Advances and Applications in Mathematical Sciences
Volume 19, Issue 12, October 2020, Pages 1227-1239

APPLICATIONS OF INCOMPLETE *I*-FUNCTIONS IN INITIAL BOUNDARY VALUE PROBLEMS

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Abstract

The incomplete *I*-functions are the extension of *I*-function [17] which is an extension of familiar Fox's *H*-function ([9], [14]). In this paper we find the solutions of one dimensional Heat flow equation in terms of incomplete *I*-functions. Further, numerous special cases are also obtained from our main results.

1. Introduction

In the last decade, many authors (see, e.g. [1-7], [12-13], [15-16], [18]) have developed numerous integral formulas involving a variety of incomplete hypergeometric functions. Such integral formulas have many applications in potential field of physics, applied sciences, engineering and chemical sciences.

Recently, Bansal et al. [1] introduce new incomplete *I*-functions and gave certain interesting integral formulas and transform of these functions, which are expressed in terms of generalized (Wright) hypergeometric function.

In order to derive our main results we recall here the following definitions of some well known special functions:

The incomplete Gamma functions (IGFs) $\gamma(p, y)$ and $\Gamma(p, y)$ [19] are defined as follows

2010 Mathematics Subject Classification: 33E20, 44A40.
Keywords: Incomplete *I*-functions; Incomplete Gamma functions and Mellin- Barnes type integral.
Received May 1, 2020; Accepted July 31, 2020

$$\gamma(p,\ y) = \int_0^y e^{-t} t^{p-1} dt,\ (\text{Re}\,(p) \geq 0;\ y \geq 0) \tag{1.1}$$

and

$$\Gamma(p,\ y) = \int_0^y e^{-t} t^{p-1} dt,\ (\text{Re}\,(p) \geq 0;\ y \geq 0) \tag{1.2}$$

respectively, holds the subsequent relation

$$\gamma(p,\ y) + \Gamma(p,\ y) = \Gamma(p),\ (\text{Re}\,(p) > 0). \tag{1.3}$$

The incomplete I-functions (IIFs) ${}^{(\Gamma)}I^{m,n}_{p_l, q_l; r}(z)$ and ${}^{(\gamma)}I^{m,n}_{p_l, q_l; r}(z)$ [1] are defined as follows

$$ {}^{(\Gamma)}I^{m,n}_{p_l, q_l; r}(z) = {}^{(\Gamma)}I^{m,n}_{p_l, q_l; r}\left[z \,\middle|\, \begin{matrix} (g_j,\ G_j,\ y),\ (g_j,\ G_j)_{2,\ n},\ (g_{jl},\ G_{jl})_{n+1,\ p_l} \\ (h_j,\ H_j)_{1,\ m},\ (h_{jl},\ H_{jl})_{m+1,\ ql} \end{matrix} \right] $$

$$ = \frac{1}{2\pi\omega} \int_L \theta_1(\xi,\ y) z^{-\xi} d\xi $$

where

$$\theta_1(\xi,\ y) = \frac{\Gamma(1 - g_1 - G_1\xi,\ y) \prod_{j=1}^{m} \Gamma(h_j + H_j\xi) \prod_{j=2}^{n} \Gamma(1 - g_j - G_j\xi)}{\sum_{l=1}^{r} \left[\prod_{j=m+1}^{q_l} \Gamma(1 - h_{jl} - H_{jl}\xi) \prod_{j=n+1}^{p_l} \Gamma(g_{jl} + G_{jl}\xi) \right]} \tag{1.5}$$

and

$$ {}^{(\gamma)}I^{m,n}_{p_l, q_l; r}(z) = {}^{(\gamma)}I^{m,n}_{p_l, q_l; r}\left[z \,\middle|\, \begin{matrix} (g_j,\ G_j,\ y),\ (g_j,\ G_j)_{2,\ n},\ (g_{jl},\ G_{jl})_{n+1,\ p_l} \\ (h_j,\ H_j)_{1,\ m},\ (h_{jl},\ H_{jl})_{m+1,\ ql} \end{matrix} \right] $$

$$ = \frac{1}{2\pi\omega} \int_L \theta_2(\xi,\ y) z^{-\xi} d\xi \tag{1.6}$$

where

$$\theta_2(\xi, y) = \frac{\gamma(1-g_1-G_1\xi, y)\prod_{j=1}^{m}\Gamma(h_j+H_j\xi)\prod_{j=2}^{n}\Gamma(1-g_j-G_j\xi)}{\sum_{l=1}^{r}\left[\prod_{j=m+1}^{q_l}\Gamma(1-h_{jl}-H_{jl}\xi)\prod_{j=n+1}^{p_l}\Gamma(g_{jl}+G_{jl}\xi)\right]}. \quad (1.7)$$

The incomplete *I*-functions ${}^{(\Gamma)}I_{p_l, q_l; r}^{m, n}(z)$ and ${}^{(\gamma)}I_{p_l, q_l; r}^{m, n}(z)$ in (1.4) and (1.6) exists for $y \geq 0$ under the following set of conditions satisfied.

The contour L in the complex ξ-plane extends from $c - i\infty$ to $c + i\infty$, $c \in \text{Re}$, and poles of the gamma functions $\Gamma(1 - g_l - G_j\xi)$, $j = \overline{1, n}$ do not exactly match with the poles of the gamma functions $\Gamma(h_j + H_j\xi)$, $j = \overline{1, m}$. The parameters m, n, p_l, q_l are non negative integers satisfying $0 \leq n \leq p_l$, $0 \leq m \leq q_l$, $l = \overline{1, r}$. The parameters G_j, H_j, G_{jl}, H_{jl} are positive integers and g_j, h_j, g_{jl}, h_{jl} are complex. All poles of $\theta_1(\xi, y)$ and $\theta_2(\xi, y)$ are supposed to be simple and the empty product is treated as unity.

$$\lambda_l = \sum_{j=2}^{n} G_j + \sum_{j=2}^{n} H_j - \sum_{j=n+1}^{p_l} G_{jl} - \sum_{j=m+1}^{q_l} H_{jl}, \quad (1.8)$$

$$\mu_l = \sum_{j=1}^{n} g_j - \sum_{j=2}^{n} h_j + \sum_{j=n+1}^{p_l} g_{jl} - \sum_{j=m+1}^{q_l} h_{jl} + \frac{1}{2}(p_l - q_l),\ l = \overline{1, r}. \quad (1.9)$$

On setting $y = 0$, the incomplete *I*-functions ${}^{(\Gamma)}I_{p_l, q_l; r}^{m, n}(z)$ and ${}^{(\gamma)}I_{p_l, q_l; r}^{m, n}(z)$ reduce to Saxena's *I*-function [18]:

$$\begin{aligned} {}^{(\Gamma)}I_{p_l, q_l; r}^{m, n}&\left[z \,\middle|\, \begin{matrix} (g_j, G_j, 0), (g_j, G_j)_{2, n}, (g_{jl}, G_{jl})_{n+1, p_l} \\ (h_j, H_j)_{1, m}, (h_{jl}, H_{jl})_{m+1, ql} \end{matrix}\right] \\ &= I_{p_l, q_l; r}^{m, n}\left[z \,\middle|\, \begin{matrix} (g_j, G_j)_{1, n}, (g_{jl}, G_{jl})_{n+1, p_l} \\ (h_j, H_j)_{1, m}, (h_{jl}, H_{jl})_{m+1, ql} \end{matrix}\right], \end{aligned} \quad (1.10)$$

and

$$ {}^{(\gamma)}I^{m,n}_{p_l,q_l;r}\left[z \,\middle|\, \begin{matrix}(g_1, G_1, 0), (g_j, G_j)_{2,n}, (g_{jl}, G_{jl})_{n+1,p_l} \\ (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1,ql}\end{matrix}\right] $$

$$ = I^{m,n}_{p_l,q_l;r}\left[z \,\middle|\, \begin{matrix}(g_j, G_j)_{1,n}, (g_{jl}, G_{jl})_{n+1,p_l} \\ (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1,ql}\end{matrix}\right]. \tag{1.11} $$

2. Preliminaries

Here, we find the following interesting results involving IIFs ${}^{(\Gamma)}I^{m,n}_{p_l,q_l;r}(z)$ and ${}^{(\gamma)}I^{m,n}_{p_l,q_l;r}(z)$ by using [10, p.72, Equation (2.2.1.8)] for next section:

$$ (a)\int_0^{\pi}(\sin x)^{\rho-1}\cos mx\; {}^{(\Gamma)}I^{m,n}_{p_l,q_l;r} $$

$$ \left[z(\sin x)^{2\sigma} \,\middle|\, \begin{matrix}(g_1, G_1, y), (g_j, G_j)_{2,n}, (g_{jl}, G_{jl})_{n+1,p_l} \\ (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1,ql}\end{matrix}\right] $$

$$ = \sqrt{\pi}\cos\frac{m\pi}{2}\; {}^{(\Gamma)}I^{m+2,n}_{p_l+2,q_l+2;r} $$

$$ \left[z \,\middle|\, \begin{matrix}(g_1, G_1, y), (g_j, G_j)_{2,n'}, (g_{jl}, G_{jl})_{n+1,p_l}, \left(\frac{\rho \pm m+1}{2}, \sigma\right) \\ \left(\frac{\rho}{2}, \sigma\right), \left(\frac{\rho+1}{2}, \sigma\right), (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1,ql}\end{matrix}\right] \tag{2.1} $$

and

$$ \int_0^{\pi}(\sin x)^{\rho-1}\cos mx\; {}^{(\gamma)}I^{m,n}_{p_l,q_l;r} $$

$$ \left[z(\sin x)^{2\sigma} \,\middle|\, \begin{matrix}(g_1, G_1, y), (g_j, G_j)_{2,n}, (g_{jl}, G_{jl})_{n+1,p_l} \\ (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1,ql}\end{matrix}\right] dx $$

$$ = \sqrt{\pi}\cos\frac{m\pi}{2}\; {}^{(\gamma)}I^{m+2,n}_{p_l+2,q_l+2;r} $$

$$\left[\begin{matrix} z \mid (g_1, G_1, y), (g_j, G_j)_{2,n}(g_{jl}, G_{jl})_{n+1, p_l}, \left(\frac{\rho \pm m+1}{2}, \sigma\right) \\ \left(\frac{\rho}{2}, \sigma\right), \left(\frac{\rho+1}{2}, \sigma\right), (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1, ql} \end{matrix}\right]. \tag{2.2}$$

(b) $$\int_0^{\pi} (\sin x)^{\rho-1} \sin mx \ {}^{(\Gamma)}I_{p_l, q_l; r}^{m, n}$$

$$\left[\begin{matrix} z(\sin x)^{2\sigma} \mid (g_1, G_1, y.), (g_j, G_j)_{2, n}, (g_{jl}, G_{jl})_{n+1, p_l} \\ (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1, ql} \end{matrix}\right] dx$$

$$= \sqrt{\pi} \sin \frac{m\pi}{2} \ {}^{(\Gamma)}I_{p_l+2, q_l+2; r}^{m+2, n}$$

$$\left[\begin{matrix} z \mid (g_1, G_1, y), (g_j, G_j)_{2,n}, (g_{jl}, G_{jl})_{n+1, p_l}, \left(\frac{\rho \pm m+1}{2}, \sigma\right) \\ \left(\frac{\rho}{2}, \sigma\right), \left(\frac{\rho+1}{2}, \sigma\right), (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1, ql} \end{matrix}\right], \tag{2.3}$$

$$\int_0^{\pi} (\sin x)^{\rho-1} \sin mx \ {}^{(\gamma)}I_{p_l, q_l; r}^{m, n}$$

$$\left[\begin{matrix} z(\sin x)^{2\sigma} \mid (g_1, G_1, y.), (g_j, G_j)_{2, n}, (g_{jl}, G_{jl})_{n+1, p_l} \\ (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1, ql} \end{matrix}\right] dx$$

$$= \sqrt{\pi} \cos \frac{m\pi}{2} \ {}^{(\gamma)}I_{p_l+2, q_l+2; r}^{m+2, n}$$

$$\left[\begin{matrix} z \mid (g_1, G_1, y.), (g_j, G_j)_{2,n}, (g_{jl}, G_{jl})_{n+1, p_l}, \left(\frac{\rho \pm m+1}{2}, \sigma\right) \\ \left(\frac{\rho}{2}, \sigma\right), \left(\frac{\rho+1}{2}, \sigma\right), (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1, ql} \end{matrix}\right]$$

provided that

(i) $\operatorname{Re}(\rho) > 0$, $\operatorname{Re}(\rho) + 2\sigma \max_{1 \leq j \leq n}\left[\operatorname{Re} \frac{(1-g_j)}{G_j}\right] > 0$.

(iii) Other conditions are same as given in equations (1.4)-(1.9).

3. Main results

3.1. Solution of Non Homogeneous Heat Equation

Let us start with the heat equation [8], consider a wire (or a thin metal rod) of length L that is insulated except at the endpoints. Let x denote the position along the wire and t denote time

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{3.1.1}$$

where c^2 is constant diffusivity of material and satisfying boundary and initial conditions

$$u(0, t) = u(L, t) = 0, \tag{3.1.2}$$

$$u(x, 0) = \Omega(x), \tag{3.1.3}$$

Initially, we assume

$$u(x, 0) = \Omega(x)$$

$$= (\sin x)^{\rho-1 (\Gamma)} I^{m,n}_{p_l, q_l, r}\left[z(\sin x)^{2\sigma} \,\middle|\, \begin{matrix} (g_1, G_1, y), (g_j, G_j)_{2,n}, (g_{jl}, G_{jl})_{n+1, p_l} \\ (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1, ql} \end{matrix} \right]$$

$$at\, t = 0,\ 0 \le x \le L, \tag{3.1.4}$$

where ρ, σ are positive numbers and ${}^{(\Gamma)}I^{m,n}_{p_l, q_l, r}(z)$ is defined by (1.4).

Since I-function contains as particular cases almost all known special functions like Bessel, Mittag-Leffer, Legendre, Hermite, hyper geometric functions etc. so this assumption is fairly general.

Let the solution of equation (3.1.1)-(3.1.3) is obtained by separation of variables and represented in the following general form

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin K_n x\, e^{-K_n{}^2 c^2 t}, \tag{3.1.5}$$

Where B_n is arbitrary constant and $K_n = \frac{N\pi}{L}$, $(N = 0, 1, 2, \ldots)$.

From equations (3.1.4) and (3.1.5), we obtain

$$(\sin x)^{\rho-1}{}^{(\Gamma)}I^{m,n}_{p_l,q_l,r}\left[\begin{matrix} z(\sin x)^{2\sigma} \mid (g_1, G_1, y), (g_j, G_j)_{2,n}, (g_{jl}, G_{jl})_{n+1,p_l} \\ (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1,ql} \end{matrix}\right]$$

$$= \sum_{n=1}^{\infty} B_n \sin x. \tag{3.1.6}$$

Multiplying on both sides of (3.1.6) by $e^{iK_m x}$, m is any positive integer and integration with respect to x from 0 to L. Separating real and imaginary parts and using (2.1) and (2.3), we have

$$B_n = \frac{2K_n\sqrt{\pi}\cos K_n \frac{\pi}{2}}{\sin K_n^2 L}\,{}^{(\Gamma)}I^{m+2,n}_{p_l+2,q_l+2;r}\left[z \,\middle|\, \begin{matrix} A^{\cdot} \\ B^{\cdot} \end{matrix}\right]$$

$$= \frac{2K_n\sqrt{\pi}\sin K_n \frac{\pi}{2}}{(K_n L - \sin K_n L\cos K_n L)}\,{}^{(\Gamma)}I^{m+2,n}_{p_l+2,q_l+2;r}\left[z \,\middle|\, \begin{matrix} A^{\cdot} \\ B^{\cdot} \end{matrix}\right], \tag{3.1.7}$$

where

$$A^{\cdot} = (g_j, G_j, y), (g_j, G_j)_{2,n}, (g_{jl}, G_{jl})_{n+1,p_l}, (\rho \pm K_n + 1/2, \sigma) \tag{3.1.8}$$

and

$$B^{\cdot} = (\rho/2, \sigma), (\rho + 1/2, \sigma), (h_j, H_j)_{1,m}, (h_{jl}, H_{jl})_{m+1,q_l}. \tag{3.1.9}$$

Hence, we arrive at the desired solution of equation (3.1.1)-(3.1.3) is given by

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2K_n\sqrt{\pi}\cos K_n \frac{\pi}{2}\sin K_n x e^{-K_n^2 c^2 t}}{\sin K_n^2 L}\,{}^{(\Gamma)}I^{m+2,n}_{p_l+2,q_l+2;r}\left[z \,\middle|\, \begin{matrix} A^{\cdot} \\ B^{\cdot} \end{matrix}\right]$$

$$= \sum_{n=1}^{\infty} \frac{2K_n \sqrt{\pi} \sin K_n \frac{\pi}{2} \sin K_n \, x \, e^{-K_n^2 c^2 t}}{(K_n L - \sin K_n L \cos K_n L)} \, {}^{(\Gamma)}I^{m+2, n}_{p_l+2, q_l+2; r}\left[z \,\middle|^{A^{\cdot}}_{B^{\cdot}} \right]. \quad (3.1.10)$$

Now, the solution of equation (3.1.1)-(3.1.3) in IIF ${}^{(\gamma)}I^{m, n}_{p_l, q_l, r}(z)$, is given by

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2K_n \sqrt{\pi} \cos K_n \frac{\pi}{2} \sin K_n x \, e^{-K_n^2 c^2 t}}{\sin K_n^2 L} \, {}^{(\gamma)}I^{m+2, n}_{p_l+2, q_l+2; r}\left[z \,\middle|^{A^{\cdot}}_{B^{\cdot}} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2K_n \sqrt{\pi} \sin K_n \frac{\pi}{2} \sin K_n \, x \, e^{-K_n^2 c^2 t}}{(K_n L - \sin K_n L \cos K_n L)} \, {}^{(\gamma)}I^{m+2, n}_{p_l+2, q_l+2; r}\left[z \,\middle|^{A^{\cdot}}_{B^{\cdot}} \right]. \quad (3.1.11)$$

IIFs ${}^{(\Gamma)}I^{m, n}_{p_l, q_l, r}(z)$ and ${}^{(\gamma)}I^{m, n}_{p_l, q_l, r}(z)$ are satisfied the conditions which are given in (1.4) and (1.6) respectively.

3.2. Solution of Homogeneous Wave Equation

The wave equation [8] is of second order with respect to the space variable x and time t, and takes of the form

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x}, \; (0 < x < L, \; t > 0). \quad (3.2.1)$$

Here the constant c^2 is called the wave speed, with initial and boundary conditions

$$u(x, 0) = u(L, t) = 0, \quad (3.2.2)$$

$$u(x, 0) = \emptyset(x) \text{ and } \frac{\partial u}{\partial t} = 0 \text{ at } t = 0. \quad (3.2.3)$$

Then, we consider the general solution of (3.2.1)-(3.2.3) by separation of variables method is given by

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin K_n x \cos \frac{K_n c}{t}, \quad (3.2.4)$$

where B_n is arbitrary constant and $K_n = \frac{N\pi}{L}$, $(N = 0, 1, 2, \ldots)$.

Now apply the same lines of section 3.1, then we arrive at the desire solutions of (3.2.1)-(3.2.3) in terms of IIFs ${}^{(\Gamma)}I^{m,n}_{p_l,q_l,r}(z)$ and ${}^{(\gamma)}I^{m,n}_{p_l,q_l,r}(z)$ are given by

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2K_n\sqrt{\pi}\cos K_n \frac{\pi}{2}\sin K_n x \cos \frac{K_n c}{t}}{\sin K_n^2 L} {}^{(\Gamma)}I^{m+2,n}_{p_l+2,q_l+2;r}\left[z \,\middle|\, {}^{A^{\cdot}}_{B^{\cdot}}\right]$$

$$= \sum_{n=1}^{\infty} \frac{2K_n\sqrt{\pi}\sin K_n \frac{\pi}{2}\sin K_n\, x \cos \frac{K_n c}{t}}{(K_n L - \sin K_n L \cos K_n L)} {}^{(\Gamma)}I^{m+2,n}_{p_l+2,q_l+2;r}\left[z \,\middle|\, {}^{A^{\cdot}}_{B^{\cdot}}\right], \quad (3.2.5)$$

and

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2K_n\sqrt{\pi}\cos K_n \frac{\pi}{2}\sin K_n x \cos \frac{K_n c}{t}}{\sin K_n^2 L} {}^{(\gamma)}I^{m+2,n}_{p_l+2,q_l+2;r}\left[z \,\middle|\, {}^{A^{\cdot}}_{B^{\cdot}}\right]$$

$$= \sum_{n=1}^{\infty} \frac{2K_n\sqrt{\pi}\sin K_n \frac{\pi}{2}\sin K_n\, x \cos \frac{K_n c}{t}}{(K_n L - \sin K_n L \cos K_n L)} {}^{(\gamma)}I^{m+2,n}_{p_l+2,q_l+2;r}\left[z \,\middle|\, {}^{A^{\cdot}}_{B^{\cdot}}\right], \quad (3.2.6)$$

Where $A^{\cdot}$ and $B^{\cdot}$ are defined in (3.1.8) and (3.1.9) and $K_n = \frac{N\pi}{L}$, $(N = 0, 1, 2, \ldots)$. IIFs ${}^{(\Gamma)}I^{m,n}_{p_l,q_l,r}(z)$ and ${}^{(\gamma)}I^{m,n}_{p_l,q_l,r}(z)$ are satisfied the conditions which are given in (1.4) and (1.6) respectively.

3.3. Special cases. Now, we find certain interesting cases of solution of heat equation (3.1.1).

(i) On setting $y = 0$, the incomplete *I*-functions ${}^{(\Gamma)}I^{m,n}_{p_l,q_l,r}(z)$ reduce to Saxena's *I*-Function [17]. Then solution is

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2K_n\sqrt{\pi}\cos K_n \frac{\pi}{2}\sin K_n x\, e^{-K_n^2 c^2 t}}{\sin K_n^2 L} I^{m+2,n}_{p_l+2,q_l+2;r}\left[z \,\middle|\, {}^{A^{\cdot}}_{B^{\cdot}}\right]$$

$$= \sum_{n=1}^{\infty} \frac{2K_n \sqrt{\pi} \sin K_n \frac{\pi}{2} \sin K_n \; x e^{-K_n^2 c^2 t}}{(K_n L - \sin K_n L \cos K_n L)} I^{m+2, n}_{p_l+2, q_l+2; r}\left[z \,|^{A^{\cdot}}_{B^{\cdot}} \right], \tag{3.3.1}$$

where $A^{\cdot} = (g_j, G_j)_{1, n}, (g_j, G_j)_{n+1, p_l}, (\rho \pm K_n + 1/2, \sigma)$ and $B^{\cdot} = (\rho/2, \sigma), (\rho + 1/2, \sigma), (h_j, H_j)_{1, m}, (h_{jl}, H_{jl})_{m+1, q_l}$, provided that each member in (3.1) exists.

(ii) On setting $y = 0$, the incomplete I-functions ${}^{(\Gamma)}I^{m, n}_{p_l, q_l, r}(z)$ reduce to I^*- Function [11]. Then

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2K_n \sqrt{\pi} \cos K_n \frac{\pi}{2} \sin K_n x \, e^{-K_n^2 c^2 t}}{\sin K_n^2 L} I^{m+2, n}_{p_l+2, q_l; r}\left[z \,|^{A^{\cdot}}_{B^{\cdot}} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2K_n \sqrt{\pi} \sin K_n \frac{\pi}{2} \sin K_n \; x e^{-K_n^2 c^2 t}}{(K_n L - \sin K_n L \cos K_n L)} I^{m+2, n}_{p_l+2, q_l; r}\left[z \,|^{A^{\cdot}}_{B^{\cdot}} \right], \tag{3.3.1}$$

where $A^{\cdot} = (g_j, G_j)_{1, n}, (g_{jl}, G_{jl})_{1, p_l}, (\rho \pm K_n + 1/2, \sigma)$ and $B^{\cdot} = (\rho/2, \sigma), (\rho + 1/2, \sigma), (h_j, H_j)_{1, m}, (h_{jl}, H_{jl})_{1, q_l}$, provided that each member in (3.1) exists.

(iii) On setting $r = 1$ in (1.4) and (1.6), it reduces to IHF introduced by Srivastava [19]. Then

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2K_n \sqrt{\pi} \cos K_n \frac{\pi}{2} \sin K_n x \, e^{-K_n^2 c^2 t}}{\sin K_n^2 L} \Gamma^{m+2, n}_{p+2, q+2}\left[z \,|^{A^{\cdot}}_{B^{\cdot}} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2K_n \sqrt{\pi} \sin K_n \frac{\pi}{2} \sin K_n \; x e^{-K_n^2 c^2 t}}{(K_n L - \sin K_n L \cos K_n L)} \Gamma^{m+2, n}_{p+2, q+2}\left[z \,|^{A^{\cdot}}_{B^{\cdot}} \right], \tag{3.3.2}$$

where $A^{\cdot} = (g_j, G_j, y), (g_j, G_j)_{2, n}, (g_{jl}, G_{jl})_{n+1, p}, (\rho \pm K_n + 1/2, \sigma)$ and $B^{\cdot} = (\rho/2, \sigma), (\rho + 1/2, \sigma), (h_j, H_j)_{1, q}$, provided that each member in (3.2) exists.

(iv) If we put $r = 1$ and $y = 0$ in (1.4) and (1.6), it reduces to Fox's H-function [9]:

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2K_n \sqrt{\pi} \cos K_n \frac{\pi}{2} \sin K_n x \, e^{-K_n^2 c^2 t}}{\sin K_n^2 L} H_{p+2, q+2}^{m+2, n}\left[z \,|_{B^{\cdot}}^{A^{\cdot}} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2K_n \sqrt{\pi} \sin K_n \frac{\pi}{2} \sin K_n \, x e^{-K_n^2 c^2 t}}{(K_n L - \sin K_n L \cos K_n L)} H_{p+2, q+2}^{m+2, n}\left[z \,|_{B^{\cdot}}^{A^{\cdot}} \right], \tag{3.3.3}$$

where $A^{\cdot} = (g_j, G_j)_{1, n}, (g_j, G_j)_{n+1, p}, (\rho \pm K_n + 1/2, \sigma)$ and $B^{\cdot} = (\rho/2, \sigma), (\rho + 1/2, \sigma), (h_j, H_j)_{1, q}$, provided that each member in (3.3) exists.

(v) On setting $y = 0$ and $t = 0$, the incomplete I-functions ${}^{(\Gamma)}I_{p_l, q_l, r}^{m, n}(z)$ reduce to Anharmonic Fourier series for I-function [17]. Then solution is

$$\Omega(x) = \sum_{n=1}^{\infty} \frac{2K_n \sqrt{\pi} \cos K_n \frac{\pi}{2} \sin K_n x}{\sin K_n^2 L} I_{p+2, q_l+2; r}^{m+2, n}\left[z \,|_{B^{\cdot}}^{A^{\cdot}} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2K_n \sqrt{\pi} \sin K_n \frac{\pi}{2} \sin K_n \, x}{(K_n L - \sin K_n L \cos K_n L)} \Gamma_{p+2, q_l+2; r}^{m+2, n}\left[z \,|_{B^{\cdot}}^{A^{\cdot}} \right], \tag{3.3.4}$$

where $A^{\cdot} = (g_j, G_j)_{1, n}, (g_{jl}, G_{jl})_{n+1, p_l}, (\rho \pm K_n + 1/2, \sigma)$ and $B^{\cdot} = (\rho/2, \sigma), (\rho + 1/2, \sigma), (h_j, H_j)_{1, m}, (h_{jl}, H_{jl})_{m+1, ql}$, provided that each member in (3.3.4) exists. Similarly, we can find some other special cases of solution of wave equation (3.2.1).

4. Conclusions

In this paper, we have used new incomplete I-functions [1], which is an extension of Saxena's I-function [17]. Next, we gave certain integrals involving incomplete *I*-functions. Further, we find the solutions of heat and wave equations in terms of incomplete *I*-functions and also we obtained numerous special cases from our main result. The outcomes of this work are very helpful in the study of physics, engineering and applied sciences.

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