J. Math. And Appl. 6(2-A) (2018), 271-274.

[15] F. M. Zeyada, G. H. Hassan and M. A. Ahmed, A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces, The Arabian Journal for Science and Engineering 31(1A) (2005), 111-114.