Advances and Applications in Mathematical Sciences
Volume 22, Issue 3, January 2023, Pages 751-767

RELATIONS ON BI-PERIODIC MERSENNE AND MERSENNE-LUCAS SEQUENCES

B. MALINI DEVI and S. DEVIBALA

Department of Mathematics
Sri Meenakshi Govt. Arts College for Women (A)
Madurai, India
E-mail: bmalini22.mdu@gmail.com
devibala27@yahoo.com

Abstract

In this paper, we define bi-periodic Mersenne sequence and bi-periodic Mersenne-Lucas sequence. The Binet's formula and generating functions for these sequences are given. Some of the identities like Cassini, Catalan and d'Ocagne and some related formulas are given.

Introduction

In [6, 7, 8], we can notice many different varieties of sequences whose applications take part in many fields of science and arts. In 1988, Horadam introduced Jacobsthal and Jacobsthal-Lucas sequences [5]. In [6], he demonstrated the properties of the same in detail. In [7], Khoshy elaborated Fibonacci and Lucas numbers. In [4, 10], Edson and Yayenie defined the bi-periodic Fibonacci sequences. In [1], Bilgici defined the bi-periodic Lucas sequences and also, he found some interesting properties of sequences [2, 3, 9], these results have motivated as to work on this area.

Now, in this paper we define bi-periodic Mersenne and bi-periodic Mersenne-Lucas sequences. We will then proceed to find the generating functions as well as Binet's formula. Some of the identities like Cassini, Catalan and d'Ocagne and some related formulas are given.

2010 Mathematics Subject Classification: 11B99.
Keywords: Bi-periodic Mersenne sequence, Bi-periodic Mersenne-Lucas sequence, Binet's formula.
Received February 26, 2022; Accepted June 6, 2022

Method of Analysis

The bi-periodic Mersenne sequence is defined recursively by

$$\mathfrak{m}_n = \begin{cases} 3a\mathfrak{m}_{n-1} - 2\mathfrak{m}_{n-2} & \text{if } n \text{ is even} \\ 3b\mathfrak{m}_{n-1} - 2\mathfrak{m}_{n-2} & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2$$

The first few elements of the bi-periodic Mersenne sequence are

$$\mathfrak{m}_0 = 0,\ \mathfrak{m}_1 = 1,\ \mathfrak{m}_2 = 3a,\ \mathfrak{m}_3 = 9ab - 2,\ \mathfrak{m}_4 = 27a^2b^2 - 12a,$$

$$\mathfrak{m}_5 = 81a^2b^2 54ab + 4,\ \mathfrak{m}_6 = 243a^3b^2 - 216a^2b + 36a,\ \ldots$$

The bi-periodic Mersenne-Lucas sequence is defined by

$$\mathfrak{M}_n = \begin{cases} 3b\mathfrak{M}_{n-1} - 2\mathfrak{M}_{n-2} & \text{if } n \text{ is even} \\ 3a\mathfrak{M}_{n-1} - 2\mathfrak{M}_{n-2} & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2$$

The first few elements of the bi-periodic Mersenne-Lucas sequence are

$$\mathfrak{M}_0 = 2,\ \mathfrak{M}_1 = 3a,\ \mathfrak{M}_2 = 9ab - 4,\ \mathfrak{M}_3 = 27a^2b - 18a,\ \mathfrak{M}_4 = 81a^2b^2$$

$$- 72ab + 8,\ \mathfrak{M}_5 = 243a^3b^2 - 270a^2b + 60a,\ \mathfrak{M}_6 = 729a^3b^3 - 972a^2b^2$$

$$+ 324ab + 16,\ \ldots$$

The recurrence equation for the bi-periodic Mersenne sequence and the bi-periodic Mersenne-Lucas sequence are given as

$$x^2 - 3abx + 2ab = 0$$

with roots $\alpha = \frac{3ab + \sqrt{9a^2b^2 - 8ab}}{2}$ and $\beta = \frac{3ab - \sqrt{9a^2b^2 - 8ab}}{2}$

The Binet's formula for the bi-periodic Mersenne sequence and the bi-periodic Mersenne-Lucas sequence are

$$\mathfrak{m}_n = \frac{a^{1-\epsilon(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$$

and

$$\mathfrak{M}_n = \frac{a^{\epsilon(n)}}{(ab)^{\left\lfloor \frac{n+1}{2} \right\rfloor}} (\alpha^n + \beta^n)$$

where $\lfloor a \rfloor$ is the floor function of a, and $\epsilon(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function.

The bi-periodic Mersenne sequence satisfies the following relations

$$\mathfrak{m}_{n+4} = (9ab - 4)\mathfrak{m}_{n+2} - 4\mathfrak{m}_n$$

$$\mathfrak{m}_n + 2\mathfrak{m}_{n-2} = 3a^{1-\epsilon(n)} b^{\epsilon(n)} \mathfrak{m}_{n-1}$$

Also, α and β satisfies the following properties

- $\alpha + \beta = 3ab$, $\alpha\beta = 2ab$
- $3\alpha - 2 = \frac{\alpha^2}{ab}$, $3\beta - 2 = \frac{\beta^2}{ab}$
- $(3\alpha - 2)(3\beta - 2) = 4$
- $2\alpha = (3\alpha - 2)\beta$, $2\beta = (3\beta - 2)\alpha$

The generating function for the bi-periodic Mersenne sequence is given by

$$\mathfrak{m}(x) = \frac{x + 3ax^2 + 2x^3}{1 - (9ab - 4)x^2 + 4x^4}$$

The generating function for the bi-periodic Mersenne-Lucas sequence is given by

$$\mathfrak{M}(x) = \frac{2 + 3ax - (9ab - 4)x^2 - 6ax^3}{1 - (9ab - 4)x^2 + 4x^4}$$

Main Results

Theorem 1. *For any integer* $n > 0$, *we have*

$$\mathfrak{m}_{2n}\mathfrak{M}_{2n} = \mathfrak{m}_{4n}.$$

Proof.

$$\mathfrak{m}_{2n}\mathfrak{M}_{2n} = \frac{a^{1-\epsilon(2n)}}{(ab)^{\left\lfloor \frac{2n}{2} \right\rfloor}}\left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)\frac{a^{\epsilon(2n)}}{(ab)^{\left\lfloor \frac{2n+1}{2} \right\rfloor}}(\alpha^{2n} + \beta^{2n})$$

$$= \frac{a^{-\epsilon(2n)+\epsilon(2n)}(\alpha^{4n} - \beta^{4n} - \alpha^{2n}\beta^{2n} + \alpha^{2n}\beta^{2n})}{(ab)^{\left\lfloor \frac{2n}{2} \right\rfloor + \left\lfloor \frac{2n+1}{2} \right\rfloor}(\alpha - \beta)}$$

$$= \frac{a^{1-\epsilon(4n)}(\alpha^{4n} - \beta^{4n})}{(ab)^{\left\lfloor \frac{4n}{2} \right\rfloor}(\alpha - \beta)} = \mathfrak{m}_{4n}.$$

Theorem 2. *For any integer $n > 0$, we have*

$$\mathfrak{m}_{2n}\mathfrak{M}_{2n+1} = \frac{\mathfrak{m}_{4n+1} - 2^{2n}}{ab}.$$

Proof.

$$\mathfrak{m}_{2n}\mathfrak{M}_{2n+1} = \frac{a^{1-\epsilon(2n)}}{(ab)^{\left\lfloor \frac{2n}{2} \right\rfloor}}\left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)\frac{a^{\epsilon(2n+1)}}{(ab)^{\left\lfloor \frac{2n+1}{2} \right\rfloor}}(\alpha^{2n+1} + \beta^{2n+1})$$

$$= \frac{a^{1-\epsilon(2n)+\epsilon(2n+1)}(\alpha^{4n+1} - \beta^{4n+1} - \alpha^{2n+1}\beta^{2n} + \alpha^{2n}\beta^{2n+1})}{(ab)^{\left\lfloor \frac{4n+2}{2} \right\rfloor}(\alpha - \beta)}$$

$$= \frac{a^{1-\epsilon(4n+1)}(\alpha^{4n+1} - \beta^{4n+1})}{(ab)(ab)^{\left\lfloor \frac{4n+1}{2} \right\rfloor}(\alpha - \beta)} - \frac{a^{1-\epsilon(2n)+\epsilon(2n+1)}\alpha^{2n} - \beta^{2n}(\alpha - \beta)}{(ab)^{(2n+1)}(\alpha - \beta)}$$

$$= \frac{\mathfrak{m}_{4n+1} - 2^{2n}}{ab}.$$

Theorem 3. *For any integer $n > 0$, we have*

$$\mathfrak{m}_{2n}\mathfrak{M}_{2n+2} = \mathfrak{m}_{4n+2} - 3a(2^{2n}).$$

Proof.

$$\mathfrak{m}_{2n}\mathfrak{M}_{2n+1} = \frac{a^{1-\epsilon(2n)}}{(ab)^{\left\lfloor \frac{2n}{2} \right\rfloor}}\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right)\frac{a^{\epsilon(2n+2)}}{(ab)^{\left\lfloor \frac{2n+3}{2} \right\rfloor}}(\alpha^{2n+2}+\beta^{2n+2})$$

$$= \frac{a^{1-\epsilon(2n)+\epsilon(2n+1)}(\alpha^{4n+2}-\beta^{4n+2}-\alpha^{2n+2}\beta^{2n}+\alpha^{2n}\beta^{2n+2})}{(ab)^{\left\lfloor \frac{2n}{2} \right\rfloor+\left\lfloor \frac{2n+3}{2} \right\rfloor}(\alpha-\beta)}$$

$$= \frac{a^{1-\epsilon(4n+2)}(\alpha^{4n+2}-\beta^{4n+2})}{(ab)^{\left\lfloor \frac{4n+3}{2} \right\rfloor}(\alpha-\beta)} - \frac{a\alpha^{2n}\beta^{2n}(\alpha^2-\beta^2)}{(ab)^{\left\lfloor \frac{4n+3}{2} \right\rfloor}(\alpha-\beta)}$$

$$= \mathfrak{m}_{4n+2} - \frac{a(2)^{2n}(3ab)}{ab}$$

$$= \mathfrak{m}_{4n+2} - 3a(2)^{2n}.$$

Theorem 4. *For any integer* $n > 0$, *we have*

$$\mathfrak{m}_{2n-1}\mathfrak{M}_{2n+1} = \mathfrak{m}_{4n} - 3a(2)^{2n-1}.$$

Proof.

$$\mathfrak{m}_{2n-1}\mathfrak{M}_{2n+1} = \frac{a^{1-\epsilon(2n-1)}}{(ab)^{\left\lfloor \frac{2n-1}{2} \right\rfloor}}\left(\frac{\alpha^{2n-1}-\beta^{2n-1}}{\alpha-\beta}\right)\frac{a^{\epsilon(2n+1)}}{(ab)^{\left\lfloor \frac{2n+2}{2} \right\rfloor}}(\alpha^{2n+1}+\beta^{2n+1})$$

$$= \frac{a^{1-\epsilon(2n-1)+\epsilon(2n+1)}(\alpha^{4n}-\beta^{4n}-\alpha^{2n+1}\beta^{2n-1}+\alpha^{2n-1}\beta^{2n+1})}{(ab)^{\left\lfloor \frac{4n+1}{2} \right\rfloor}(\alpha-\beta)}$$

$$= \frac{a^{1-\epsilon(4n)}(\alpha^{4n}-\beta^{4n})}{(ab)^{\left\lfloor \frac{4n}{2} \right\rfloor}(\alpha-\beta)} - \frac{a\alpha^{2n-1}\beta^{2n-1}(\alpha^2-\beta^2)}{(ab)^{2n}(\alpha-\beta)}$$

$$= \mathfrak{m}_{4n} - \frac{a(2)^{2n-1}(3ab)}{ab}$$

$$= \mathfrak{m}_{4n} - 3a(2)^{2n-1}.$$

Theorem 5. *For any integer* $n > 0$, *we have*

$$\mathfrak{m}_{2n}\mathfrak{M}_{2n} = \mathfrak{m}_{3n} - 2^n \mathfrak{m}_n.$$

Proof.

$$\mathfrak{m}_n\mathfrak{M}_{2n} = \frac{a^{1-\epsilon(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) \frac{a^{\epsilon(2n)}}{(ab)^{\left\lfloor \frac{2n+1}{2} \right\rfloor}} (\alpha^{2n} + \beta^{2n})$$

$$= \frac{a^{1-\epsilon(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{2n+1}{2} \right\rfloor}} \frac{(\alpha^{3n} - \beta^{3n} - \alpha^{2n}\beta^n + \alpha^n\beta^{2n})}{(\alpha - \beta)}$$

$$= \frac{a^{1-\epsilon(3n)}[\alpha^{3n} - \beta^{3n}]}{(ab)^{\left\lfloor \frac{3n}{2} \right\rfloor}(\alpha - \beta)} - \frac{a^{1-\epsilon(n)}(2ab)^n[\alpha^n - \beta^n]}{(ab)^{\left\lfloor \frac{3n}{2} \right\rfloor}(\alpha - \beta)}$$

$$= \mathfrak{m}_{3n} - 2^n \mathfrak{m}_n.$$

Theorem 6. *For any integer* $n > 0$, *we have*

$$\mathfrak{m}_{2n}\mathfrak{M}_n = \left(\frac{a}{b}\right)^{\epsilon(n)} \mathfrak{m}_{3n} - 2^n \left(\frac{a}{b}\right)^{\epsilon(n)} \mathfrak{m}_n.$$

Proof.

$$\mathfrak{m}_{2n}\mathfrak{M}_n = \frac{a^{1-\epsilon(2n)}}{(ab)^{\left\lfloor \frac{2n}{2} \right\rfloor}} \left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right) \frac{a^{\epsilon(n)}}{(ab)^{\left\lfloor \frac{n+1}{2} \right\rfloor}} (\alpha^n + \beta^n)$$

$$= \frac{a^{1-\epsilon(n)}[\alpha^{3n} - \beta^{3n} - \alpha^n\beta^n(\alpha^n - \beta^n)]}{(ab)^{\left\lfloor \frac{3n+1}{2} \right\rfloor}(\alpha - \beta)}.$$

If n is odd,

$$\mathfrak{m}_{2n}\mathfrak{M}_n = \frac{a^2[\alpha^{3n} - \beta^{3n} + \alpha^n\beta^n(\alpha^n - \beta^n)]}{(ab)^{\left\lfloor \frac{3n}{2} \right\rfloor + 1}(\alpha - \beta)}$$

$$= \frac{a(\alpha^{3n} - \beta^{3n})}{b(ab)^{\left\lfloor \frac{3n}{2} \right\rfloor}(\alpha - \beta)} + \frac{a(2ab)^n(\alpha^n - \beta^n)}{b(ab)^{\left\lfloor \frac{3n}{2} \right\rfloor}(\alpha - \beta)}$$

$$= \frac{aa^{1-\epsilon(3n)}(\alpha^{3n}-\beta^{3n})}{b(ab)^{\left\lfloor \frac{3n}{2}\right\rfloor}(\alpha-\beta)} + \frac{a(2)^n a^{1-\epsilon(n)}(\alpha^n-\beta^n)}{b(ab)^{\left\lfloor \frac{3n}{2}\right\rfloor - n}(\alpha-\beta)}$$

$$= \left(\frac{a}{b}\right)^{\epsilon(n)} \mathfrak{m}_{3n} - 2^n\left(\frac{a}{b}\right)^{\epsilon(n)} \mathfrak{m}_n.$$

If n is even,

$$\mathfrak{m}_{2n}\mathfrak{M}_n = \frac{a[\alpha^{3n}-\beta^{3n}+\alpha^n\beta^n(\alpha^n-\beta^n)]}{(ab)^{\left\lfloor \frac{3n}{2}\right\rfloor}(\alpha-\beta)}$$

$$= \frac{a(\alpha^{3n}-\beta^{3n})}{(ab)^{\left\lfloor \frac{3n}{2}\right\rfloor}(\alpha-\beta)} + \frac{a(2ab)^n(\alpha^n-\beta^n)}{(ab)^{\left\lfloor \frac{3n}{2}\right\rfloor}(\alpha-\beta)}$$

$$= \frac{a^{1-\epsilon(3n)}(\alpha^{3n}-\beta^{3n})}{(ab)^{\left\lfloor \frac{3n}{2}\right\rfloor}(\alpha-\beta)} + \frac{(2)^n a^{1-\epsilon(n)}(\alpha^n-\beta^n)}{(ab)^{\left\lfloor \frac{3n}{2}\right\rfloor}(\alpha-\beta)}$$

$$= \left(\frac{a}{b}\right)^{\epsilon(n)} \mathfrak{m}_{3n} - 2^n\left(\frac{a}{b}\right)^{\epsilon(n)} \mathfrak{m}_n.$$

Theorem 7. *Let m and n be any two positive integers, we have*

$$\mathfrak{m}_m\mathfrak{M}_{m+n} = \left(\frac{a}{b}\right)^{\epsilon(m+1)\epsilon(n)}(\mathfrak{m}_{2m+n} - 2^m\mathfrak{m}_n).$$

Proof.

$$\mathfrak{m}_m\mathfrak{M}_{m+n} = \frac{a^{1-\epsilon(m)}}{(ab)^{\left\lfloor \frac{m}{2}\right\rfloor}}\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)\frac{a^{\epsilon(m+n)}}{(ab)^{\left\lfloor \frac{n+n+1}{2}\right\rfloor}}(\alpha^{m+n}+\beta^{n+n})$$

$$= \frac{a^{1-\epsilon(m)+\epsilon(m+n)}}{(ab)^{\left\lfloor \frac{m}{2}\right\rfloor + \left\lfloor \frac{m+n+1}{2}\right\rfloor}}\frac{\alpha^{2m+n}-\beta^{2m+n}-\alpha^m\beta^m(\alpha^n+\beta^n)}{(\alpha-\beta)}.$$

If m and n are even, $m+n$ is even,

$$\mathfrak{m}_m\mathfrak{M}_{m+n} = \frac{a}{(ab)^{\left\lfloor \frac{2m+n}{2} \right\rfloor}} \frac{\alpha^{2m+n} - \beta^{2m+n}}{(\alpha - \beta)} - \frac{a(2)^m}{(ab)^{\left\lfloor \frac{2m+n}{2} \right\rfloor - m}} \frac{\alpha^n - \beta^n}{(\alpha - \beta)}$$

$$= \frac{a^{1-\epsilon(2m+n)}}{(ab)^{\left\lfloor \frac{2m+n}{2} \right\rfloor}} \frac{\alpha^{2m+n} - \beta^{2m+n}}{(\alpha - \beta)} - \frac{a^{1-\epsilon(n)}(2)^m}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \frac{\alpha^n - \beta^n}{(\alpha - \beta)}$$

$$= \mathfrak{m}_{2m+n} - 2^m \mathfrak{m}_n.$$

If m and n are odd, $m+n$ is even,

$$\mathfrak{m}_m\mathfrak{M}_{m+n} = \frac{a}{(ab)^{\left\lfloor \frac{2m+n}{2} \right\rfloor}} \frac{\alpha^{2m+n} - \beta^{2m+n}}{(\alpha - \beta)} - \frac{(2)^m}{(ab)^{\left\lfloor \frac{2m+n}{2} \right\rfloor - m}} \frac{\alpha^n - \beta^n}{(\alpha - \beta)}$$

$$= \frac{a^{1-\epsilon(2m+n)}}{(ab)^{\left\lfloor \frac{2m+n}{2} \right\rfloor}} \frac{\alpha^{2m+n} - \beta^{2m+n}}{(\alpha - \beta)} - \frac{a^{1-\epsilon(n)}(2)^m}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \frac{\alpha^n - \beta^n}{(\alpha - \beta)}$$

$$= \mathfrak{m}_{2m+n} - 2^m \mathfrak{m}_n.$$

If m is odd and n is even then $m+n$ is odd,

$$\mathfrak{m}_m\mathfrak{M}_{m+n} = \frac{a}{(ab)^{\left\lfloor \frac{2m+n}{2} \right\rfloor}} \frac{\alpha^{2m+n} - \beta^{2m+n}}{(\alpha - \beta)} - \frac{(2)^m}{(ab)^{\left\lfloor \frac{2m+n}{2} \right\rfloor - m}} \frac{\alpha^n - \beta^n}{(\alpha - \beta)}$$

$$= \frac{a^{1-\epsilon(2m+n)}}{(ab)^{\left\lfloor \frac{2m+n}{2} \right\rfloor}} \frac{\alpha^{2m+n} - \beta^{2m+n}}{(\alpha - \beta)} - \frac{a^{1-\epsilon(n)}(2)^m}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \frac{\alpha^n - \beta^n}{(\alpha - \beta)}$$

$$= \mathfrak{m}_{2m+n} - 2^m \mathfrak{m}_n.$$

Finally if m is even and n is odd then $m+n$ is odd,

$$\mathfrak{m}_m\mathfrak{M}_{m+n} = \frac{a^2}{(ab)^{\left\lfloor \frac{2m+n}{2} \right\rfloor + 1}} \frac{\alpha^{2m+n} - \beta^{2m+n}}{(\alpha - \beta)} - \frac{a^2(\alpha\beta)^m}{(ab)^{\left\lfloor \frac{2m+n}{2} \right\rfloor + 1}} \frac{\alpha^n - \beta^n}{(\alpha - \beta)}$$

$$= \frac{a}{b}\frac{a^{1-\epsilon(2m+n)}}{(ab)^{\left\lfloor \frac{2m+n}{2} \right\rfloor}}\frac{\alpha^{2m+n}-\beta^{2m+n}}{(\alpha-\beta)} - \frac{a}{b}\frac{a^{1-\epsilon(n)}(2)^m}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}}\frac{\alpha^n-\beta^n}{(\alpha-\beta)}$$

$$= \frac{a}{b}\mathfrak{m}_{2m+n} - (2^m)\frac{a}{b}\mathfrak{m}_n.$$

Hence, $\mathfrak{m}_n\mathfrak{M}_{m+n} = \left(\frac{a}{b}\right)^{\epsilon(m+1)\epsilon(n)}(\mathfrak{m}_{2m+n} - 2^m\mathfrak{m}_n)$

Theorem 8. *For any integer* $n > 0$, *we have*

$$(\alpha-\beta)^2\mathfrak{m}_{2n+3}\mathfrak{m}_{2n-3} = \mathfrak{M}_{4n} - (2)^{2n-3}\mathfrak{M}_6.$$

Proof.

$$\mathfrak{m}_{2n+3}\mathfrak{M}_{2n-3} = \frac{a^{1-\epsilon(2n+3)}}{(ab)^{\left\lfloor \frac{2n+3}{2} \right\rfloor}}\left(\frac{\alpha^{2n+3}-\beta^{2n+3}}{\alpha-\beta}\right) - \frac{a^{1-\epsilon(2n+3)}}{(ab)^{\left\lfloor \frac{2n-3}{2} \right\rfloor}}\left(\frac{\alpha^{2n-3}-\beta^{2n-3}}{\alpha-\beta}\right)$$

$$= \frac{1}{(ab)^{\left\lfloor \frac{2n+3}{2} \right\rfloor + \left\lfloor \frac{2n-3}{2} \right\rfloor}}\frac{(\alpha^{4n}-\beta^{4n}-\alpha^{2n-3}\beta^{2n+3}-\alpha^{2n+3}+\beta^{2n-3})}{(\alpha-\beta)^2}.$$

Hence, $(\alpha-\beta)^2\mathfrak{m}_{2n+3}\mathfrak{m}_{2n-3} = \dfrac{\alpha^{4n}+\beta^{4n}-\alpha^{2n-3}\beta^{2n+3}(\alpha^6+\beta^6)}{(ab)^{2n}}$

$$= \frac{a^{\epsilon(4n)}}{(ab)^{\left\lfloor \frac{4n+1}{2} \right\rfloor}}(\alpha^{4n}-\beta^{4n})\frac{(2ab)^{2n-3}a^{\epsilon(6)}(\alpha^6-\beta^6)}{(ab)^{2n}}$$

$$= \mathfrak{M}_{4n} - (2)^{2n-3}\mathfrak{M}_6.$$

Theorem 9. *For any integer* $n > 0$, *we have*

$$2\mathfrak{m}_{2n+1} = 3b\mathfrak{m}_{n+2}\mathfrak{M}_n - \left(\frac{b}{a}\right)^{\epsilon(n)}\mathfrak{m}_{n+1}\mathfrak{M}_{n+2} - 2^n(9ab+2).$$

Proof. $\left(\frac{b}{a}\right)^{\epsilon(n)} m_{n+1}M_{n+2}$

$$= \left(\frac{b}{a}\right)^{\epsilon(n)} \left[\frac{a^{1-\epsilon(n+1)}}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}} \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right)\right]\left[\frac{a^{\epsilon(n+2)}}{(ab)^{\left\lfloor\frac{n+3}{2}\right\rfloor}} (\alpha^{n+2} + \beta^{n+2})\right]$$

$$= \left(\frac{b}{a}\right)^{\epsilon(n)} \frac{a^{2\epsilon(n)}}{(ab)^{2\left\lfloor\frac{n+1}{2}\right\rfloor+1}} \frac{\alpha^{2n+3} - (\alpha\beta)^{n+1}(\alpha - \beta) - \beta^{2n+3}}{\alpha - \beta}$$

$$= \frac{(ab)^{\epsilon(n)}}{(ab)^{n+1+\epsilon(n)}} \frac{\alpha^{2n+3} - \beta^{2n+3}}{\alpha - \beta} - \frac{(ab)^{\epsilon(n)}(\alpha\beta)^{n+1}}{(ab)^{n+1+\epsilon(n)}}$$

$$= \frac{a^{1-\epsilon(2n+3)}}{(ab)^{\left\lfloor\frac{2n+3}{2}\right\rfloor}} \left(\frac{\alpha^{2n+3} - \beta^{2n+3}}{\alpha - \beta}\right) - 2^{n+1}$$

$$= \mathfrak{m}_{2n+3} - 2^{n+1}$$

$$3b\mathfrak{m}_{n+2}\mathfrak{M}_n = 3b\left[\frac{a^{1-\epsilon(n+2)}}{(ab)^{\left\lfloor\frac{n+2}{2}\right\rfloor}} \left(\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\right)\right]\left[\frac{a^{\epsilon(n)}}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}} (\alpha^{n} + \beta^{n})\right]$$

$$= \frac{3ab}{(ab)^{n+1}}\left[\frac{\alpha^{2n+2} - (\alpha\beta)^{n}(\alpha^2 - \beta^2) - \beta^{2n+2}}{\alpha - \beta}\right]$$

$$= \frac{3a^{1-\epsilon(n+2)}b}{(ab)^{n+1}}\left(\frac{\alpha^{2n+2} - \beta^{2n+2}}{\alpha - \beta}\right) + \frac{3ab(2ab)^{n}(\alpha + \beta)(\alpha - \beta)}{(ab)^{n+1}(\alpha - \beta)}$$

$$= 3b\mathfrak{m}_{2n+2} + 2^{n}(9ab)$$

$$\therefore \left(\frac{b}{a}\right)^{\epsilon(n)} \mathfrak{m}_{n+1}\mathfrak{M}_{n+2} - 3b\mathfrak{m}_{n+2}\mathfrak{M}_n + 2^{n}(9ab + 2)$$

$$= \mathfrak{m}_{2n+3} - 3b\mathfrak{m}_{2n+2} = -2\mathfrak{m}_{2n+1}.$$

Hence $2\mathfrak{m}_{2n+1} = 3b\mathfrak{m}_{n+2}\mathfrak{M}_n - \left(\frac{b}{a}\right)^{\epsilon(n)} \mathfrak{m}_{n+1}\mathfrak{M}_{n+2} - 2^{n}(9ab + 2).$

Theorem 10. *For any nonnegative integer n, we have*

$$\mathfrak{m}_{n+6} = 3(\mathfrak{M}_2 - 2)a^{1-\epsilon(n)}b^{\epsilon(n)}\mathfrak{m}_{n+3} - 8\mathfrak{m}_n.$$

Proof.

$$\mathfrak{m}_{n+6} = (9ab - 4)\mathfrak{m}_{n+4} - 4\mathfrak{m}_{n+2}$$

$$= (9ab - 4)(3a^{1-\epsilon(n)}b^{\epsilon(n)}\mathfrak{m}_{n+3} - 2\mathfrak{m}_{n+2}) - 4\mathfrak{m}_{n+2}$$

$$= 3(9ab - 4)a^{1-\epsilon(n)}b^{\epsilon(n)}\mathfrak{m}_{n+3} - 2(9ab - 4)\mathfrak{m}_{n+2} - 4\mathfrak{m}_{n+2}$$

$$= 3(9ab - 6)a^{1-\epsilon(n)}b^{\epsilon(n)}\mathfrak{m}_{n+3} - (18ab - 4)\mathfrak{m}_{n+2} + 6a^{1-\epsilon(n)}b^{\epsilon(n)}\mathfrak{m}_{n+3}$$

$$= 3(9ab - 6)a^{1-\epsilon(n)}b^{\epsilon(n)}\mathfrak{m}_{n+3} - (18ab - 4)\mathfrak{m}_{n+2}$$

$$+ 6a^{1-\epsilon(n)}b^{\epsilon(n)}(3a^{1-\epsilon(n+3)}b^{\epsilon(n+3)}\mathfrak{m}_{n+2} - 2\mathfrak{m}_{n+1})$$

$$= 3(9ab - 6)a^{1-\epsilon(n)}b^{\epsilon(n)}\mathfrak{m}_{n+3} - 18ab\mathfrak{m}_{n+2} + 18a^{2-\epsilon(n)-\epsilon(n+3)}b^{\epsilon(n)+\epsilon(n+3)}\mathfrak{m}_{n+2}$$

$$-8\mathfrak{m}_n$$

$$= 3(\mathfrak{M}_2 - 2)a^{1-\epsilon(n)}b^{\epsilon(n)}\mathfrak{m}_{n+3} - 8\mathfrak{m}_n.$$

Theorem 11. *For any integer* $n > 2,$ *we have*

$$\mathfrak{m}_n = \frac{a^{\epsilon(n+1)}}{2^{n+1}} \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} 3^{m-2j-1}(ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - j}(\mathfrak{M}_2 - 4)^j.$$

Proof.

Since $2\alpha = 3ab + \sqrt{ab(9ab - 8)}$ and $2\beta = 3ab - \sqrt{ab(9ab - 8)}$

We get $(2\alpha)^n = (3ab + \sqrt{ab(9ab - 8)})^n$

$$= \sum_{k=0}^{n} \binom{n}{k} (ab)^{n-\frac{k}{2}}(3)^{n-k}(9ab - 8)^{\frac{k}{2}}$$

$$(2\beta)^n = (3ab - \sqrt{ab(9ab - 8)})^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} (ab)^{n-\frac{k}{2}} (3)^{n-k} (-1)^k (9ab - 8)^{\frac{k}{2}}.$$

Therefore, we obtain

$$(2\alpha)^n - (2\beta)^n = \sum_{k=0}^{n} \binom{n}{k} (ab)^{n-\frac{k}{2}} (3)^{n-k} (9ab - 8)^{\frac{k}{2}} [1 - (-1)^k]$$

$$2^n (\alpha^n - \beta^n) = \left(\frac{\alpha - \beta}{2^{n-1}}\right) \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} (ab)^{n-j-1} (3)^{n-2j-1} (9ab - 8)^j.$$

By using Binet's formula for bi-periodic Mersenne sequence

$$m_n = \frac{a^{1-\epsilon(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$$

$$= \left(\frac{a^{1-\epsilon(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}}\right) \left(\frac{1}{2^{n-1}}\right) \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} (3)^{n-2j-1} (ab)^{n-j-1} (\mathfrak{M}_2 - 4)^j$$

$$= \frac{a^{\epsilon(n+1)}}{2^{n-1}} \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} (3)^{n-2j-1} (ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - j} (\mathfrak{M}_2 - 4)^j.$$

Theorem 12 (Catalan's Identity)**.** *For every positive integers n and r, with $n \geq r$, we have*

$$a^{\epsilon(n+r)} b^{1-\epsilon(n-r)} \mathfrak{m}_{n-r} \mathfrak{m}_{n+r} - a^{\epsilon(n)} b^{1-\epsilon(n)} \mathfrak{m}_n^2 = -2^{n-r} a^{\epsilon(r)} b^{1-\epsilon(r)} \mathfrak{m}_r^2$$

$$\left(\frac{b}{a}\right)^{\epsilon(n+r)} \mathfrak{M}_{n-r} \mathfrak{M}_{n+r} - \left(\frac{a}{b}\right)^{\epsilon(n)} \mathfrak{M}_n^2 = \frac{2^{n-r}}{(ab)^r} (\alpha^r - \beta^r)^2.$$

Proof. By using Binet's formula for bi-periodic Mersenne sequence

$$a^{\epsilon(n-r)} b^{1-\epsilon(n-r)} \mathfrak{m}_{n-r} \mathfrak{m}_{n+r}$$

$$= a^{\epsilon(n-r)}b^{1-\epsilon(n-r)}\left(\frac{a^{1-\epsilon(n-r)}}{(ab)^{\left\lfloor\frac{n-r}{2}\right\rfloor}}\right)\left(\frac{a^{1-\epsilon(n+r)}}{(ab)^{\left\lfloor\frac{n+r}{2}\right\rfloor}}\right)\left(\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\right)\left(\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}\right)$$

$$= \frac{a^{2-\epsilon(n-r)}b^{1-\epsilon(n-r)}}{(ab)^{n-\epsilon(n-r)}}\left[\frac{\alpha^{2n}-\alpha^{n+r}\beta^{n-r}-\alpha^{n-r}\beta^{n+r}+\beta^{2n}}{(\alpha-\beta)^2}\right]$$

$$= \frac{a}{(ab)^{n-1}}\left[\frac{\alpha^{2n}-(\alpha\beta)^{n-r}(\alpha^{2r}+\beta^{2r})+\beta^{2n}}{(\alpha-\beta)^2}\right]$$

$$a^{\epsilon(n)}b^{1-\epsilon(n)}m_n^2 = a^{\epsilon(n)}b^{1-\epsilon(n)}\left[\left(\frac{a^{1-\epsilon(n)}}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\right)\left(\frac{\alpha^n-\beta^\alpha}{\alpha-\beta}\right)\right]^2$$

$$= a^{\epsilon(n)}b^{1-\epsilon(n)}\left(\frac{a^{2-2\epsilon(n)}}{(ab)^{2\left\lfloor\frac{n}{2}\right\rfloor}}\right)\left[\frac{\alpha^{2n}-2(\alpha\beta)^n+\beta^{2n}}{(\alpha-\beta)^2}\right]$$

$$= \frac{a}{(ab)^{n-1}}\left[\frac{\alpha^{2n}-2(\alpha\beta)^n+\beta^{2n}}{(\alpha-\beta)^2}\right]$$

$$a^{\epsilon(n-r)}b^{1-\epsilon(n-r)}\mathfrak{m}_{n-r}\mathfrak{m}_{n+r} - a^{\epsilon(n)}b^{1-\epsilon(n)}\mathfrak{m}_n^2$$

$$= \frac{a}{(ab)^{n-1}}\left[\frac{\alpha^{2n}-(\alpha\beta)^{n-r}(\alpha^{2r}+\beta^{2r})+\beta^{2n}}{(\alpha-\beta)^2}\right] - \frac{a}{(ab)^{n-1}}\left[\frac{\alpha^{2n}-2(\alpha\beta)^n+\beta^{2n}}{(\alpha-\beta)^2}\right]$$

$$= \left[\frac{a}{(ab)^{n-1}}\right]\left[\frac{(\alpha\beta)^{n-r}(-\alpha^{2r}-\beta^{2r}+2(\alpha\beta)^r)}{(\alpha-\beta)^2}\right]$$

$$= \left[\frac{-a}{(ab)^{n-1}}\right](2ab)^{n-r}\frac{(ab)^{2\left\lfloor\frac{r}{2}\right\rfloor}}{a^{2-2\epsilon(r)}}\mathfrak{m}_r^2$$

$$= -2^{n-r}a^{-1+2\epsilon(r)}(ab)^{-n}(ab)^{n-r}(ab)^{2\left\lfloor\frac{r}{2}\right\rfloor}\mathfrak{m}_r^2$$

$$= -2^{n-r}a^{-1+2\epsilon(r)}(ab)^{1-\epsilon(r)}\mathfrak{m}_r^2$$

$$= -2^{n-r}a^{\epsilon(r)}b^{1-\epsilon(r)}\mathfrak{m}_r^2.$$

Similarly, by using Binet's formula for bi-periodic Mersenne-Lucas sequence we obtain

$$\left(\frac{b}{a}\right)^{\epsilon(n+r)}\mathfrak{M}_{n-r}\mathfrak{M}_{n+r} = (ab)^{-n}(\alpha^{2n} + \beta^{2n} + \alpha^{n+r}\beta^{n-r} + \alpha^{n-r}\beta^{n+r})$$

$$\left(\frac{b}{a}\right)^{\epsilon(n)}\mathfrak{m}_n^2 = (ab)^{-n}[\alpha^{2n} + 2(\alpha\beta)^n + \beta^{2n}]$$

$$\left(\frac{b}{a}\right)^{\epsilon(n+r)}\mathfrak{M}_{n-r}\mathfrak{M}_{n+r} - \left(\frac{b}{a}\right)^{\epsilon(n)}\mathfrak{m}_n^2$$

$$= (ab)^{-n}[\alpha^{2n} + \beta^{2n} + \alpha^{n+r}\beta^{n-r} + \alpha^{n-r}\beta^{n+r} - \alpha^{2n} - 2(\alpha\beta)^n - \beta^{2n}]$$

$$= (ab)^{-n}(2ab)^n\left[\frac{\beta^r}{\alpha^r} + \frac{\alpha^r}{\beta^r} - 2\right]$$

$$= \frac{2^{n-r}}{(ab)^r}[\alpha^r - \beta^r]^2.$$

Theorem 13 (Cassini's Property)**.** *For every positive integer n, we have*

$$\left(\frac{b}{a}\right)^{\epsilon(n-1)}\mathfrak{m}_{n-1}\mathfrak{m}_{n+1} - \left(\frac{a}{b}\right)^{\epsilon(n)}\mathfrak{m}_n^2 = -\left(\frac{a}{b}\right)2^{n-1}$$

$$\left(\frac{b}{a}\right)^{\epsilon(n+1)}\mathfrak{M}_{n-1}\mathfrak{M}_{n+1} - \left(\frac{a}{b}\right)^{\epsilon(n)}\mathfrak{M}_n^2 = 2^{n-1}(9ab - 8).$$

Proof. The proof of Cassini's Property is obtained by substituting $r = 1$ in Catalan's identity.

Theorem 14 (d'Ocagne's property)**.** *For any positive integer m and n, with $m \geq n$,*

$$a^{\epsilon(mn+m)}b^{\epsilon(mn+n)}\mathfrak{m}_m\mathfrak{m}_{n+1} - a^{\epsilon(mn+n)}b^{\epsilon(mn+m)}\mathfrak{m}_{m+1}\mathfrak{m}_n = 2^n a^{\epsilon(m-n)}\mathfrak{m}_{m-n}$$

$$a^{\epsilon(mn+m)}b^{\epsilon(mn+n)}\mathfrak{M}_{m+1}\mathfrak{M}_n - a^{\epsilon(mn+n)}b^{\epsilon(mn+m)}\mathfrak{M}_m\mathfrak{M}_{n+1}$$

$$= 2^n a^{\epsilon(m-n)}(9ab - 8)\mathfrak{M}_{m-n}.$$

Proof. We have the following equalities

- $\epsilon(m)+\epsilon(n+1)-2\epsilon(mn+m)=\epsilon(m+1)+\epsilon(n)-2\epsilon(mn+n)$
  $=1-\epsilon(m-n)$
- $\epsilon(m-n)=\epsilon(mn+m)+\epsilon(mn+n)$
- $\dfrac{m-n-\epsilon(m-n)}{2}=\left\lfloor\dfrac{m}{2}\right\rfloor+\left\lfloor\dfrac{n+1}{2}\right\rfloor-\epsilon(mn+n)-n$
- $\dfrac{m-n-\epsilon(m-n)}{2}=\left\lfloor\dfrac{m+1}{2}\right\rfloor+\left\lfloor\dfrac{n}{2}\right\rfloor-\epsilon(mn+n)-n$
- $\dfrac{m-n-\epsilon(m-n)}{2}=\left\lfloor\dfrac{m-n}{2}\right\rfloor$

By using the Binet's formula for bi-periodic Mersenne sequence and the above equalities we have

$$a^{\epsilon(mn+m)}b^{\epsilon(mn+n)}\mathfrak{m}_n\mathfrak{m}_{n+1}$$

$$=a^{\epsilon(mn+m)}b^{\epsilon(mn+n)}\left(\frac{a^{1-\epsilon(m)}}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor}}\right)\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)\left(\frac{a^{1-\epsilon(n+1)}}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\right)\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)$$

$$=\frac{ab^{\epsilon(mn+n)}a^{\epsilon(mn+n)}}{(ab)^{\frac{m-n-\epsilon(m-n)}{2}+\epsilon(mn+n)+n}}\left(\frac{\alpha^{m+n+1}+\beta^{m+n+1}-\beta^m\alpha^{n+1}-\alpha^m\beta^{n+1}}{(\alpha-\beta)^2}\right)$$

$$=\frac{a(ab)^{-n}}{(ab)^{\frac{m-n-\epsilon(m-n)}{2}}}\left(\frac{\alpha^{m+n+1}+\beta^{m+n+1}-(\alpha\beta)^n(\alpha\beta^{m-n}+\alpha^{m-n}\beta)}{(\alpha-\beta)^2}\right)$$

$$=\frac{a(ab)^{-n}}{(ab)^{\left\lfloor\frac{m-n}{2}\right\rfloor}}\left(\frac{\alpha^{m+n+1}+\beta^{m+n+1}-(\alpha\beta)^n(\alpha\beta^{m-n}+\alpha^{m-n}\beta)}{(\alpha-\beta)^2}\right)$$

$$a^{\epsilon(mn+n)}b^{\epsilon(mn+m)}\mathfrak{m}_{m+1}\mathfrak{m}_n$$

$$= a^{\epsilon(mn+n)}b^{\epsilon(mn+m)}\left(\frac{a^{1-\epsilon(m+1)}}{(ab)^{\left\lfloor\frac{m+1}{2}\right\rfloor}}\right)\left(\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}\right)\left(\frac{a^{1-\epsilon(n)}}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\right)\left(\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\right)$$

$$= \frac{ab^{\epsilon(mn+n)}a^{\epsilon(mn+n)-\epsilon(mn+n)}}{(ab)^{\frac{m-n-\in(m-n)}{2}+\epsilon(mn+n)+n}}\left[\frac{\alpha^{m+n+1}+\beta^{m+n+1}-\beta^{m+1}\alpha^{n}-\alpha^{m+1}\beta^{n}}{(\alpha-\beta)^2}\right]$$

$$= \frac{a(ab)^{-n}}{(ab)^{\left\lfloor\frac{m-n}{2}\right\rfloor}}\left[\frac{\alpha^{m+n+1}+\beta^{m+n+1}-(\alpha\beta)^{n}(\alpha^{m-n+1}+\beta^{m-n+1})}{(\alpha-\beta)^2}\right]$$

$$\alpha^{m+n+1}+\beta^{m+n+1}\mathfrak{m}_m\mathfrak{m}_{n+1}-a^{\epsilon(mn+n)}b^{\epsilon(mn+n)}\mathfrak{m}_{m+1}\mathfrak{m}_n$$

$$= \frac{a(ab)^{-n}}{(ab)^{\left\lfloor\frac{m-n}{2}\right\rfloor}}\left[\frac{(\alpha\beta)^{n}(-\alpha\beta^{m-n}-\beta\alpha^{m-n}+\alpha^{m-n+1}+\beta^{m-n+1})}{(\alpha-\beta)^2}\right]$$

$$= \frac{a(ab)^{-n}}{(ab)^{\left\lfloor\frac{m-n}{2}\right\rfloor}}(2ab)^{n}\left[\frac{\alpha^{m-n}(\alpha-\beta)-\beta^{m-n}(\alpha-\beta)}{(\alpha-\beta)^2}\right]$$

$$= \frac{2^{n}a}{(ab)^{\left\lfloor\frac{m-n}{2}\right\rfloor}}\left(\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}\right)$$

$$= 2^{n}a^{\epsilon(m-n)}\mathfrak{m}_{m-n}.$$

Similarly, by using the Binet's formula for bi-periodic Mersenne-Lucas sequences we obtain

$$a^{\epsilon(mn+m)}b^{\epsilon(mn+n)}\mathfrak{M}_{m+1}\mathfrak{M}_n$$

$$= \frac{1}{b(ab)^{\frac{m+n-\epsilon(m-n)}{2}}}(\alpha^{m+n+1}+\beta^{m+n+1}+\beta^{n}\alpha^{m+1}+\alpha^{n}\beta^{m+1})$$

$$a^{\epsilon(mn+m)}b^{\epsilon(mn+n)}\mathfrak{M}_m\mathfrak{M}_{n+1}$$

$$= \frac{1}{b(ab)^{\frac{m+n-\epsilon(m-n)}{2}}}[\alpha^{m+n+1}+\beta^{m+n+1}+\alpha^{n+1}\beta^{m}+\beta^{n+1}\alpha^{m}]$$

$$a^{\epsilon(mn+m)}b^{\epsilon(mn+n)}\mathfrak{M}_{m+1}\mathfrak{M}_n - a^{\epsilon(mn+n)}b^{\epsilon(mn+m)}\mathfrak{M}_m\mathfrak{M}_{n+1}$$

$$= \frac{1}{b(ab)^{\frac{m+n-\epsilon(m-n)}{2}}}[\alpha^n\beta^{m+1} + \beta^n\alpha^{m+1} - \alpha^{n+1}\beta^m - \beta^{n+1}\alpha^m]$$

$$= \frac{1}{b(ab)^{\frac{m+n-\epsilon(m-n)}{2}}}(\alpha\beta)^n[\beta^{m-n+1} + \alpha^{m-n+1} - \alpha\beta^{m-n} - \beta\alpha^{m-n}]$$

$$= \frac{2^n}{b(ab)^{\left\lfloor\frac{m-n}{2}\right\rfloor}}(\alpha - \beta)(\alpha^{m-n} - \beta^{m-n})$$

$$= 2^n a^{\epsilon(m-n)}(9ab - 8)\mathfrak{M}_{m-n}.$$

References

[1] G. Bilgici, Two generalizations of Lucas sequences, Applied Mathematics and Computations 245 (2014), 526-538.

[2] Y. Choo, On the generalization of Fibonacci identities, Results in Mathematics 71 (2017), 347-356.

[3] Y. Choo, Some identities on generalized bi-periodic Fibonacci sequences, International Journal of Mathematical Analysis 13(6) (2019), 259-267.

[4] M. Edson and O. Yayenie, A new generalization of Fibonacci sequence and extended Binet's formula, Integers 9 (2009), 639-654.

[5] A. F. Horadam, Jacobsthal and Pell curves, The Fibonacci Quaterly 26(1) (1988), 79-83.

[6] A. F. Horadam, Jacobsthal Representation Numbers, the Fibonacci Quaterly 37(2) (1996), 40-54.

[7] T. Khoshy, Fibonacci and Lucas numbers with applications, John Wiley and sons Inc., New York, 2001.

[8] N. J. Sloane, A Hand Book of Integer Sequences, New York Academic Press, 1973.

[9] S. Uygan, H. Karatas and E. Akinci, Relations on Bi-periodic Jacobsthal sequence, Transylvanian Journal of Mathematics and Mechanics 10(2) (2018), 141-151.

[10] O. Yayenie, A note on generalized Fibonacci sequences, Applied Mathematics and Computations 217 (2011), 5603-5611.