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RESULTS ON PRIME LABELING AND PRIME DISTANCE LABELING OF GRAPHS

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Abstract

A graph $G(V(G), E(G))$ is known as a prime distance graph (PDG) if it allows a prime distance labeling (PDL) defined as an injective function $h : V(G) \to Z$ such that for each pair of adjacent nodes x and y in G, the integer $|h(x) - h(y)|$ is a prime. Similarly, a graph $G'(V(G'), E(G'))$ with p-nodes is said to permit a prime labeling (PL) if there is an injection $g : V(G') \to \{1, 2, \ldots, p\}$ such that for every line $e = xy$, $\gcd\{g(x), g(y)\} = 1$, where gcd refers to the greatest common divisor. A graph G becomes a prime graph (PG) if it allows a PL. In this paper PL and PDL of certain graphs in the context of extension of nodes, super subdivision, and one point union are investigated.

1. Introduction

Every graph discussed in this article is simple, finite, and undirected. R. Entringer gave the idea of PL and the same was discussed in an article by Tout [6]. Two integers x and y are co-prime if their gcd is 1. A graph $G(V, E)$ is a PDG if there is an injective map $g : V(G) \to Z$ such that for every pair

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of adjacent nodes x and y, the integer $|g(x) - g(y)|$ is a prime and g is known as a PDL of G [9]. For a further study on PDL of graphs, see [1, 2, 3, 4, 5, and 9].

Definition 1. A closed path on n nodes in $G = (V, E)$ is known as a cycle and represented by C_n.

Definition 2. A connected graph containing exactly one C_n is known as a unicyclic graph.

Definition 3. [9]. $G(V, E)$ becomes a PDG if there is $g : V(G) \rightarrow Z$ such that for every pair of adjacent nodes x and y, $|g(x) - g(y)|$ is a prime.

Definition 4. [12]. $G' = (V(G'), E(G'))$ with n nodes is said to permit a PL if there is an injection $g : V(G') \rightarrow \{1, 2, \ldots, n\}$ such that for every line $e = st$, $\gcd\{g(s), g(t)\} = 1$.

Definition 5. $G = (V, E)$ is known to be bipartite if $V(G)$ can be partitioned into 2 sets Y_1 and Y_2 in such a way that each line of G has one end node in Y_1 and the other in Y_2.

2. PDL of Graphs

This section is dedicated for deriving PDL of various classes of graphs.

Theorem 1. [9]. (i) *Any bipartite graph admits a PDL.* (ii) *Every cycle is a PDG.*

Theorem 2. *A graph G with maximum degree $\Delta(G) > 2$ cannot have a PDL with all its nodes labelled odd (even).*

Proof. Since $\Delta(G) > 2$, therefore there is $v \in V$ such that $d(v) > 2$. If possible, let $f : V(G) \rightarrow Z$ be a PDL of G. Without loss of generality (WLG), let $f(v) = m$, where m is an odd integer. Let v_1, v_2, v_3 be adjacent to v. Then $|f(v_1) - m| = p_1, |f(v_2) - m| = p_2, |f(v_3) - m| = p_3$, where p_1, p_2, p_3 are primes. Now one can observe that p_1, p_2, p_3 are even primes and hence equal to 2. But there can be at most two odd numbers at a distance 2 from one odd number, a contradiction. So, therefore any graph G with $\Delta(G) > 2$ cannot have a PDL with all its nodes labelled odd. Similar argument holds good for other nodes of G. The possibility of assigning even labels to all the nodes of G is ruled out in a similar fashion.

Observations 1. Only a path P_n can have a PDL with all its nodes labelled odd or even.

Example 1. Let P_n be path on n-nodes. Then, we have

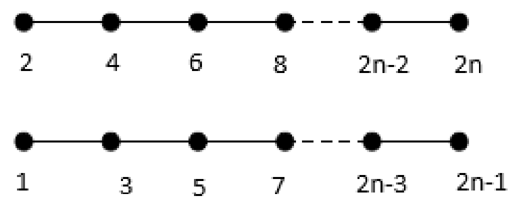

Figure 1. A PDL of P_n.

Observation 2. A cycle C_n cannot have a PDL with all its nodes labeled only odd (even).

Definition 6. [10] The super subdivision of G, $SS(G)$ is formed from G by replacing each line of G by a complete bipartite graph $K_{2,m}$, m a positive integer.

Example 2. One can see $SS(P_n)$ in Figure 2.

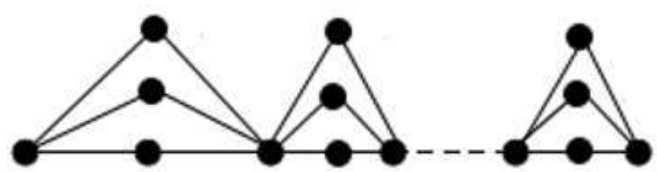

Figure 2. $SS(P_n)$.

Definition 7 [10]**.** The arbitrary super subdivision of G, $ASS(G)$ is constructed from G by replacing each line of G by K_{2,m_i} (m_i is a +ve integer) in such a way that the end nodes of every e_i are joined with the two nodes of 2-nodes part of K_{2,m_i}.

Lemma 1. *$SS(G)$ is bipartite.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ be the node set of a given graph G and $H = SS(G)$ with $V(H) = \{w_i^1, w_i^2, \ldots, w_i^m\}$ be the nodes of H corresponding to the i^{th} line of G. Then $|V(H)| = p + mq$ and $|E(H)| = 2mq$. Further, if $W_1 = V(G), W_2 = \{w_i^j : 1 \leq i \leq q, 1 \leq j \leq m\}$, then $V(H) = W_1 \cup W_2$ is the required bipartition.

Theorem 3. *$SS(G)$ is a PDG.*

Proof. The result is direct from Theorem 2 and Lemma 1.

Remark 1. Theorem 3 is true for $ASS(G)$ also.

Plane Coloring Problem. The plane coloring problem requires the least number of colors to color the plane such that any two nodes at a unit distance are given the different colors. The solution to this problem is not known but the chromatic number of plane has been narrowed down to either one of 5, 6 or 7.

Theorem 4. [1]. *G with $V(G) \subseteq Z$ and $\chi(G) \geq 5$ is not a PDG.*

Theorem 5. *The graph G whose node set consists of every point in the plane and there exists a line between two points if they are at a unit distance cannot have a PDL.*

Proof. The proof follows from the fact that $5 \leq \chi(G) \leq 7$ and Theorem 4.

Remark 2. There may be some finite subgraphs H of G defined in the above theorem which may or may not admit a PDL whose $\chi(H) \leq 4$.

Theorem 6. *The graph formed from taking a finite copies of a PDG H and joining i^{th}-node of every copy of H by a line admits a PDL.*

Proof. Since H is a PDG, therefore there is a PDL $f : V(H) \to Z$ of H. Let G be formed by using m copies of H and connecting i^{th}-node of each copy of H by a line. Now, define $h : V(G) \to Z$ as follows: $h(H_1) = f(H)$. Let k be the greatest label assigned by h to H_1 and p_1 be a prime such that $p_1 > k$. Now assign labels to nodes of H_2 by adding p_1 to the labels of corresponding nodes of H_1. Let l be the greatest label assigned to H_2 and p_2 be a prime such that $p_2 > l$. Again assign labels to nodes of H_3 by adding p_2 to the labels of corresponding nodes of H_2. Continuing in this way, we get primes $p_3, p_4, \ldots, p_{m-1}$ such that each succeeding copy H_r of H is assigned the labels by adding p_{r-1} to the labels of corresponding nodes of preceding H_{r-1} of H. This induces a PDL of G.

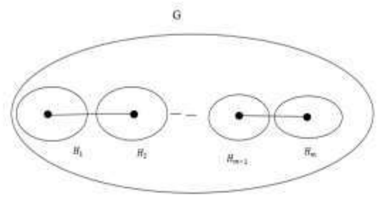

Figure 3. A graph G formed from taking m copies of a PDG H and joining i^{th}-node of each copy of H.

Lemma 2. *One point union of finite copies of bipartite graphs is again bipartite.*

Proof. Since the graphs are bipartite, therefore they do not contain any odd cycle. Moreover, one point union of these graphs again does not give rise to any odd cycle in them and so a bipartite graph.

Conjecture 1 (Goldbach) [9]**.** Every $2n > 2, n > 2$ is a sum of two primes.

Theorem 7. *The graph constructed by taking one point union of any two* 2*-regular graphs admits a PDL if Conjecture* 1 *is true.*

Proof. Let G_1 and G_2 be two 2-regular graphs. Then there are three cases:

Case (1). When G_1 and G_2 both are bipartite.

The proof is clear from Lemma 2 and Theorem 2.

Case (2). When none of G_1 or G_2 is bipartite.

Let G_1 and G_2 be the given 2-regular graphs with node sets $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$, respectively and G be the one point union of G_1 and G_2.

Let the nodes v_1 and u_1 be identified to obtain G. Since G_1 is 2-regular and by Theorem 2, there exists a PDL, say f. Similarly, since G_2 is also a 2-regular graph, there exists a PDL, say g. Define $h : V(G) \to Z$ as follows: let $h(u_1) =$ Results on Prime Labeling and Prime Distance Labeling of Graphs $h(v_1) = 0$. Then $h(u_i) = 2(i-1)$, for $2 \leq i \leq m-1$. By Conjecture 1, $(u_{m-1}) = p_1 + p_2$ and $h(u_m) = p_1$. Similarly, let $h(v_j) = -2(j-1)$, for $2 \leq j \leq n-1$. Now again by Conjecture 1, $h(v_{n-1} = -2(n-1) = -(p_3 + p_4)$ and $h(v_n) = -p_3$. Evidently, h is the required PDL of G.

Case (3). When either G_1 or G_2 is not bipartite.

This case can be dealt with that of Case 2.

Conjecture 2. Any unicyclic graph admits a PDL.

Definition 8 [2]**.** An extension of v by a new u in G produces G_1 such that $N(u) = N[v]$.

Theorem 8. *The graph constructed by taking an extension of an arbitrary node in* K_n, $n \geq 4$ *is not a PDG.*

Proof. Since a graph is PDG if and only if all of its subgraphs PDGs and the graph formed by taking extension of a node in K_n, $n \geq 4$ contains K_5 as a subgraph. Therefore in view of Theorem 4, the proof follows.

3. PL of Graphs

This section is devoted for establishing PL of a few classes of graphs.

Theorem 9 [11]**.** K_n *is not a PG for* $n \geq 4$.

Theorem 10. *If* K_n, $n \geq 3$ *is a complete graph, then the graph formed by taking extension of a node in* K_n *is not a PG.*

Proof. Since the extension of a node in K_n gives rise to K_{n+1} and therefore in view of Theorem 9, the graph formed by taking extension of any arbitrary node in K_n, $n \geq 3$ is not a PG.

Theorem 11. *If* P_n *is a path on n-nodes, then the graph obtained by taking extension of any node of* P_n *is a PG.*

Proof. Let P_n be a path on n-nodes, say $v_1, v_2, v_3, \ldots, v_n$ and extension of v_k in P_n is formed by inserting a node v'_k such that $N(v'_k) = N[v_k]$. Then the node set of new graph having $(n+1)-$nodes, namely, is $\{v_1, v_2, v_{k-1}, v_k, v'_k, v_{k+1}, \ldots, v_n\}$. Define a function $f : \{v_1, v_2, \ldots, v_{k-1}, v_k, v'_k, v_{k+1}, \ldots, v_n\} \to \{1, 2, 3, 4, \ldots, k-1, k, k+1, \ldots, n-1, n, n+1\}$ as $f(v'_k) = 1$ and $f(v_i) = i+1, 1 \leq i \leq n.$ Then *f* induces the required PL.

Theorem 12. *If* P_n *is a path on n-nodes where n is odd, then the graph obtained by taking extension of pendant nodes of* P_n *is a PG.*

Proof. Let P_n be a path on n-nodes, say $v_1, v_2, v_3, \ldots, v_n$ and extensions of the pendant nodes v_1 and v_n are respectively taken by adding new nodes v'_1 and v'_n such that $N(v'_1) = N[v_1]$ and $N(v'_n) = N[v_n]$. Then the node set of new graph having $(n+2)$-nodes is $\{v_1, v'_1, v_2, \ldots, v_n, v'_n\}$. Define $f : \{v_1, v'_2, v_2, \ldots, v'_n, v_n\} \to \{1, 2, 3, 4, \ldots, n-1, n, n+1, n+2\}$ as $f(v'_1) = 1$, $f(v'_n) = n+2$ and $f(v'_1) = i+1, 1 \leq i \leq n.$ Then *f* induces the required PL.

4. Open Problems

Open Problems 1. Is Theorem 12 true if *n* is even?

Open Problem 2. Is the graph formed by taking extension of all the nodes of P_n a PG?

Open Problem 3. Does the graph formed by taking one node union of two 3-regular or 4-regular graphs allow a PDL?

Open Problem 4. Is there a characterization Theorem for PDL or PL?

Open Problem 5. Let *G* be an *n*-cyclic graph. What is the least value of *n* for which *G* ceases to be a PDG?

Conclusion

Study of prime numbers and relatively prime numbers has always been fascinating and challenging. In this paper we have investigated PL of path graphs in the context of extension of nodes and PDL of certain graphs in the context of maximum degree, super subdivision, extension of nodes and one point union.

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