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ON PARTIAL SUMS OF MOCK THETA FUNCTIONS OF ORDER FIVE

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Abstract

In this paper, we investigate relations between fifth order mock theta functions and their partial sums. Further, we obtain mock theta representations for Roger-Ramanujan identities.

1. Introduction

In his last letter to G. H Hardy, Ramanujan introduced a list of seventeen functions called "Mock theta functions". He classified these functions into order third, fifth and seventh without assigning any reasons. All the results related to order third and fifth were proved by Watson [9, 10]. Andrews rediscovered Ramanujan's unpublished works (now published as the "Lost Notebook", [6]), has offered a new platform for further development in mock theta functions. Further several researches on mock theta functions in the lost notebook was proved by Andrews and Garvan [1], Andrews and Hickerson [2]. Two of the eight listed identities of tenth order mock theta functions are proved by Choi [4].

In [7], A. K. Srivastava defined partial mock theta functions of order three and five by taking a partial sum of the series-defining these functions. A perusal of literature shows that several mathematicians contributed to the domain of mock theta functions. Recently, M. Pathak and P. Srivastava [5]

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established identities for partial mock theta functions of order eight. Aim of this paper is to obtain relations between fifth order mock theta and their partial sum. We conclude the paper with exceptional cases of our main results, analogues to Roger-Ramanujan identities.

2. Notations

In this section, we will adopt the following definitions and notations.

For $|q| < 1$, q-shifted factorial is defined by,

$$(a)_n = \prod_{n=0}^{n-1} (1 - aq^n),\ n \geqslant 1,$$

$$(a)_\infty = \prod_{n=0}^{\infty} (1 - aq^n),$$

$$(a)_0 = 1,$$

$$(a_1, a_2, a_3, \ldots, a_r)_n = (a_1)_n (a_2)_n \ldots (a_r)_n.$$

Roger-Ramanujan identities are given by,

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q : q)_n} = \frac{1}{(q;\ q^5)_\infty (q^4;\ q^5)_\infty} \tag{2.1}$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q : q)_n} = \frac{1}{(q^2;\ q^5)_\infty (q^3;\ q^5)_\infty} \tag{2.2}$$

Following are the definitions and notation of mock theta functions and partial mock theta functions that we employ in our paper.

Ramanujan's fifth order mock theta functions, (Watson [9]):

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;\ q)_n}, \quad f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;\ q)_n},$$

$$\phi_0(q) = \sum_{n=0}^{\infty} q^{n^2} (-q,\ q^2), \quad \phi_1(q) = \sum_{n=0}^{\infty} q^{(n+1)^2} (-q,\ q^2)_n,$$

$$\psi_0(q) = \sum_{n=0}^{\infty} q^{\frac{(n+1)(n+2)}{2}} (-q, q)_n, \quad \psi_1(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} (-q, q)_n,$$

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q^2)_n}, \quad F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2(n+1)}}{(q; q^2)_{n+1}},$$

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n (q; q)_n}{(q; q)_{2n}}, \quad \chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n (q; q)_n}{(q; q)_{2n+1}}.$$

If $F(q) = \sum_{n=0}^{\infty} f(q, n)$ is mock theta function, then corresponding partial mock theta function is denoted by the truncated series $F_k(q) = \sum_{n=0}^{\infty} f(q, n)$.

3. Relation between Mock Theta Functions and Partial Mock Theta Functions

In this section, we obtain the relation between mock theta functions and partial mock theta functions using the following Lemma.

Lemma 3.1 [8, eqn. 4.1],

$$\sum_{\gamma=0}^{p} \alpha_\gamma \beta_\gamma = \beta_{p+1} \sum_{\gamma=0}^{p} \alpha_\gamma + \sum_{m=0}^{p} (\beta_m - \beta_{m+1}) \sum_{\gamma=0}^{m} \alpha_\gamma. \tag{3.1}$$

Theorem 3.1.

$$f_1(q) = (1-q) \sum_{m=0}^{\infty} q^m f_{0,m}(q), \tag{3.2}$$

$$= \frac{1}{(-q; q)_\infty} \left[\sum_{\gamma=0}^{\infty} q^{\gamma(\gamma+1)} + \sum_{m=0}^{\infty} q^{m+1} (-q; q)_m f_{1,m}(q) \right]. \tag{3.3}$$

Proof. Substitute $\alpha_\gamma = \frac{q^{\gamma^2}}{(-q; q)_\gamma}$ and $\beta_\gamma = q^\gamma$ in (3.1), we obtain

$$f_{1,\,q}(q) = q^{p+1} f_{0,\,p}(q) + (1-q)\sum_{m=0}^{p} q^{m} f_{0,\,m}(q). \tag{3. 4}$$

then by letting $p \to \infty$, we obtain (3.2).

Similarly by taking $\alpha_\gamma = \dfrac{q^{\gamma(\gamma+1)}}{(-q;\, q)_\gamma}$ and $\beta_\gamma = (-q;\, q)_\gamma$, we obtain

$$\sum_{\gamma=0}^{p} q^{\gamma(\gamma+1)} = (-q;\, q)_{p+1} f_{1,\,p}(q) - \sum_{m=0}^{p} q^{m+1}(-q;\, q)_m f_{1,\,m}(q). \tag{3. 5}$$

then by letting $p \to \infty$, we obtain (3.3).

Theorem 3.2.

$$f_0(q) = \frac{1}{(-q;\, q)_\infty}\left[\sum_{\gamma=0}^{\infty} q^{\gamma^2} + \sum_{m=0}^{\infty} q^{m+1}(-q;\, q)_m f_{0,\,m}(q)\right]. \tag{3.6}$$

Proof. Substitute $\alpha_\gamma = \dfrac{q^{\gamma^2}}{(-q;\, q)_\gamma}$ and $\beta_\gamma = (-q;\, q)_\gamma$ in (3.1), we obtain

$$\sum_{\gamma=0}^{p} q^{\gamma^2} = (-q;\, q)_{p+1} f_{0,\,p}(q) - \sum_{m=0}^{p} q^{m+1}(-q;\, q)_m f_{0,\,m}(q). \tag{3.7}$$

then by letting $p \to \infty$, we obtain (3.6).

Theorem 3.3.

$$\phi_1(q) = (1-q^2)\sum_{m=0}^{\infty} q^{2m+1}\phi_{0,\,m}(q), \tag{3.8}$$

$$= \frac{1}{(-q;\, q^2)_\infty}\left[\sum_{\gamma=0}^{\infty} q^{(\gamma+1)^2} - \sum_{m=0}^{\infty}\frac{q^{2m}}{(-q;\, q^2)_{m+1}}\,\phi_{1,\,m}(q)\right]. \tag{3.9}$$

Proof. Substitute $\alpha_\gamma = q^{\gamma^2}(-q;\, q^2)_\gamma$ and $\beta_\gamma = q^{2\gamma+1}$ in (3.1), we obtain

$$\phi_{1,\,p}(q) = q^{2p+3}\phi_{0,\,p}(q) + (1-q^2)\sum_{m=0}^{p} q^{2m+1}\phi_{0,\,m}(q). \tag{3.10}$$

the by letting $p \to \infty$, in (3.10), we obtain (3.8).

Similarly by taking $\alpha_\gamma = q^{(\gamma+1)^2}(-q;\,q^2)_\gamma$ and $\beta_\gamma = \frac{1}{(-q;\,q^2)_\gamma}$, we obtain

$$\sum_{\gamma=0}^{p} q^{(\gamma+1)^2} = \frac{1}{(-q;\,q^2)_{p+1}}\phi_{1,\,p}(q) + \sum_{m=0}^{p}\frac{q^{2m}}{(-q;\,q^2)_{m+1}}\phi_{1,\,m}(q). \tag{3.11}$$

then by letting $p \to \infty$, we obtain (3.9).

Theorem 3.4.

$$\phi_0(q) = (-q;\,q^2)_\infty\left[\sum_{\gamma=0}^{\infty} q^{\gamma^2} - \sum_{m=0}^{\infty}\frac{q^{2m}}{(-q;\,q^2)_{m+1}}\phi_{0,\,m}(q)\right]. \tag{3.12}$$

Proof. Substitute $\alpha_\gamma = q^{\gamma^2}(-q;\,q^2)_\gamma$ and $\beta_\gamma = \frac{1}{(-q;\,q^2)_\gamma}$ in (3.1), we obtain

$$\sum_{\gamma=0}^{p} q^{\gamma^2} = \frac{1}{(-q;\,q^2)_{p+1}}\phi_{0,\,p}(q) + \sum_{m=0}^{p}\frac{q^{2m}}{(-q;\,q^2)_{m+1}}\phi_{0,\,m}(q). \tag{3. 13}$$

then by letting $p \to \infty$, we obtain (3.12).

Theorem 3.5.

$$\psi_0(q) = (1-q)\sum_{m=0}^{\infty} q^m \psi_{1,\,m}(q), \tag{3.14}$$

$$= (-q;\,q)_\infty\left[\sum_{\gamma=0}^{\infty} q^{\frac{(\gamma+1)(\gamma+2)}{2}} - \sum_{m=0}^{\infty}\frac{q^{m+1}}{(-q;\,q^2)_{m+1}}\psi_{1,\,m}(q)\right]. \tag{3.15}$$

Proof. Substitute $\alpha_\gamma = q^{\frac{\gamma(\gamma+1)}{2}}(-q; q)_\gamma$ and $\beta_\gamma = q^{\gamma+1}$ in (3.1), we obtain

$$\psi_{0,\,p}(q) = q^{2p+3}\psi_{1,\,p}(q) + (1-q)\sum_{m=0}^{p} q^m \psi_{1,\,m}(q). \tag{3.16}$$

then by letting $p \to \infty$, in, we obtain (3.14).

Similarly by taking $\alpha_\gamma = q^{\frac{(\gamma+1)(\gamma+2)}{2}}(-q; q)_\gamma$ and $\beta_\gamma = \frac{1}{(-q; q)_\gamma}$, we obtain

$$\sum_{\gamma=0}^{p} q^{\frac{(\gamma+1)(\gamma+2)}{2}} = \frac{1}{(-q; q)_{p+1}}\psi_{0,\,p}(q) + \sum_{m=0}^{p}\frac{q^{m+1}}{(-q; q)_{m+1}}\psi_{0,\,m}(q). \tag{3.17}$$

then by letting $p \to \infty$, we obtain (3.15).

Theorem 3.6.

$$\psi_1(q) = (-q; q)_\infty\left[\sum_{\gamma=0}^{\infty} q^{\frac{\gamma(\gamma+1)}{2}} - \sum_{m=0}^{\infty}\frac{q^{m+1}}{(-q; q)_{m+1}}\psi_{1,\,m}(q)\right]. \tag{3.18}$$

Proof. Substitute $\alpha_\gamma = q^{\frac{\gamma(\gamma+1)}{2}}(-q; q)_\gamma$ and $\beta_\gamma = \frac{1}{(-q; q)_\gamma}$ in (3.1), we obtain

$$\sum_{\gamma=0}^{p} q^{\frac{\gamma(\gamma+1)}{2}} = \frac{1}{(-q; q)_{p+1}}\psi_{1,\,p}(q) + \sum_{m=0}^{p}\frac{q^{m+1}}{(-q; q)_{m+1}}\psi_{1,\,m}(q). \tag{3.19}$$

then by letting $p \to \infty$, in (3.19), we obtain (3.18).

Theorem 3.7.

$$F_0(q) = \frac{1}{(-q; q^2)_\infty}\left[\sum_{\gamma=0}^{\infty} q^{2\gamma^2} - \sum_{m=0}^{\infty} q^{2m}(q; q^2)_m F_{0,\,m}(q)\right]. \tag{3.20}$$

Proof. Substitute $\alpha_\gamma = \dfrac{q^{2\gamma^2}}{(q;\, q^2)_\gamma}$ and $\beta_\gamma = (q;\, q^2)_\gamma$ in (3.1), we obtain

$$\sum_{\gamma=0}^{p} q^{2\gamma^2} = (q;\, q^2)_{p+1} F_{0,\, p}(q) + \sum_{m=0}^{p} q^{2m} (q;\, q^2)_m F_{0,\, m}(q). \tag{3.21}$$

then by letting $p \to \infty$, we obtain (3.20).

Theorem 3.8.

$$F_1(q) = \frac{1}{(q;\, q^2)_\infty}\left[\sum_{\gamma=0}^{\infty} q^{2\gamma(\gamma+1)} - \sum_{m=0}^{\infty} q^{2m+3}(q;\, q^2)_{m+1} F_{1,\, m}(q)\right]. \tag{3.22}$$

$$= \sum_{m=0}^{\infty}\left(\frac{q^{2m}}{1+q^{2m}} - \frac{q^{2(m+1)}}{1+q^{2(m+1)}}\right) F_{0,\, m}(q). \tag{3.23}$$

Proof. Substitute $\alpha_\gamma = \dfrac{q^{2\gamma(\gamma+1)}}{(q;\, q^2)_{\gamma+1}}$ and $\beta_\gamma = (q;\, q^2)_{\gamma+1}$ in (3.1), we obtain

$$\sum_{\gamma=0}^{p} q^{2\gamma(\gamma+1)} = (q;\, q^2)_{p+1} F_{1,\, p}(q) + \sum_{m=0}^{p} q^{2m+3} (q;\, q^2)_{m+1} F_{1,\, m}(q). \tag{3.24}$$

then by letting $p \to \infty$ in we obtain (3.22).

Similarly by taking $\alpha_\gamma = \dfrac{q^{2\gamma^2}}{(q;\, q^2)_\gamma}$ and $\beta_\gamma = \dfrac{q^{2\gamma}}{1+q^{2\gamma+1}}$, we obtain

$$F_{1,\, p}(q) = \frac{q^{2(p+1)}}{1+q^{2(p+2)}} F_{0,\, p}(q) + \sum_{m=0}^{p}\left(\frac{q^{2m}}{1+q^{2m}} - \frac{q^{2(m+1)}}{1+q^{2(m+1)}}\right) F_{0,\, m}(q). \tag{3.25}$$

then by letting $p \to \infty$, we obtain (3.23).

Theorem 3.9.

$$\chi_0(q) = \sum_{\gamma=0}^{\infty} q^{\gamma} - \sum_{m=0}^{\infty}[(q^{m+1};\, q)_m - (q^{m+2};\, q)_{m+1}]\chi_{0,\, m}(q). \tag{3. 26}$$

Proof. Substitute $\alpha_\gamma = \frac{q^\gamma}{(q^{\gamma+1}; q)_\gamma}$ and $\beta_\gamma = (q^{\gamma+1}; q)_\gamma$ in (3.1), we obtain

$$\sum_{\gamma=0}^{\infty} q^\gamma = \chi_{0,\,p}(q) + \sum_{m=0}^{\infty} [(q^{m+1}; q)_m - (q^{m+2}; q)_{m+1}]\chi_{0,\,m}(q). \tag{3.27}$$

then by letting $p \to \infty$, we obtain (3.26).

Theorem 3.10.

$$\chi_1(q) = \sum_{\gamma=0}^{\infty} q^\gamma - \sum_{m=0}^{\infty} [(q^{m+1}; q)_{m+1} - (q^{m+2}; q)_{m+2}]\chi_{1,\,m}(q). \tag{3.28}$$

Proof. Substitute $\alpha_\gamma = \frac{q^\gamma}{(q^{\gamma+1}; q)_\gamma}$ and $\beta_\gamma = (q^{\gamma+1}; q)_{\gamma+1}$ in (3.1), we obtain

$$\sum_{\gamma=0}^{\infty} q^\gamma = \chi_{1,\,p}(q) + \sum_{m=0}^{\infty} [(q^{m+1}; q)_{m+1} - (q^{m+2}; q)_{m+2}]\chi_{1,\,m}(q). \tag{3.29}$$

then by letting $p \to \infty$, we obtain (3.28).

Theorem 3.11.

$$\chi_1(q) - \chi_0(q) = \sum_{m=0}^{\infty} \left(\frac{1}{1 + q^{2m}} - \frac{1}{1 + q^{2m+2}} \right) \chi_{0,\,m}(q). \tag{3.30}$$

Proof. Substitute $\alpha_\gamma = \frac{q^\gamma}{(q^{\gamma+1}; q)_\gamma}$ and $\beta_\gamma = \frac{1}{(1 + q^{2\gamma})}$ in (3.1), we obtain

$$\chi_1(q) = \chi_0(q) + \sum_{m=0}^{\infty} \left(\frac{1}{1 + q^{2m}} - \frac{1}{1 + q^{2m+2}} \right) \chi_{0,\,m}(q). \tag{3.31}$$

then by letting $p \to \infty$, we obtain (3.30).

4. Special Cases

In this section, we obtain mock theta representation for Roger-Ramanujan identities.

Theorem 4.1.

$$G(q) = \frac{(-q;\, q)_\infty f_0(q) - \sum_{m=0}^{\infty} q^{m+1}(-q;\, q)_m f_{0,\, m}(q)}{(q;\, q)_\infty} - \sum_{m=0}^{\infty} \frac{q^{m+1}}{(q;\, q)_{m+1}} \sum_{\gamma=0}^{m} q^{\gamma^2} \tag{4.1}$$

$$= \frac{\frac{1}{(-q;\, q^2)_\infty} \phi_0(q) + \sum_{m=0}^{\infty} \frac{q^{2m}}{(-q;\, q^2)_{m+1}} \phi_{0,\, m}(q)}{(q;\, q)_\infty} - \sum_{m=0}^{\infty} \frac{q^{m+1}}{(q;\, q)_{m+1}} \sum_{\gamma=0}^{m} q^{\gamma^2} \tag{4.2}$$

Proof. Substitute $\alpha_\gamma = q^{\gamma^2}$ and $\beta_\gamma = \frac{1}{(q;\, q)_\gamma}$ in (3.1), then letting $p \to \infty$, we obtain

$$\sum_{\gamma=0}^{\infty} \frac{q^{\gamma^2}}{(q;\, q)_\infty} = \frac{1}{(q;\, q)_\infty} \sum_{m=0}^{\infty} \frac{q^{m+1}}{(q;\, q)_{m+1}} \sum_{\gamma=0}^{m} q^{\gamma^2}. \tag{4.3}$$

Substituting (3.6) in (4.3) and (3.12) in (4.3), we obtain (4.1) and (4.2) respectively.

Theorem 4.2.

$$H(q) = \frac{(-q : q)_\infty f_1(q) - \sum_{m=0}^{\infty} q^{m+1}(-q;\, q)_m f_{1,\, m}(q)}{(q;\, q)_\infty} - \sum_{m=0}^{\infty} \frac{q^{m+1}}{(q;\, q)_{m+1}} \sum_{\gamma=0}^{m} q^{\gamma(\gamma+1)}. \tag{4.4}$$

Proof. Substitute $\alpha_\gamma = q^{\gamma(\gamma+1)}$ and $\beta_\gamma = \frac{1}{(q;\, q)_\gamma}$ in (3.1), then letting $p \to \infty$ we obtain

$$\sum_{\gamma=0}^{\infty} \frac{q^{\gamma(\gamma+1)}}{(q;\, q)_\infty} = \frac{1}{(q;\, q)_\infty} \sum_{\gamma=0}^{\infty} q^{\gamma(\gamma+1)} - \sum_{m=0}^{\infty} \frac{q^{m+1}}{(q;\, q)_{m+1}} \sum_{\gamma=0}^{m} q^{\gamma(\gamma+1)}. \tag{4.5}$$

Substitute (3.3) in (4.5), we obtain (4.4).

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