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AN ANALYTIC SOLUTION OF THE ONE-DIMENSIONAL SOLUTE TRANSPORT EQUATION FOR UNIFORM UNSTEADY FLOW IN A HOMOGENEOUS MEDIUM

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Abstract

The solution of a one-dimensional solute transport equation in a homogeneous medium has been obtained. Flow has been taken as uniform and unsteady. A present paper deals with temporal variation in dispersion. Two types of variation in the dispersion with time are assumed. Initially the medium is not solute free, i.e. assumed that initially, concentration is exponentially decreasing with space variable. The velocity is considered proportional to the dispersion. Dispersion is exponentially increase in the first case while sinusoidally varies in the second case. The analytic solution using the Laplace transform variation iteration method (LVIM) are obtained. The obtained results are presented in terms of temporal and spatial variation of concentration. The Graphical representation of the developed model have been obtained using MATHEMATICA coding.

1. Introduction

The increasing demand for groundwater has forced the human being to put efforts for further development of ground resources (Todd and Mays, [19]). This results in the advanced investigation of the occurrence and

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movement of groundwater and extension in under-standing through research (Todd and Mays, [19]). The quality and quantity of groundwater are the two inseparable entities which must be taken into account at the time of study of water resource management (Bear, [2]; Bear and Cheng, [3]). Increasing pollution and hence, the mismanagement of waste have unconfined the research in groundwater resources which was earlier limited to only water flow with few water quality aspects. The type and behavior of the contaminant source are additional aspects that need to be incorporated during the modeling of solute transport through the groundwater. Many efforts have been done to model the transport phenomena in groundwater with various physical conditions by many researchers (Desai, [5]; Joshi, Desai, and Mehta, [14]; Patel and Dhodiya, [15]). A. E. Scheidegger (Scheidegger, [17]) has studied the general theory of dispersion in a porous medium. The physical concept of flow through porous media has been given by A. E. Scheidegger. (Scheidegger, [16]). Warrick et al., (Warrick, Kichen, and Thames, [20]) have derived the solution of miscible displacement of soil water with temporal velocity and dispersion coefficient. Authors (Warrick et al., [20]) have used one dimensional flow with semi-infinite solute free medium. Jaiswal et al. (Jaiswal and Kumar, n.d.) developed one dimensional model with space dependent variable coefficient advection dispersion equation using Laplace transform. In which authors (Jaiswal and Kumar, n.d.) have used varying pulse type input point source and dispersion coefficient changes with the square of velocity. Using Laplace transform technique, the solute transport equation has been solved by Yadav and Kumar (R. R. Yadav and Kumar, [22]) considering dispersion as an exponentially decreasing function of space and velocity is directly proposed to dispersion coefficient. A convective-dispersion equation has been analytically solved by Singh et al. (Singh, Mahato, Singh, et al., [18]) with the assumption that the aquifer is solute free and uniform source concentration with pulse type boundary condition. Longitudinal dispersion is taken into account in their article. Numerical solution of one-dimensional advection diffusion equation found by Yadav and Roy (R. R. Yadav and Roy, [23]) using implicit Crank-Nicolson finite difference method. In their work medium of the aquifer has been considered as homogeneous. Either temporally or constant velocity was assumed by the author (R. R. Yadav and Roy, [23]). Yadav et al. (S. K. Yadav, Kumar, Jaiswal, and Kumar, [24]) have studied phenomena of contaminant transport in a homogeneous medium with unsteady flow. Authors (S. K. Yadav et al., [24]) used Laplace integral transform to obtain the result and assume that the velocity is increasing function of space variable.

A non-linear partial differential equation has been solved by the Variation Iteration Method (VIM) by Jihuan He (J. He, [6]). In his study, using variation theory, Lagrange's multiplier was found. He (J.-H. He, [8], [7], [9], [10]; J.-H. He and Wu, [11]) has proposed the VIM for solving linear and non-linear equations. The method has been incorporated by many researchers for various mathematical modeling. Further, to make the VIM less complected, a Laplace transform is included along with a VIM to propose a more effective Laplace transform variational iteration method (LVIM). The applications of the LVIM by many researchers show its sound efficiency (Anjum and He, [1]; Biala, Asim, and Afolabi, [4]; Hesameddini and Latizadeh, [12]; Wu and Baleanu, [21]). In the present research, the concentration of solute transport in the homogeneous medium with the uniform unsteady flow has been studied. The current work focuses on the analytic solution of the solute transport equation that governs the solute transport in a homogeneous porous medium. In many practical aspects, it is observed that the contaminant concentration varies with time due to temporal variation in the dispersion. Hence, to obtain a clear idea of the temporal and spatial pattern of the contaminant concentration, it is required to have an analytic solution of the governing equation in such cases. Two different types of cases are discussed in the present paper. In case-1 dispersion is exponentially increasing with time while in a second case it is sinusoidally vary with time. The velocity of the containment depends upon the dispersion and it is considered in proportional to the dispersion. Hence, the effect of the temporal variation in the dispersion also reflects on the velocity. Initially, concentration is exponentially decreasing with space variables. Analytic solution of solute transport equation is obtained using LVIM. Using the obtained solution, the temporal and spatial variation of the concentration for each case are presented graphically. A detailed description of each of the plots are provided in the respective sections of given below.

2. Mathematical Formulation

The contaminant concentration in the porous medium at any time $t[T]$ is denoted by $c(x, t)$ with dimension $[ML^{-3}]$. The solute particles are also transported along with groundwater. The velocity of groundwater is $u[LT^{-1}]$ in the porous medium and D be the solute dispersion $[L^2T^{-1}]$. It is considered that dispersion is directly proportional to the velocity (u).

With these notations the contaminant transport equation can be expressed as,

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_y \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(D_z \frac{\partial c}{\partial z}\right)$$

$$- \frac{\partial}{\partial x}(u_x c) - \frac{\partial}{\partial y}(u_y c) - \frac{\partial}{\partial z}(u_z c) \tag{1}$$

where D_x, D_y and D_z are dispersion coefficients along x, y and z directions respectively. u_x, u_y and u_z are the velocities of flow along x, y and z directions respectively and c is the solute concentration.

One-dimensional solute transport equation with variable coefficient and initially concentration is exponentially decreasing with space can be written as,

$$\frac{\partial c}{\partial t} = D(x, t)\frac{\partial^2 c}{\partial x^2} - u(x, t)\frac{\partial c}{\partial x} \tag{2}$$

with initial condition,

$$c(x, 0) = c_0 e^{-mx};\ 0 < x < 1,\ t = 0. \tag{3}$$

Where, $m[L^{-1}]$ is a constant parameter that decide exponential decreasing rate and $c_0[M/L^3]$ is initial constant concentration.

3. Methodology

Consider nonlinear differential equation,

$$lu(x, t) + Nu(x, t) = f(x, t) \tag{4}$$

where l is a linear operator, N is a nonlinear operator and $f(x, t)$ is a known analytical function.

According to the variational iteration method, we can construct a correction functional as follows:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \bar{\lambda}(x, \xi)[lu_n(x, \xi) + N\bar{u}_n(\xi) - f(x, \xi)]d\xi \tag{5}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory. The subscript n denotes the n^{th} approximation and $\bar{u}_n$ is a restricted variation $\delta\bar{u}_n = 0$.

Operating with Laplace transform on both sides of equation (5) the correction functional will be constructed in the following manner:

$$L[u_{n+1}(x, t)] = L[u_n(x, t)] + L[\int_0^t \bar{\lambda}(x, t-\xi)[lu_n(x, \xi) + N\bar{u}_n(x, \xi) - f(x, \xi)]d\xi] \tag{6}$$

$n = 0, 1, 2, 3, \ldots$

Using convolution theorem,

$$L[u_{n+1}(x, t)] = L[u_n(x, t)] + L[\bar{\lambda}(x, t)[lu_n(x, t) + N\bar{u}_n(x, t) - f(x, t)] \tag{7}$$

Take the variation with respect to $u_n(x, t)$ and hence upon applying the variation this simplifies to

$$L[\delta u_{n+1}(x, t)] = L[\delta u_n(x, t)] + \delta[L[\bar{\lambda}(x, t)]L[u_n(x, t)]] \tag{8}$$

Applying the inverse Laplace transform to equation (8) and using variational theory, the value of Lagrange multiplier is obtained.

4. Analytic Solution Using LVIM

4.1 Case-1 exponentially increasing dispersion

Dispersion coefficient is in proportional to the velocity. Also it is exponentially increasing with time can be written as,

$$D(x, t) = D_0 e^{at}$$

$$u(x, t) = u_0 e^{at}$$

where; $D_0[L^2T^{-1}]$ is initial dispersion coefficient and $u_0[LT^{-1}]$ is initial ground water velocity which are constant. Parameter a $[T^{-1}]$ is coefficient that decide exponential increasing rate which is fixed.

Equation (2) can be written as,

$$\frac{\partial c}{\partial t} = D_0 e^{at} \frac{\partial^2 c}{\partial x^2} - u_0 e^{at} \frac{\partial c}{\partial x} \tag{9}$$

Using Dimensionless variables,

$$\zeta = \frac{u_0^2 t}{D_0}, \ \eta = \frac{u_0 x}{D_0}, \ C = \frac{c}{c_0}$$

$$A = \frac{aD_0}{u_0^2}, \ M = \frac{mD_0}{u_0}, \ B = \frac{bD_0}{u_0^2} \tag{10}$$

Equation (9) can be written as,

$$\frac{\partial C}{\partial \zeta} = e^{A\zeta} \frac{\partial^2 C}{\partial \eta^2} - e^{A\zeta} \frac{\partial C}{\partial \eta} \tag{11}$$

with initial condition,

$$C(\eta, \zeta) = e^{-M\eta} \text{ at } \zeta = 0, \ 0 < \eta < \frac{u_0}{D_0}$$

The Laplace variational iteration correction functional will be constructed as,

$$L[C_{n+1}(\eta, \zeta)] = L[C_n(\eta, \zeta)] + L[\int_0^t \overline{\lambda}(\eta, \zeta - \xi)[(C_n)_\zeta(\eta, \xi) - e^{A\zeta}((\overline{C}_n)_{\eta\eta}(\eta, \xi)$$

$$+ e^{A\zeta}[(\overline{C}_n)_\eta(\eta, \xi)]d\xi] \tag{12}$$

$$L[C_{n+1}(\eta, \zeta)] = L[C_n(\eta, \zeta)] + L[\overline{\lambda}(\eta, \zeta)]L[(C_n)_\zeta(\eta, \zeta) - e^{A\zeta}((\overline{C}_n)_{\eta\eta}(\eta, \zeta)$$

$$+ e^{A\zeta}[(\overline{C}_n)_\eta(\eta, \zeta)] \tag{13}$$

Taking the variation with respect to $C_n(\eta, \zeta)$ of equation (13), and use extreme condition $\delta C_{n+1}(\eta, \zeta) = 0$ obtain result is,

$$\overline{\lambda}(\eta, \zeta) = -1 \tag{14}$$

Using equations (13) and (14) obtained result is,

$$L[C_{n+1}(\eta, \zeta)] = L[C_n(\eta, \zeta)] + L[(-1)]L[(C_n)_\zeta(\eta, \zeta) - e^{A\zeta}((\overline{C}_n)_{\eta\eta}(\eta, \zeta)$$

$$+ e^{A\zeta}[(\overline{C}_n)_\eta(\eta, \zeta)] \tag{15}$$

Let the initial approximation is,

$$C_0(\eta, \zeta) = C(\eta, 0) = e^{-M\eta} \tag{16}$$

Using equations (15) and (16),

$$L[C_1(\eta, \zeta)] = L[C_0(\eta, \zeta)] + L[-1]L[(C_0)_\zeta - e^{A\zeta}(C_0)_{\eta\eta} + e^{A\zeta}(C_0)_\eta] \tag{17}$$

$$C_1(\eta, \zeta) = e^{-M\eta}\left[1 - \frac{M}{A}(1 + M)(1 - e^{A\zeta})\right] \tag{18}$$

$$C_2(\eta, \zeta) = e^{-M\eta}\left[1 - \frac{M}{A}(1 + M)(1 - e^{A\zeta}) + \frac{M^2}{2!\,A^2}(1 + M)^2(1 - e^{A\zeta})^2\right] \tag{19}$$

Using equations (15) and (19) we obtained,

$$C_3(\eta, \zeta) = e^{-M\eta}\left[1 - MT(1 + M)(1 - e^{A\zeta}) + \frac{M^2}{2!\,A^2}(1 + M)^2(1 - e^{A\zeta})^2\right.$$

$$\left. + \frac{M^3}{3!\,A^3}(1 + M)^3(1 - e^{A\zeta})^3\right]$$

By continuing this procedure, the exact solution is given by,

$$C(\eta, \zeta) = \lim_{n\to\infty} C_n(\eta, \zeta)$$

Then the solution of equation (11) is given by,

$$C(\eta, \zeta) = \exp\left[-M\eta - \frac{M}{A}(1 + M)(1 - e^{A\zeta})\right] \tag{20}$$

The solution of equation (9) is,

$$c(x, t) = c_0 \exp\left[-mx - \frac{m}{a}(u_0 + mD_0)(1 - e^{at})\right] \tag{21}$$

4.2 Case-2 sinusoidally varies dispersion

The dispersion varies sinusoidally with time. It is directly proposal to the groundwater velocity (Scheidegger, [16]) which can be written as,

$$D(x, t) = D_0 \sin bt$$

$$u(x, t) = u_0 \sin bt$$

Where; parameter $b[T^{-1}]$ is the constant coefficient that decides the rate of frequency of sinusoidal variation in dispersion.

Equation (2) can be written as,

$$\frac{\partial c}{\partial t} = D_0 \sin bt \frac{\partial^2 c}{\partial x^2} - u_0 \sin bt \frac{\partial c}{\partial x} \tag{22}$$

with initial condition,

$$c(x, 0) = c_0 e^{-mx};\ 0 < x < 1,\ t = 0 \tag{23}$$

Using dimensionless variables from equation (10), in equation (22 and 23),

$$\frac{\partial C}{\partial \zeta} = \sin B\zeta \frac{\partial^2 C}{\partial \eta^2} - \sin B\zeta \frac{\partial C}{\partial \eta} \tag{24}$$

with initial condition,

$$C(\eta, \zeta) = e^{-M\eta} \quad \text{at} \quad \zeta = 0,\ 0 < \eta < \frac{u_0}{D_0}$$

The Laplace variational iteration correction functional will be constructed as,

$$L[C_{n+1}(\eta, \zeta)] = L[C_n(\eta, \zeta)] + L[\int_0^{\zeta} \overline{\lambda}(\eta, \zeta - \xi)[(C_n)_{\zeta}(\eta, \xi) - \sin B\zeta((\overline{C}_n)_{\eta\eta}(\eta, \xi)$$

$$+ \sin B\zeta[(\overline{C}_n)_{\eta}(\eta, \xi)]d\xi] \tag{25}$$

$$L[C_{n+1}(\eta, \zeta)] = L[C_n(\eta, \zeta)] + L[\overline{\lambda}(\eta, \zeta)[(C_n)_{\zeta}(\eta, \zeta) - \sin B\zeta((\overline{C}_n)_{\eta\eta}(\eta, \zeta)$$

$$+ \sin B\zeta[(\overline{C}_n)_{\eta}(\eta, \zeta)] \tag{26}$$

Taking the variation with respect to $C_n(\eta, \zeta)$ of equation (26) and using extreme condition of $C_n(\eta, \zeta)$ the value of Lagrange's Multiplier $(\lambda = -1)$ is obtained and equation (26) can be written as,

$$L[C_{n+1}(\eta, \zeta)] = L[C_n(\eta, \zeta)] + L[\overline{\lambda}(-1)]L[(C_n)_{\zeta}(\eta, \zeta) - \sin B\zeta(C_n)_{\eta\eta}(\eta, \zeta)$$

$$+ \sin B\zeta(C_n)_{\eta}(\eta, \zeta)] \tag{27}$$

Let,

$$C_0(\eta, \zeta) = C(\eta, 0) = e^{-M\eta} \tag{28}$$

Using equation (27) and (28) and applying the inverse Laplace transform, first approximation is given by,

$$C_1(\eta, \zeta) = e^{-M\eta}\left[1 - \frac{M}{1!\, B}(1 + M)(1 - \cos A\zeta)\right] \tag{29}$$

In similar manner,

$$C_2(\eta, \zeta) = e^{-M\eta}\left[1 + \frac{M}{1!\, B}(1 + M)(1 - \cos B\zeta) + \frac{M^2}{1!\, B^2}(1 + M)^2(1 - \cos B\zeta)^2\right] \tag{30}$$

$$C_3(\eta, \zeta) = e^{-M\eta}\left[1 + \frac{M}{1!\, B}(1 + M)(1 - \cos B\zeta) + \frac{M^2}{1!\, B^2}(1 + M)^2(1 - \cos B\zeta)^2\right.$$

$$\left. + \frac{M^3}{1!\, B^3}(1 + M)^3(1 - \cos B\zeta)^3\right] \tag{31}$$

With the same procedure, the solution of equation (24) is given by,

$$C(\eta, \zeta) = \exp\left[-M\eta - \frac{M}{A}(1+M)(1-\cos B\zeta)\right] \tag{32}$$

Then the solution of equation (22) is given by,

$$c(x, t) = c_0 \exp\left[-mx + \frac{m}{b}(u_0 + mD_0)(1-\cos bt)\right] \tag{33}$$

5. Result and Discussion

The nature of the solution discussed in above section of the paper is studied in terms of spatial and temporal variation of concentration (c).

Figures (1) and (2) show the spatial variation of concentration (c). In the present study, the spatial variation of the concentration (c) is obtained with four different values of $D_0(km^2/year)$ (i.e. 0.15, 0.20, 0.25, 0.30). All the plots of spatial variation in case-1 and case-2 have been derived at fixed time $t = 0.3$ year. The initial value of concentration $c_0 = 1$ and velocity $u_0 = 0.25$ have been taken as input for the calculation. The parameter a and m are considered $1.2(year^{-1})$ and $4(km^{-1})$ respectively in case-1 and case-2 in the calculation of spatial and temporal variation. The spatial variation is shown in the finite domain of $x = 0.0001 - 1.0(km)$. Figure (1) shows that the concentration c is decreasing exponentially with x for all D_0s. The minimum concentration (c) is obtained with $D_0 = 0.15$ and the maximum value of the concentration (c) is obtained with $D_0 = 0.30$. This proves that increase in the dispersion co-efficient, increases the concentration (c).

It is observed that the concentration lines for different dispersion are more overlapping in the case of sinusoidal then in the exponential. This shows that the effect of variation of dispersion on concentration is more in the exponential case than in the sinusoidal case. It is also observed that the concentration values at $t = 0.3$ for low values of x are lower in sinusoidal case than the exponential one. The numerical values of spatial variation in case-1 and case-2 are given in Table (1) and Table (2), respectively.

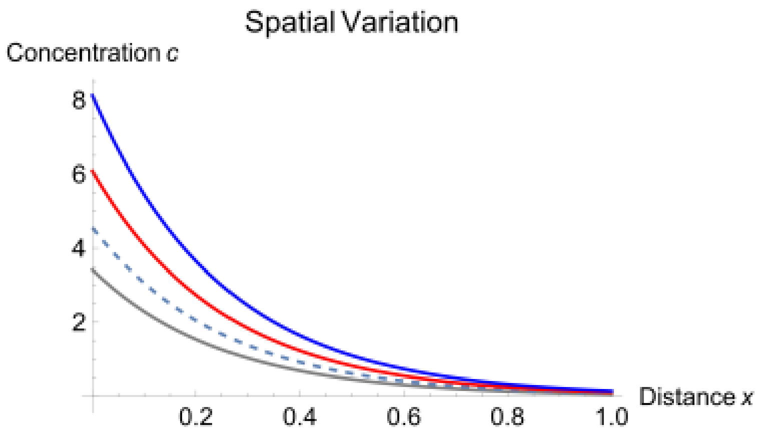

Figure 1. Spatial variation of concentration (c) (Case-1).

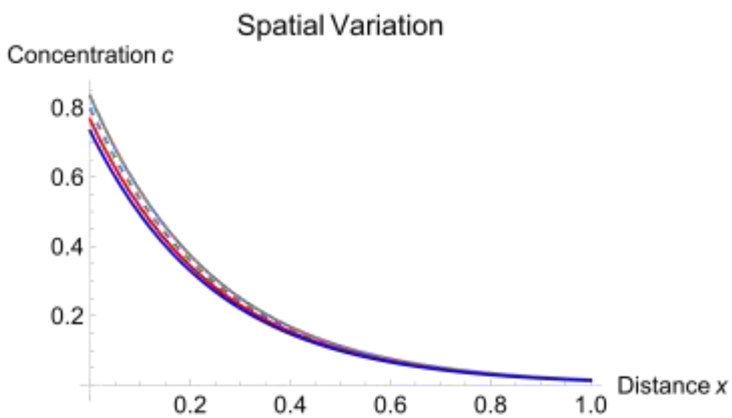

Figure 2. Spatial variation of concentration (c) (Case-2).

Table 1. Contaminant concentration at various values of dispersion coefficient. Case 1(Spatial variation).

	Concentration(c)			
x	$D_0 = 0.15$	$D_0 = 0.20$	$D_0 = 0.25$	$D_0 = 0.35$
0.0001	3.412232	4.555125	6.080818	8.117527
0.0501	2.793699	3.729421	4.978553	6.646069
0.1001	2.287288	3.053392	4.076094	5.441341
0.1501	1.872673	2.499906	3.337224	4.454993

0.2001	1.533215	2.046750	2.732288	3.647440
0.2501	1.255290	1.675737	2.237008	2.986271
0.3001	1.027745	1.371977	1.831507	2.444952
0.3501	0.841446	1.123280	1.499511	2.001757
0.4001	0.688918	0.919664	1.227696	1.638900
0.4501	0.564038	0.752957	1.005152	1.341818
0.5001	0.461795	0.616469	0.822949	1.098588
0.5501	0.378086	0.504722	0.673774	0.899448
0.6001	0.309551	0.413232	0.551639	0.736405
0.6501	0.253439	0.338325	0.451644	0.602918
0.7001	0.207498	0.276997	0.369775	0.493627
0.7501	0.169885	0.226786	0.302746	0.404148
0.8001	0.139090	0.185677	0.247868	0.330888
0.8501	0.113877	0.152019	0.202937	0.270908
0.9001	0.093235	0.124463	0.166151	0.221801
0.9501	0.076334	0.101902	0.136033	0.181595

Figure (3) and figure (4) show the temporal variation of the concentration (c). To maintain consistency, the temporal variation is also obtained for the same $D_0 s$ as in the case of spatial variation. Further, input parameters a and m are also taken with the same values as in the spatial variation. The space variable x is fixed with 0.2km in temporal variation. The temporal variation of concentration is calculated in the time interval $0.0001 \leqslant t(year) \leqslant 0.2$. An exponential increase is found in concentration (c) with time for all four $D_0 s$ under study. The concentration (c) increases with increase in D_0. It can be seen that the concentration values at $x = 0.2$ for higher t are higher in the exponential case than in the sinusoidal case. Table (3) and Table (4) show the contaminant concentration for temporal variation for case-1 and case-2, respectively.

Table 2. Contaminant concentration at various values of dispersion coefficient. Case 2(Spatial variation).

	Concentration(c)			
x	$D_0 = 0.15$	$D_0 = 0.20$	$D_0 = 0.25$	$D_0 = 0.35$
0.0001	0.833580	0.798707	0.765293	0.733277
0.0501	0.682478	0.653926	0.626569	0.600356
0.1001	0.558765	0.535389	0.512991	0.491530
0.1501	0.457478	0.438340	0.420002	0.402431
0.2001	0.374552	0.358882	0.343868	0.329483
0.2501	0.306657	0.293828	0.281536	0.269757
0.3001	0.251069	0.240566	0.230502	0.220859
0.3501	0.205558	0.196959	0.188719	0.180824
0.4001	0.168297	0.161256	0.154510	0.148046
0.4501	0.137790	0.132025	0.126502	0.121210
0.5001	0.112813	0.108093	0.103571	0.099238
0.5501	0.092363	0.088499	0.084797	0.081249
0.6001	0.075621	0.072457	0.069426	0.066521
0.6501	0.061913	0.059323	0.056841	0.054463
0.7001	0.050690	0.048569	0.046538	0.044591
0.7501	0.041502	0.039765	0.038102	0.036508
0.8001	0.033979	0.032557	0.031195	0.029890
0.8501	0.027819	0.026655	0.025540	0.024472
0.9001	0.022777	0.021824	0.020911	0.020036
0.9501	0.018648	0.017868	0.017120	0.016404

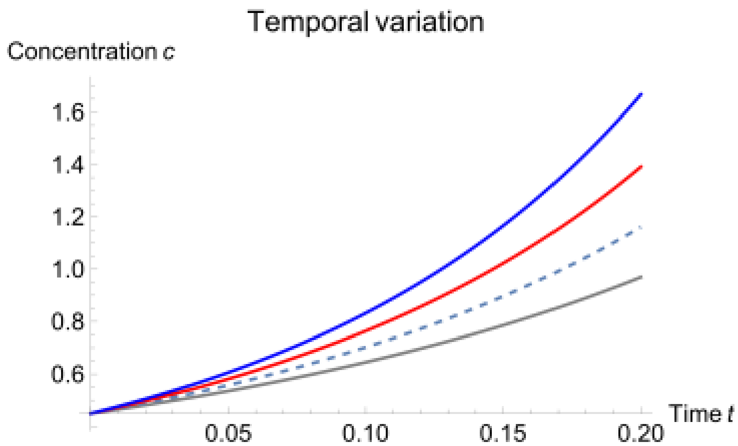

Figure 3. Temporal variation of concentration (c)(Case-1).

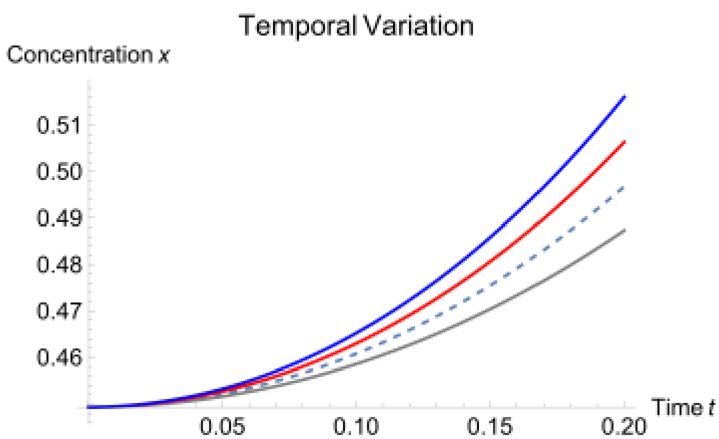

Figure 4. Temporal variation of concentration (c) (Case-2).

Table 3. Contaminant concentration at various values of dispersion coefficient. Case 1(Temporal variation).

	Concentration(c)			
t	$D_0 = 0.15$	$D_0 = 0.20$	$D_0 = 0.25$	$D_0 = 0.35$
0.0001	0.449482	0.449518	0.449554	0.449590
0.0101	0.465124	0.468921	0.472748	0.476607
0.0201	0.481510	0.489411	0.497441	0.505603

0.0301	0.498681	0.511060	0.523746	0.536747
0.0401	0.516683	0.533945	0.551785	0.570220
0.0501	0.535564	0.558151	0.581690	0.606223
0.0601	0.555375	0.583766	0.613608	0.644976
0.0701	0.576172	0.610888	0.647695	0.686719
0.0801	0.598014	0.639620	0.684121	0.731719
0.0901	0.620962	0.670075	0.723074	0.780264
0.1001	0.645083	0.702375	0.764755	0.832676
0.1101	0.670451	0.736651	0.809387	0.889305
0.1201	0.697140	0.773043	0.857210	0.950541
0.1301	0.725233	0.811706	0.908489	1.016811
0.1401	0.754819	0.852804	0.963510	1.088586
0.1501	0.785991	0.896518	1.022589	1.166388
0.1601	0.818850	0.943042	1.086071	1.250792
0.1701	0.853504	0.992586	1.154332	1.342436
0.1801	0.890071	1.045379	1.227788	1.442025
0.1901	0.928674	1.101669	1.306890	1.550340

Table 4. Contaminant concentration at various values of dispersion coefficient. Case 2(Temporal variation).

	Concentration(c)			
t	$D_0 = 0.15$	$D_0 = 0.20$	$D_0 = 0.25$	$D_0 = 0.35$
0.0001	0.301194	0.301194	0.301194	0.301194
0.0101	0.301132	0.301117	0.301102	0.301087
0.0201	0.300946	0.300888	0.300829	0.300771
0.0301	0.300638	0.300507	0.300377	0.300246

0.0401	0.300208	0.299976	0.299745	0.299514
0.0501	0.299656	0.299296	0.298935	0.298576
0.0601	0.298984	0.298466	0.297949	0.297434
0.0701	0.298192	0.297490	0.296789	0.296090
0.0801	0.297281	0.296367	0.295457	0.294549
0.0901	0.296252	0.295101	0.293954	0.292812
0.1001	0.295107	0.293693	0.292286	0.1001
0.1101	0.293848	0.292146	0.290453	0.288770
0.1201	0.292476	0.290461	0.288460	0.286474
0.1301	0.290992	0.288642	0.286311	0.283999
0.1401	0.289400	0.286692	0.284010	0.281353
0.1501	0.287700	0.284614	0.281561	0.278540
0.1601	0.285896	0.282410	0.278968	0.275567
0.1701	0.283989	0.280085	0.276236	0.272439
0.1801	0.281982	0.277642	0.273370	0.269163
0.1901	0.279878	0.275085	0.270375	0.265746

6. Conclusion

The LVIM method has been applied to propose an analytic solution of one dimensional solute transport equation for uniform unsteady flow in homogeneous medium. The applied method LVIM has been found suitable to obtain the analytic solution for the said physical conditions. The presently adopted method is less complected method than those which require discretization or linearization. A rapidly converging more realistic series solution obtained by LVIM reduces the lengthiness of the computation. Through the present work, it becomes possible to express the close form of a solution with appropriate initial condition. Initially concentration is exponentially decreasing with space variable. Exponentially increasing and sinusoidal temporal variation in dispersion is directly proportional to the flow velocity. An exponentially decreasing nature of concentration variation with distance is predicted for the solution obtained, whereas the exponentially increasing and sinusoidal nature of temporal variation is proposed by LVIM. The presented nature of both of the variations may help as an effective predictive tool in many groundwater hydrological problems, since it provides the concentration at required time and position.

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