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A STUDY ON $L(2, 1)$ LABELLING PROBLEM FOR COPPER-OXIDE AND ITS EXTENDED NETWORKS

KINS YENOKE[1], T. SHIBA NEENU[1], CHARLES ROBERT KENNETH[1], D. FRANCIS XAVIER[1], M. PETER[1] and K. LEO LAWRENCE[2]

[1]Department of Mathematics
Loyola College, Chennai-034, India
E-mail: kins@loyolacollege.edu

[2]Department of Chemistry
Loyola College, Chennai-034, India
E-mail: leolawrence@loyolacollege.edu

Abstract

An $L(2, 1)$-labeling of a simple connected graph $G = (V, E)$ is a mapping $\varrho : V(G) \rightarrow N \cup \{0\}$ satisfies the condition $d(x, y) + |\varrho(x) - \varrho(y)| \geq 3 \; \forall \, x, y \in V(G)$, where $d(x, y)$ is the distance between y and x in G. The $L(2, 1)$-*labeling number* of G, denoted by $\lambda_{2,1}(G)$, is the least number l such that there is an $L(2, 1)$ labeling with maximum label l. In this paper, we have newly constructed certain extended networks from Copper-Oxide structure $CuO(m, n)$ containing m rows and n columns of octagons. Further, we have investigated the $L(2, 1)$ labeling number for Copper-Oxide network $CuO(m, n)$ and its extended networks.

1. Introduction

In the channel assignment problem, transmitters at several landmarks within a geographical region must be allotted distinct frequencies or channels to avoid channel co-inference. Channels (whole numbers) are allocated to every radio transmitter (nodes) so that interfering (adjacent) transmitters get distinct channels. Keeping the radio transmitter analogy in mind, in 1992,

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Griggs et al. [5] introduced a labeling technique called $L(2, 1)$ labeling in which a graph must to be labeled so that "very close" vertices (at distance 1) get labels that are farther apart while "close" vertices (at distance 2) get different labels. This quantification led to the definition of an $L(2, 1)$ labeling as follows: An $L(2, 1)$ labeling of a simple connected graph $G = (V, E)$ is a mapping $\varrho : V(G) \to N \cup \{0\}$ satisfies the condition $d(x, y) + |\varrho(x) - \varrho(y)| \geq 3 \; \forall x, y \in V(G)$, where $d(x, y)$ is the distance between x and y in G. The $L(2, 1)$-*labeling number* of G, denoted by $\lambda_{2,1}(G)$, is the least number l such that there is an $L(2, 1)$ labeling with maximum label l.

Griggs et al. [5] observed that for any graph G with maximum degree Δ, a greedy algorithm yields $\lambda_{2,1}(G) \leq 1 + 2\Delta + \Delta^2$. In addition, for $\Delta \geq 2$, they posed a conjecture that $\lambda_{2,1}(G) \leq \Delta^2$ and proved that $\lambda_{2,1}(G) \leq 2\Delta + \Delta^2$. Chang et al. [2] improved the bound to $\Delta^2 + \Delta$ and gave a polynomial-time algorithm on trees and linear-time algorithm for cographs. Kral et al. [7] further reduced the bound to $\Delta^2 + \Delta - 1$. In 2008, Goncalves [4] improved the bound by further reducing to $\Delta^2 + \Delta - 2$. For general graphs, the $L(2, 1)$-labeling problem is NP-hard [1]. But it has been studied for several graphs, such as hypercubes [7], tree dendrimers [14], circular arc graphs [10], block graphs [1], generalized Petersen graphs [6], permutation graphs [9], Silicate and oxide networks [13] and chordal graphs [11]. Xavier et al. [12] presented that $\lambda_{2,1}(WS(n, r)) \leq 2r + n - 1$, $\lambda_{2,1}(S(n, r)) \leq r + n - 1$, $r > 3$ and $\lambda_{2,1}(SW(n, r)) \leq 2 + n - 1$.

In this paper, we have newly designed four copper-oxide derived networks from the existing copper-oxide structure $CuO(m, n)$ containing m rows and n columns of octagons. Further, we have shown that $\lambda_{2,1}(CuO(m, n)) \leq 7$, $\lambda_{2,1}(CuO(m, n)) \leq 11$, $\lambda_{2,1}(CuO(m, n)) \leq 9$, $\lambda_{2,1}(CuO(m, n)) \leq 11$ and $\lambda_{2,1}(CuO(m, n)) \leq 10$.

2. Copper-Oxide and its Extended Networks

In this section we have constructed certain derived networks from the known Copper-Oxide structures.

2.1 Coper-Oxide Network. The chemical structure of Copper (II) Oxide [3, 8] is constructed as follows: The octagon structure of Copper (II) Oxide is linked to each other in rows and columns. The connection between two octagon is attained by forming each $Cu4$ bond between two octagons. The resulting network structure named as Copper-Oxide network with m rows and n columns of octagons. It is denoted by $CuO(m, n)$. See in figure 1. The cardinality of vertex and edge sets present in $CuO(m, n)$ are $4mn + m + 3m$ and $2n(m + 1)$ respectively.

2.2 Diagonal Copper-Oxide networks. In $CuO(m, n)$, if the oxide nodes in each octagon are joined diagonally, then the network constructed is called O-diagonal Copper-Oxide network. It is denoted $CuO_{OD}(m, n)$, where m and n are the number of row octagons and column octagons in $CuO(m, n)$. In the similar manner, if the copper nodes in each octagon are joined diagonally, then the resulting network obtained is named as Cu -diagonal Copper-Oxide network. It is symbolized by $CuO_{CuD}(m, n)$. The number of vertices and their names for both the diagonal Copper-Oxide networks are same as that of Copper-Oxide network. Also, the number of edges in both $CuO_{OD}(m, n)$ and $CuO_{CuD}(m, n)$ is $4mn + 2n$ see in figure 2.

2.3 Extended Copper-Oxide networks. In the Copper-Oxide network $CuO(m, n)$, if both oxide nodes and copper nodes in each octagon are joined diagonally, then the constructed network is termed as extended Copper-Oxide network and denoted by $CuO_{EX}(m, n)$. Also, $|V(CuO_{EX}(m, n))| = 4mn + m + 3n$ and $|E(CuO_{EX}(m, n))| = 6mn + 2n$.

2.4 Enhanced Copper-Oxide networks. In the extended Copper-Oxide network $CuO_{EX}(m, n)$ if we place a vertex in each face of an octagon to form a wheel W_{8+1}, then the resulting obtained derived structure is denoted by $CuO_{EX}(m, n)$ and is named as Enhanced Coper-Oxide networks. Further, the cardinality of vertex and edge sets are $5mn + m + 3n$ and $10mn + 2n$ respectively.

Remark 1. In $CuO(m, n)$ and its extended networks with m rows and n columns of octagons we have identified that there are $m + 1$ linear rows and $3n$ linear columns. Moreover, each linear row contains $3n$ number of vertices.

In this paper, we have named the vertices of the Copper-Oxide and its extended networks containing $m + 1$ linear rows and $3n$ inear columns as follows: Let the vertex in j^{th} linear row and i^{th} linear column be named as $w_j^i, 1 \le j \le m + 1, 1 \le i \le 3n.$ Rest of the vertices which lies between odd and even linear rows are named as $u_j^i, i = 1, 2, \ldots, n + 1, j = 1, 2, \ldots, \left\lceil \frac{m}{2} \right\rceil$ and vertices which lies between even and odd linear row are named as $v_i^j, i = 1, 2, \ldots, n + 1, j = 1, 2, \ldots, \left\lfloor \frac{m}{2} \right\rfloor$. This partition of vertex set is visible in figure 1 (a).

3. $L(2, 1)$ Labeling of $CuO(m, n)$, $CuO_{OD}(m, n)$, $CuO_{CuD}(m, n)$, $CuO_{EX}(m, n)$ and $CuO_{EN}(m, n)$ Networks

In this section we have determined the $L(2, 1)$-labeling number for copper-oxide $CuO(m, n)$ and its derived networks.

Theorem 3.1. *Let $CuO(m, n)$ be a Copper-Oxide network with m rows and n columns of octagons. Then the $L(2, 1)$ labelling number of Copper-Oxide network satisfies $\lambda_{2,1}(CuO(m, n)) \le 7$.*

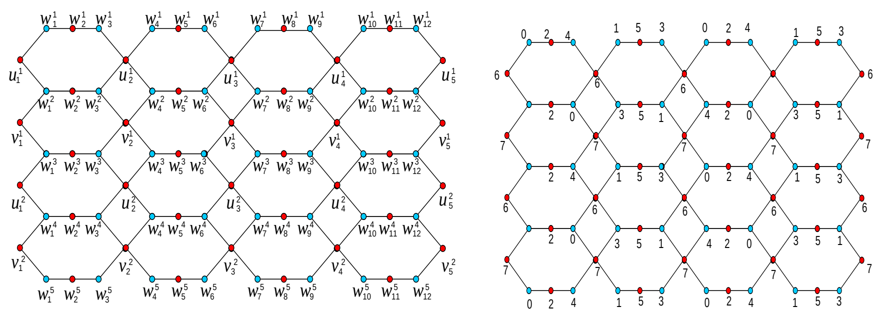
Proof of Theorem 3.1. Define a mapping $\varrho : V(CuO(m, n)) \to N \cup \{0\}$ as follows:

$$\varrho(w_i^{2j-1}) = \begin{cases} 0 & \text{if } i \equiv 1 \bmod 6 \\ 2 & \text{if } i \equiv 2 \bmod 6 \\ 4 & \text{if } i \equiv 3 \bmod 6 \\ 1 & \text{if } i \equiv 4 \bmod 6 \\ 5 & \text{if } i \equiv 5 \bmod 6 \\ 3 & \text{if } i \equiv 0 \bmod 6 \end{cases}, \; i = 1, 2, \ldots, 3n, \; j = 1, 2, \ldots, \left\lceil \frac{m+1}{2} \right\rceil,$$

$$\varrho(w_i^{2j-1}) = \begin{cases} 0 & \text{if } i \equiv 3 \bmod 6 \\ 2 & \text{if } i \equiv 2 \bmod 6 \\ 4 & \text{if } i \equiv 1 \bmod 6 \\ 1 & \text{if } i \equiv 0 \bmod 6 \\ 5 & \text{if } i \equiv 5 \bmod 6 \\ 3 & \text{if } i \equiv 4 \bmod 6 \end{cases}, \; i = 1, 2, \ldots, 3n, \; j = 1, 2, \ldots, \left\lceil \frac{m+1}{2} \right\rceil,$$

$$\varrho(v_i^j) = 6, \; i = 1, 2, \ldots, n+1, \; j = 1, 2, \ldots, \left\lceil \frac{m+1}{2} \right\rceil,$$

$\varrho(v_i^j) = 7, \; i = 1, 2, \ldots, n+1, \; j = 1, 2, \ldots, \left\lceil \frac{m}{2} \right\rceil$. It is visible in figure 1(b).



(a) (b)

Figure 1. Naming of vertices in Copper-Oxide network $CuO(4, 4)$ and its $L(2, 1)$ labelling.

Next, we verify the above labelling pattern satisfies the $L(2, 1)$ condition $d(x, y) + |\varrho(x) - \varrho(y)| \geq 3$ for every distinct pair of copper and oxide nodes in $CuO(m, n)$. Let x and y be any two vertices in $CuO(m, n)$.

Case 1. Choose the vertices x and y are of the form w_k^{2t-1} and w_s^{2t-1}, $1 \leq k, s \leq 3n, 1 \leq l, t \leq \left\lceil \frac{m+1}{2} \right\rceil$.

Case 1.1. If $l \neq t$, then x and y lie on different rows and the distance between them is at least 4. Therefore, $d(w_k^{2t-1}, w_s^{2t-1}) + |\varrho(w_k^{2t-1}) - \varrho(w_s^{2t-1})| \geq 4 > 3$.

Case 1.2. Suppose $l = t$, then x and y lie on the same row. If $k \equiv p \bmod 6$ and $s \equiv q \bmod 6$, $0 \leq p = q \leq 5$, then $d(w_k^{2t-1}, w_s^{2t-1}) \geq 8$ and $|\varrho(w_k^{2t-1}) - \varrho(w_s^{2t-1})| = 0$. Also, if $k \equiv p \bmod 2$ and $s \equiv q \bmod 3$, $0 \leq q = p+1 \leq 2$, then the distance between them is at least 4 and $|\varrho(w_k^{2t-1}) - \varrho(w_s^{2t-1})| \geq 1$. Otherwise, $d(w_k^{2t-1}, w_s^{2t-1}) \geq 1$ and $|\varrho(w_k^{2t-1}) - \varrho(w_s^{2t-1})|$. Hence, in all the options, $d(w_k^{2t-1}, w_s^{2t-1}) + |\varrho(w_k^{2t-1}) - \varrho(w_s^{2t-1})| \geq 3$.

Case 2. Assume $x = w_k^{2l}$ and $y = w_k^{2t}$, $1 \leq k, s \leq 3n$, $1 \leq l, t \leq \left\lceil \frac{m+1}{2} \right\rceil$.

Case 2.1. If l and t are distinct, then w_k^{2l} and w_s^{2l} falls on different rows so $d(w_k^{2l}, w_s^{2t}) \geq 4$. Therefore, $d(w_k^{2l}, w_s^{2t}) + |\varrho(w_k^{2l}) - \varrho(w_s^{2t})| > 3$.

Case 2.2. Supposing l and t are equal, then w_k^{2l} and w_s^{2l} falls on the same row. If $k \equiv p \bmod 6$ and $s \equiv q \bmod 6$, $0 \leq p = q \leq 5$, then w_k^{2l} and w_s^{2l} received the same labelling and $d(w_k^{2l}, w_s^{2t}) \geq 8$. Also, if $k \equiv p \bmod 2$ and $s \equiv q \bmod 3$, $0 \leq q = p+1 \leq 2$, then $d(w_k^{2l}, w_s^{2t}) \geq 4$ and $|\varrho(w_k^{2l}) - \varrho(w_s^{2t})| > 1$. Otherwise, $|\varrho(w_k^{2l}) - \varrho(w_s^{2t})|$ and $d(w_k^{2l}, w_s^{2t}) \geq 1$. Therefore, in all the possibilities, $d(w_k^{2l}, w_s^{2t}) + |\varrho(w_k^{2l}) - \varrho(w_s^{2t})| \geq 3$.

Case 3. Suppose x and y are of the form u_k^l and v_s^t, $1 \leq k, s \leq n+1$, $1 \leq l, t \leq \left\lfloor \frac{m+1}{2} \right\rfloor$. If $k \neq s$ or $l \neq t$ or both $k \neq s$ and $l \neq t$, then the distance between them is at least 4. Therefore, $d(u_k^l, u_s^t) + |\varrho(u_k^l) - \varrho(u_s^t)| \geq 4 > 3$.

Case 4. Let $x = u_k^l$ and $y = v_s^t$, where $1 < k, s \leq n+1$, $1 \leq l \leq \left\lfloor \frac{m+1}{2} \right\rfloor$,

$1 \leq t \leq \left\lceil \frac{m+1}{2} \right\rceil$. Therefore, from the mapping, $\varrho(u_k^l) = 6, \varrho(v_s^t) = 7$ and $d(u_k^l, v_s^t) \geq 2$. Hence, $d(u_k^l, v_s^t) + | \varrho(u_k^l) - \varrho(v_s^t) | \geq 1 + 2 \geq 3$.

Case 5. Assume x and y are of the form v_k^l and $v_s^t, 1 \leq k, s \leq n+1, 1 \leq l, t \leq \left\lfloor \frac{m+1}{2} \right\rfloor, 1 \leq t \leq \left\lfloor \frac{m-1}{2} \right\rfloor$. Then, $d(v_k^l, v_s^t) \geq 4$ and $\varrho(v_k^l) - \varrho(v_s^t) = 7$. Hence, $d(v_k^l, v_s^t) + | \varrho(v_k^l) - \varrho(v_s^t) | > 3$.

Case 6. If $x = u_k^l$ and $y = w_s^{2t-1}$, where $1 < k \leq n+1, 1 \leq l \leq \left\lfloor \frac{m+1}{2} \right\rfloor$, $1 \leq s \leq 3n, 1 \leq t \leq \left\lceil \frac{m+1}{2} \right\rceil$. If $\varrho(w_k^{2t-1}) = 5$, then $d(u_k^l, w_s^{2t-1}) \geq 2$ and $\varrho(u_k^l) = 6$. Otherwise, $\varrho(w_k^{2t-1}) \geq 4$ and $d(u_k^l, w_s^{2t-1}) \geq 1$. Hence in both the possibilities, $d(v_k^l, w_s^{2t-1}) + | \varrho(v_k^l) - \varrho(w_s^{2t-1}) | > 3$.

Case 7. Let $x = u_k^l$ and $y = w_s^{2t}$, where $1 \leq l \leq \left\lfloor \frac{m}{2} \right\rfloor, 1 \leq k \leq n+1$, $1 \leq s \leq 3n, 1 \leq t \leq \left\lfloor \frac{m+2}{2} \right\rfloor$, then $\varrho(u_k^l) = 6$. If $\varrho(w_k^{2t}) < 5$, then $d(u_k^l, w_s^{2t-1}) \geq 1$ else $d(u_k^l, w_s^{2t-1}) \geq 2$ and $\varrho(w_k^{2t-1}) = 5$. Hence, the inequality $d(v_k^l, w_s^{2t-1}) + | \varrho(v_k^l) - \varrho(w_s^{2t-1}) | \geq 3$ is verified in this case.

Case 8. Assume x and y takes the values $\varrho(x) = \varrho(v_k^l) = 7$ and $\varrho(x) = \varrho(w_s^{2t-1}) \leq 5$, where $1 \leq k \leq n+1, 1 \leq l \leq \left\lfloor \frac{m}{2} \right\rfloor, 1 \leq s \leq 3n, 1 \leq t \leq \left\lfloor \frac{m+2}{2} \right\rfloor$. Therefore, $d(v_k^l, w_s^{2t-1}) \geq 1$ and $| \varrho(v_k^l) - \varrho(w_s^{2t-1}) | \geq 2$. So, $d(v_k^l, w_s^{2t-1}) + | \varrho(v_k^l) - \varrho(w_s^{2t-1}) | \geq 3$.

Case 9. Take $x = v_k^l$ and $y = w_s^{2t}$, where $1 \leq l \leq \left\lfloor \frac{m}{2} \right\rfloor, 1 \leq k \leq n+1$, $1 \leq s \leq 3n, 1 \leq t \leq \left\lfloor \frac{m+2}{2} \right\rfloor$, then $\varrho(v_k^l) = 7$. If $\varrho(w_s^{2t}) = 5$, then $d(v_k^l, w_s^{2t}) \geq 2$, otherwise $d(u_k^l, w_s^{2t-1}) \geq 1$ and $\varrho(w_s^{2t}) < 5$. Therefore,

$d(v_k^l, w_s^{2t-1}) + |\varrho(v_k^l) - \varrho(w_s^{2t})| \geq 3$.

Thus, in all the possibilities, $d(x, y) + |\varrho(x) - \varrho(y)| \leq 3 \; \forall x, y \in V(CuO(m, n))$. Hence, $\lambda_{2,1}(CuO(m, n)) \leq 7$.

Theorem 3.2. *Let $CuO_{OD}(m, n)$ be a O-diagonal Copper-Oxide network with m rows and n columns respectively. Then the $L(2, 1)$ labelling number satisfies $\lambda_{2,1}(CuO_{OD}(m, n)) \leq 11$.*

Proof of Theorem 3.2. Define a mapping $\varrho : V(CuO_{OD}(m, n)) \to N \cup \{0\}$ as follows:

$$\varrho(w_i^{4j-3}) = \begin{cases} 0, \text{ if } i \equiv 1 \bmod 3 \\ 2, \text{ if } i \equiv 2 \bmod 3, \\ 4, \text{ if } i \equiv 0 \bmod 3 \end{cases} i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, \left\lceil \frac{m+1}{4} \right\rceil,$$

$$\varrho(w_i^{4j-2}) = \begin{cases} 1, \text{ if } i \equiv 1 \bmod 3 \\ 5, \text{ if } i \equiv 2 \bmod 3, \\ 3, \text{ if } i \equiv 0 \bmod 3 \end{cases} i = 1, 2, \ldots, 3n, \; j = 1, 2, \ldots, \left\lceil \frac{m}{4} \right\rceil,$$

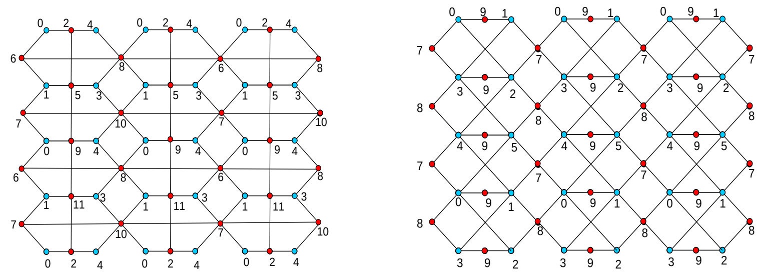
$$\varrho(w_i^{4j-1}) = \begin{cases} 0, \text{ if } i \equiv 1 \bmod 3 \\ 9, \text{ if } i \equiv 2 \bmod 3, \\ 4, \text{ if } i \equiv 0 \bmod 3 \end{cases} i = 1, 2, \ldots, 3n, \; j = 1, 2, \ldots, \left\lceil \frac{m-1}{4} \right\rceil,$$

$$\varrho(w_i^{4j}) = \begin{cases} 1, \text{ if } i \equiv 1 \bmod 3 \\ 11, \text{ if } i \equiv 2 \bmod 3, \\ 3, \text{ if } i \equiv 0 \bmod 3 \end{cases} i = 1, 2, \ldots, 3n, \; j = 1, 2, \ldots, \left\lceil \frac{m-1}{4} \right\rceil,$$

$$\varrho(u_i^j) = \begin{cases} 6, \text{ if } i \equiv 1 \bmod 2 \\ 8, \text{ if } i \equiv 0 \bmod 2, \end{cases} i = 1, 2, \ldots, 3n, \; j = 1, 2, \ldots, \left\lceil \frac{m+1}{4} \right\rceil,$$

$$\varrho(v_i^j) = \begin{cases} 7, \text{ if } i \equiv 1 \bmod 2 \\ 10, \text{ if } i \equiv 0 \bmod 2, \end{cases} i = 1, 2, \ldots, n+1, \; j = 1, 2, \ldots, \left\lceil \frac{m}{2} \right\rceil.$$

See figure 2(a).



(a) (b)

Figure 2. The diagonal Copper-Oxide networks $CuO_{OD}(4, 3)$, $CuO_{uD}(4, 3)$ and their $L(2, 1)$ labellings.

Since, verifying the $L(2, 1)$ labelling condition for the above mapping is parallel to Theorem 3.1, we ignore the remaining proof.

Theorem 3.3. *For the Cu-diagonal Copper-Oxide network $CuO_{CuD}(m, n)$ the $L(2, 1)$ labelling number satisfies* $\lambda_{2,1}(CuO_{CuD}(m, n)) \leq 9$.

Proof of Theorem 3.3. The proof of this theorem can be easily verified for the labelling pattern from the following mapping ϱ from $V(CuO_{CuD}(m, n))$ to $N \cup \{0\}$.

$$\varrho(w_i^{3j-2}) = \begin{cases} 0, \text{ if } i \equiv 1 \bmod 3 \\ 9, \text{ if } i \equiv 2 \bmod 3, \\ 1, \text{ if } i \equiv 0 \bmod 3 \end{cases} i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, \left\lceil \frac{m+1}{4} \right\rceil,$$

$$\varrho(w_i^{3j-1}) = \begin{cases} 3, \text{ if } i \equiv 1 \bmod 3 \\ 9, \text{ if } i \equiv 2 \bmod 3, \\ 2, \text{ if } i \equiv 0 \bmod 3 \end{cases} i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, \left\lceil \frac{m}{4} \right\rceil,$$

$$\varrho(w_i^{3j}) = \begin{cases} 4, \text{ if } i \equiv 1 \bmod 3 \\ 9, \text{ if } i \equiv 2 \bmod 3, \\ 5, \text{ if } i \equiv 0 \bmod 3 \end{cases} i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, \left\lceil \frac{m-1}{4} \right\rceil,$$

$$\varrho(u_i^j) = 7, \ i = 1, 2, \ldots, n+1, \ j = 1, 2, \ldots, \left\lceil \frac{m+1}{4} \right\rceil,$$

$\varrho(v_i^j) = 8, \ i = 1, 2, \ldots, n+1, \ j = 1, 2, \ldots, \left\lceil \frac{m}{2} \right\rceil$. The mapping is visible in figure 2(b).

Theorem 3.4. *Let* $CuO_{EX}(m, n)$ *be an extended Copper-Oxide network with m octagon rows and n octagon columns respectively. Then the* $L(2, 1)$ *labeling number satisfies* $\lambda_{2, 1}(CuO_{EX}(m, n)) \leq 11$.

Proof of Theorem 3.4. It can be easily proved by using the following labelling pattern.

$$\varrho(w_i^{3j-2}) = \begin{cases} 0, \text{ if } i \equiv 1 \bmod 3 \\ 6, \text{ if } i \equiv 2 \bmod 3, \; i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, \left\lceil \frac{m+1}{4} \right\rceil, \\ 1, \text{ if } i \equiv 0 \bmod 3 \end{cases}$$

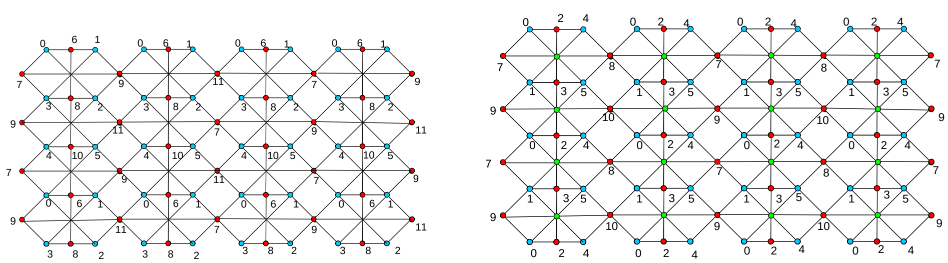
$$\varrho(w_i^{3j-1}) = \begin{cases} 3, \text{ if } i \equiv 1 \bmod 3 \\ 8, \text{ if } i \equiv 2 \bmod 3, \; i = 1, 2, \ldots, 3n, \; j = 1, 2, \ldots, \left\lceil \frac{m}{4} \right\rceil, \\ 2, \text{ if } i \equiv 0 \bmod 3 \end{cases}$$

$$\varrho(w_i^{3j}) = \begin{cases} 4, \text{ if } i \equiv 1 \bmod 3 \\ 10, \text{ if } i \equiv 2 \bmod 3, \; i = 1, 2, \ldots, 3n, \; j = 1, 2, \ldots, \left\lceil \frac{m-1}{4} \right\rceil, \\ 5, \text{ if } i \equiv 0 \bmod 3 \end{cases}$$

$$\varrho(u_i^j) = \begin{cases} 7, \text{ if } i \equiv 1 \bmod 3 \\ 9, \text{ if } i \equiv 2 \bmod 3 \;, \; i = 1, 2, \ldots, n+1, \; j = 1, 2, \ldots, \left\lceil \frac{m+1}{4} \right\rceil, \\ 11, \text{ if } i \equiv 2 \bmod 3 \end{cases}$$

$$\varrho(v_i^j) = \begin{cases} 9, \text{ if } i \equiv 1 \bmod 3 \\ 11, \text{ if } i \equiv 2 \bmod 3, \; i = 1, 2, \ldots, n+1, \; j = 1, 2, \ldots, \left\lfloor \frac{m}{2} \right\rfloor. \\ 7, \text{ if } i \equiv 0 \bmod 3 \end{cases}$$

See the figure 3 (a).



(a) (b)

Figure 3. An $L(2, 1)$ labelling of $CuO_{EX}(m, n)$ and $CuO_{EX}(m, n)$ for $m = n = 4$.

Theorem 3.5. *The* $L(2, 1)$ *labelling number of enhanced Copper-Oxide network* $CuO_{EX}(m, n)$ *satisfies* $\lambda_{2,1}(CuO_{EX}(m, n)) \leq 10$.

Proof of Theorem 3.5. As the proof is similar to previous theorems, the mapping and proof is left to the reader. The result can be verified using figure 3(b).

Conclusion

In this research work, we have introduced four different copper-oxide derived networks. Further, we have obtained the upper bounds for the $L(2, 1)$ labelling number of Copper-Oxide and its extended networks. This work can be extended by studying different graph theory problems for the newly introduced networks and also the same problem can be study for other chemical structures.

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