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INDEGREE PRIME LABELING OF SOME SPECIAL DIRECTED GRAPHS

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Abstract

Let $D(p, q)$ be a digraph. A function $f : V \to \{1, 2, 3, \ldots, p+q\}$ is said to be an in degree prime labeling of D if at each $v \in V(D)$, gcd $[f(u), f(v)] = 1 \; \forall \; \overrightarrow{uv} \in E(D)$. In this paper, we investigate in degree prime labeling of some special directed graphs. We prove that the directed graphs such as Instar $IK_{1,n}$ Inwheel IW_n, Upcomb $U_p(P_n \odot K_1)$ and Incrown $I(C_n \odot K_1)$ graphs admit Indegree prime labeling.

1. Introduction

All graphs in this paper are finite and directed. A finite graph is a graph $G(V, E)$ in which the vertex set and the edge set are finite sets. A directed graph $D(V, A)$ is an ordered pair of set of vertices (nodes) $V(D)$ and the set of arcs (directed edges) $A(D)$. Any arc (x, y) is a directed edge from x to y. The pair (x, y) is an ordered pair in which the first component (initial vertex) x is called the tail of the arc and the second component (terminal vertex) y is called the head of the arc. A digraph D is a Strict Digraph if it has no loops and no two arcs with the same end vertices. Throughout this paper we consider only strict digraphs. A digraph D with p vertices and q arcs is denoted by $D(p, q)$. The indegree $d^-(v)$ of a vertex v in a digraph D is the

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number of arcs having v as its terminal vertex. The outdegree $d^{+}(v)$ of v is the number of arcs having v as its initial vertex [6]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. The notion of a prime labeling originated with Entringer and was introduced in a paper by Hameeda Begum N. Tout, R. Rubina and M. I. Jawahar Nisha Dabboucy and Howalla. A graph with vertex set V is said to have a prime labeling of its vertices if its vertices are labeled with distinct integers $\{1, 2, 3, \ldots |V|\}$ such that for each edge xy, the labels assigned to x and y are relatively prime [4].

2. Preliminaries

For the following preliminary definitions we refer [1] and [5]

Definition 2.1. Let $D(p, q)$ be a digraph. A function $f: V \rightarrow \{1, 2, 3, \ldots, p+q\}$ is said to be an indegree prime labeling of D if at each $v \in V(D)$, gcd $[f(u), f(v)] = 1 \forall \overrightarrow{uv} \in E(D)$.

Definition 2.2. A star graph $K_{1,n}$ in which all the edges are directed towards the root vertex is called an Instar and is denoted as $IK_{1,n}$.

Definition 2.3. A wheel graph W_n in which the edges of the outer cycle are directed clockwise or anticlockwise and the spoke edges are directed towards the central vertex (Hub vertex) is called an In wheel and is denoted by IW_n or $I(C_n \odot K_{1,n})$.

Definition 2.4. A comb graph $P_n \odot K_1$ in which the path edges are directed in one direction and the pendent edges are oriented away from the end vertices is called an Upcomb and is denoted by $U_p(P_n \odot K_1)$.

Definition 2.5. A crown graph $C_n \odot K_1$ in which the edges of cycle are directed clockwise or anticlockwise and the pendent edges are directed towards the cycle is called an Incrown and is denoted by $I(C_n \odot K_1)$.

For the following remarks we refer [2]

Remark 2.6. 1 is relatively prime with all natural numbers.

Remark 2.7. 2 is relatively prime with all odd natural numbers.

Remark 2.8. Any two consecutive natural numbers is relatively prime.

Remark 2.9. Any two consecutive odd natural numbers is relatively prime.

3. Main Results

Theorem 3.1. *Instar* $\mathbb{K}_{1,n} (n \geq 3)$ *admits Indegree Prime Labeling.*

Proof. Let D be an Instar $\mathbb{K}_{1,n}$ with vertex set $V(D) = \{v, v_1, v_2, v_3, \ldots, v_n\}$, where v is the root vertex and the arc set be $E(D) = \{\overrightarrow{v_1 v}, \overrightarrow{v_2 v}, \overrightarrow{v_3 v}, \ldots, \overrightarrow{v_n v}\}$.

Then the Instar $\mathbb{K}_{1,n}$ is as in figure 3.1(a)

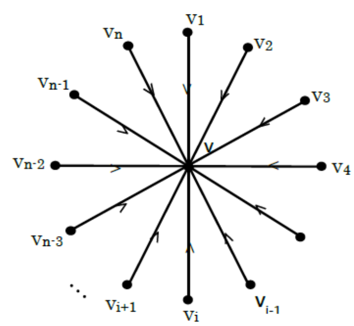

Figure 3.1(a). Instar $\mathbb{K}_{1,n}$.

Here $p = n + 1, q = n$. So, $p + q = 2n + 1$. We define a function $f : V \to \{1, 2, 3, \ldots, 2n + 1\}$ by $f(v) = 1$ and $f(v_i) = i + 1, 1 \leq i \leq n$. Indegree of v is n, $d^-(v) = n$. Indegree of v_i is zero, $d^-(v_i) = 0, \forall 1 \leq i \leq n$.

Then by remark 2.6, at $v \in V(D)$, $\gcd [f(v_i), f(v)] = 1, \forall \overrightarrow{v_i v} \in E(D)$, since $\gcd [i + 1, 1] = 1 \forall 1 \leq i \leq n$.

Therefore, In view of the above labeling pattern, it is evident that the Instar $\mathbb{K}_{1,n}$ admits an In degree Prime Labeling.

Illustration 3.1.1. Indegree Prime Labeling of $IK_{1,12}$ is shown in Figure 3.1(b)

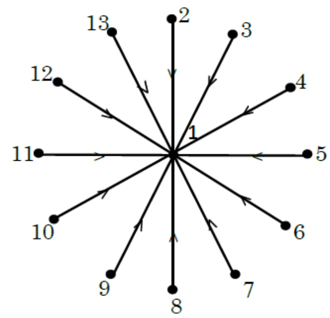

Figure 3.1(b). Instar $IK_{1,12}$.

Theorem 3.2. *In wheel $IW_n (n \geq 3)$ admits Indegree Prime Labeling.*

Proof. Let D be an Inwheel IW_n with the vertex set $V(D) = \{v, v_1, v_2, v_3, \ldots, v_n\}$, where v is the central vertex and the arc set be $E(D) = \{\overrightarrow{v_1 v}, \overrightarrow{v_2 v}, \overrightarrow{v_3 v}, \ldots, \overrightarrow{v_{n-1} v}, \overrightarrow{v_n v}\} \cup \{\overrightarrow{v_1 v_2}, \overrightarrow{v_2 v_3}, \overrightarrow{v_3 v_4}, \ldots, \overrightarrow{v_{n-1} v_n}, \overrightarrow{v_n v_1}\}$.

In general, $E(D) = \{\overrightarrow{v_i v} / 1 \leq i \leq n\} \cup \{\overrightarrow{v_i v_{i+1}} / 1 \leq i \leq n-1, \overrightarrow{v_n v_1}\}$.

Then the In wheel IW_n is as in figure 3.2(a)

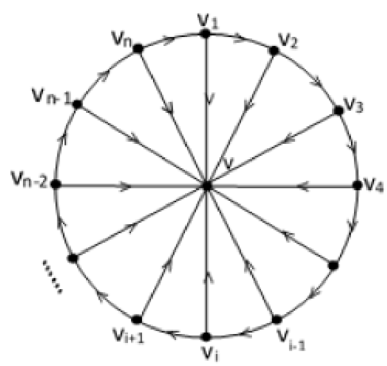

Figure 3.2(a). In wheel IW_n.

Here $p = n + 1, q = 2n$. So, $p + q = 3n + 1$. We define a function $f : V \to \{1, 2, 3, \ldots, 3n + 1\}$ by $f(v) = 2$ and $f(v_i) = 2i - 1, 1 \leq i \leq n$.

Indegree of v is n, $d^{-}(v) = n$.

Indegree of v_i is 1, $d^{-}(v_i) = 1, \forall 1 \leq i \leq n$.

Then at $v \in V(D)$, $\gcd\ [f(v_i), f(v)] = \gcd\ [2i - 1, 2] = 1, \forall \overrightarrow{v_i v} \in E(D)$, where $1 \leq i \leq n$. (by remark 2.7)

Also at $v_i \in V(D)$ for $2 \leq i \leq n$; $\gcd\ [f(v_{i-1}), f(v_i)] = \gcd\ [2i - 3, 2i - 1] = 1$, $\forall \overrightarrow{v_{i-1} v_1} \in E(D)$, where $2 \leq i \leq n$ (by remark 2.9).

Also at $v_1 \in V(D)$, $\gcd\ [f(v_n), f(v_1)] = \gcd\ [2n - 1, 1] = 1$, $\overrightarrow{v_n v_1} \in E(D)$, (by remark 2.6).

Therefore, in view of the above labeling pattern, it is evident that the Inwheel IW_n admits an Indegree Prime Labeling (irrespective of n being odd or even).

Illustration 3.2.1. Indegree Prime Labeling of IW_{13} is shown in Figure 3.2(b)

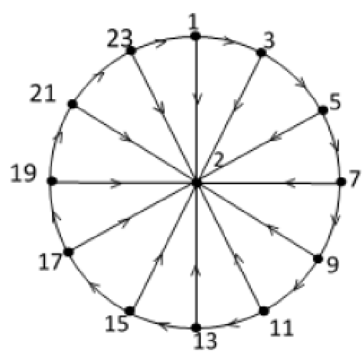

Figure 3.2(b). In wheel IW_{13}.

Theorem 3.3. *Upcomb* $U_p(P_n \odot K_1)$ *admits Indegree Prime Labeling.*

Proof. Let D be an Upcomb $U_p(P_n \odot K_1)$ with the vertex set $V(D) = \{v_1, v_2, v_3, \ldots, v_n, u_1, u_2, u_3, \ldots, u_n\}$, i.e., $V(D) = \{v_i, u_i / 1 \leq i \leq n\}$, where v_i represents the vertices of the path and u_i represents the pendent vertices corresponding to each v_i respectively and the arc set

$E(D) = \{\overrightarrow{v_1 v_2}, \overrightarrow{v_2 v_3}, \overrightarrow{v_3 v_4}, \ldots, \overrightarrow{v_{n-1} v_n}\} \cup \{\overrightarrow{u_1 v_1}, \overrightarrow{u_2 v_2}, \overrightarrow{u_3 v_3}, \ldots, \overrightarrow{u_{n-1} v_{n-1}}, \overrightarrow{u_n v_n}\}$

i.e., $E(D) = \{\overrightarrow{v_i v_{i+1}} / 1 \leq i \leq n-1\} \cup \{\overrightarrow{u_i v_i} / 1 \leq i \leq n\}$.

Then the Upcomb $U_p(P_n \odot K_1)$ is as in figure 3.3(a)

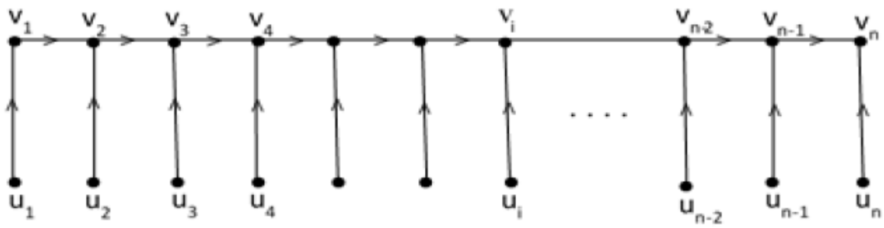

Figure 3.3(a). Upcomp $U_p(P_n \odot K_1)$.

Here $p = 2n$, $q = 2n-1$. So, $p + q = 4n - 1$. We define a function $f : V \to \{1, 2, 3, \ldots, 4n-1\}$ by $f(v_i) = 2i - 1, 1 \leq i \leq n$ and $f(u_i) = 2i$, $1 \leq i \leq n$. Indegree of u_i is 0, $d^-(u_i) = 0$. Indegree of v_1 is 1, $d^-(v_1) = 1$, since $\overrightarrow{u_1 v_1}$ is the only arc giving indegree to v_1. Indegree of v_i is 2, $d^-(v_i) = 2$; $\forall\ 2 \leq i \leq n$, since $\overrightarrow{v_{i-1} v_i}$ and $\overrightarrow{u_i v_i}$ for $2 \leq i \leq n$, are the only two arcs giving indegree to v_i.

Then at $v_1 \in V(D)$, $\gcd[f(u_1), f(v_1)] = \gcd[2, 1] = 1$, $\overrightarrow{u_1 v_1} \in E(D)$ (by remark 2.6).

Also at $v_i \in V(D)$ for $2 \leq i \leq n$, $\gcd[f(v_{i-1}), f(v_i)] = \gcd[2i-3, 2i-1] = 1$, $\forall\ \overrightarrow{v_{i-1} v_i} \in E(D)$, where $2 \leq i \leq n$. (by remark 2.9) and $\gcd[f(u_i), f(v_i)] = \gcd[2i, 2i-1] = 1$, $\forall\ \overrightarrow{u_i v_i} \in E(D)$, where $2 \leq i \leq n$ (by remark 2.8).

Therefore, in view of the above labeling pattern, it is evident that the Upcomb $U_p(P_n \odot K_1)$ admits an Indegree Prime Labeling.

Illustration 3.3.1. Indegree Prime Labeling of $U_p(P_{11} \odot K_1)$ is shown in Figure 3.3(b).

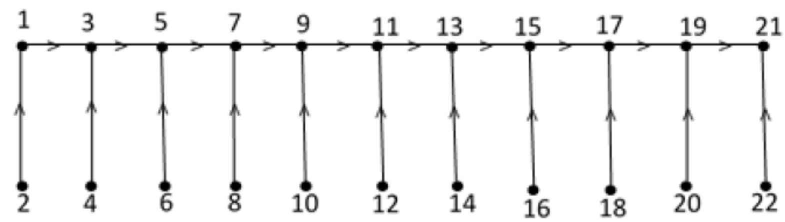

Figure 3.3(b). Upcomb $U_p(P_{11} \odot K_1)$.

Theorem 3.4. *Incrown $I(C_n \odot K_1)$ admits Indegree Prime Labeling.*

Proof. Let D be an Incrown $I(C_n \odot K_1)$ with the vertex set $V(D) = \{v_1, v_2, v_3, \ldots, v_n, u_1, u_2, u_3, u_n\}$, i.e., $V(D) = \{v_i, u_i / 1 \le i \le n\}$, where v_i represents the vertices of the cycle and u_i represents the pendent vertices corresponding to each v_i respectively and the arc set

$$E(D) = \{\overrightarrow{v_1 v_2}, \overrightarrow{v_2 v_3}, \overrightarrow{v_3 v_4}, \ldots, \overrightarrow{v_{n-1} v_n}, \overrightarrow{v_n v_1}\} \cup \{\overrightarrow{u_1 v_1}, \overrightarrow{u_2 v_2}, \overrightarrow{u_3 v_3}, \ldots, \overrightarrow{u_{n-1} v_{n-1}}, \overrightarrow{u_n v_n}\}.$$

i.e., $E(D) = \{\overrightarrow{v_i v_{i+1}} / 1 \le i \le n-1, \overrightarrow{v_n v_1}\} \cup \{\overrightarrow{u_i v_i} / 1 \le i \le n\}$.

Then the Incrown $I(C_n \odot K_1)$ is as in figure 3.4(a)

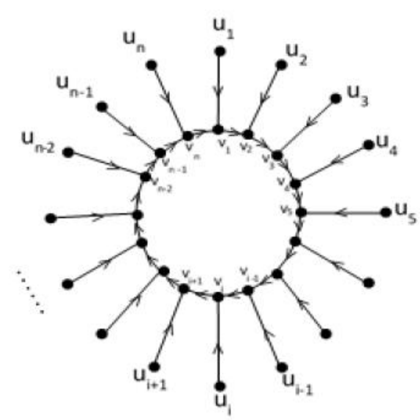

Figure 3.4(a). Incrown $I(C_n \odot K_1)$.

Here $p = 2n$, $q = 2n$. So, $p + q = 4n$. We define a function $f : V \to \{1, 2, 3, \ldots, 4n\}$ by $f(v_i) = 2i - 1$, $1 \le i \le n$ and $f(u_i) = 2i$, $1 \le i \le n$. Indegree of u_i is 0, $d^-(u_i) = 0$. Indegree of v_i is 2, $d^-(v_i) = 2$, $\forall\ 1 \le i \le n$.

Since $\overrightarrow{v_{i-1}v_i}$ and $\overrightarrow{u_i v_i}$ are the only two arcs giving indegree to v_i, for $2 \le i \le n$ and $\overrightarrow{v_n v_1}$ and $\overrightarrow{u_1 v_1}$ are the two arcs giving indegree to v_1.

Then at $v_1 \in V(D)$, $\gcd[f(v_n), f(v_1)] = \gcd[2n-1, 1] = 1, \forall \overrightarrow{v_n v_1} \in E(D)$ (by remark 2.6).

Also at $v_i \in V(D)$, $\gcd[f(u_i), f(v_i)] = \gcd[2i, 2i-1] = 1, \forall \overrightarrow{u_i v_i} \in E(D)$, where $2 \le i \le n$. (by remark 2.8) and at $v_i \in V(D)$, $\gcd[f(v_{i-1}), f(v_i)]$ $= \gcd[2i-3, 2i-1] = 1, \forall \overrightarrow{v_{i-1}v_i} \in E(D)$, (by remark 2.9).

Therefore, In view of the above labeling pattern, it is evident that the Incrown $I(C_n \odot K_1)$ admits an Indegree Prime Labeling.

Illustration 3.4.1. Indegree Prime Labeling of $I(C_{16} \odot K_1)$ is shown in Figure 3.4(b)

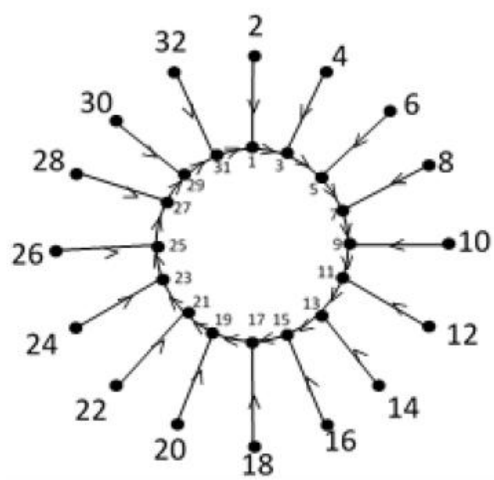

Figure 3.4(b). Incrown $I(C_{16} \odot K_1)$.

4. Conclusion

It is very interesting to investigate the directed graphs and its families which admit indegree prime labeling. In this paper we have proved that the Instar $IK_{1,n}$, Inwheel IW_n, Upcomb $U_p(P_n \odot K_1)$ and Incrown $I(C_n \odot K_1)$

graphs admit the Indegree prime labeling. To investigate similar results for other families of directed graphs is an open area of research.

References

[1] A. Catherine and K. Palani, Indegree Prime Labeling of Digraphs, International Journal of Science, Engineering and Management (IJSEM) 12(12) (2017), 358-361.

[2] https://byjus.com/maths/co-primenumbers

[3] J. Bondy and U. Murty, Graph Theory with Applications, North-Holland, New York (1976).

[4] Joseph A. Gallian, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combintorics (2018)#D56

[5] K. Palani and G. Suganya, Prime Pair Labeling of Directed Graphs, 2019 JETIR February 6(2) (2019).

[6] S. Arumugam and S. Ramachandran, Invitation to Graph Theory, SCITECH Publications (India) Pvt. Ltd.

[7] S. Ashok Kumar and S. Maragathavalli, Prime Labeling of Some Special Graphs, IOSR Journal of Mathematics (IOSR-JM) 11(1) (2015), 01-05.

[8] A. Tout, A. N. Dabboucy and K. Howalla, Prime Labeling of Graphs, National Academy Science Letters. 11 (1982), 365-368.