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PROPERTIES OF ADJACENT LINE GRAPHS

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Abstract

The adjacent line graph of a connected graph G is the graph, whose vertices are the sets of adjacent edges of G and two such sets are adjacent if an edge of one is adjacent to an edge of the other. In this paper we obtain formula for order and size of the adjacent line graphs and investigate some of its properties.

1. Introduction

The line graph of a graph G, denoted by $L(G)$, is defined to have as its vertices the edges of G, with two being adjacent if the corresponding edges have a common vertex in G. The concept of super line graph of a graph was introduced by K. S. Bagga, L. W. Beineke and B. N. Varma, in [1], which is a generalization of the notion of line graph. If G is a given graph, then its super line graph of index r, denoted by $\mathcal{L}_r(G)$, is defined as the graph whose vertices are the r-subsets of $E(G)$, and two vertices U and V are adjacent if there exist $u \in U$ and $v \in V$ such that u and v are adjacent edges in G. Properties of super line graphs were presented in [1], [2] and [3].

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Several results of the super line graph of index 2 are given in [4] and [5]. We have introduced in [8] the concept of adjacent line graph of a connected graph G.

The adjacent line graph of G, denoted by $\mathcal{AL}(G)$, is defined for any connected graph with at least two edges. Its vertices are the sets of adjacent edges of G, two such sets are adjacent if an edge of one is adjacent to an edge of the other. In [9], the diameter of adjacent line graphs of path P_n, cycle C_n and star graph $K_{1,n}$ were discussed. One may refer [6, 7] for basic concepts of graph theory.

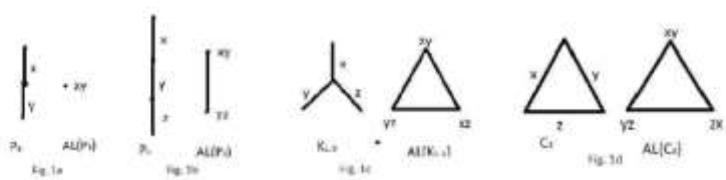
In this paper we study some of the properties of adjacent line graphs of some special graphs. In section 2 we present some results of adjacent line graphs of path (P_n), cycle (C_n) and star graph $(K_{1,n})$.

In section 3 we obtain formula for the order and size of the adjacent line graph and use it to find the order and size of a crown graph, bistar graph, friendship graph, wheel graph, complete graph and complete bipartite graph. Also we establish some properties of adjacent line graphs.

2. Preliminaries

Definition 2.1. Let $G = (V, E)$ be a simple connected graph with $|V(G)| \geq 3$. The adjacent line graph (or adjacent edge graph) of G, denoted by $\mathcal{AL}(G)$, is the graph whose vertices are the sets of adjacent edges of G and two such sets are adjacent if an edge of one is adjacent to an edge of the other.

Example 2.2



Theorem 2.3 [8]. *If G is the graph $P_n (n \geq 6)$, then $\mathcal{AL}(G)$ contains exactly two vertices of degree* 2, *two vertices of degree* 3 *and the remaining* $n - 6$ vertices are of degree 4.

Theorem 2.4 [8]**.** *The adjacent line graph of the cycle* $C_n (n \geq 5)$ *is* 4-*regular*.

Theorem 2.5 [8]**.** *The adjacent line graph of the star* $K_{1,n}$ *is the complete graph* K_{nC_2}.

Definition 2.6. Crown graphs are defined as the graphs obtained by joining m pendant edges to each vertex of C_n, denoted $C_n \odot mK_1$. Bistar is the graph obtained by joining the root(apex) vertices of two copies of Star $K_{1,n}$. The graph obtained by joining n copies of cycle C_3 to a new vertex is called Friendship graph. Wheel graph W_n is constructed by joining all vertices of a cycle C_n to a new vertex, called center. Examples of above special graphs are given below.

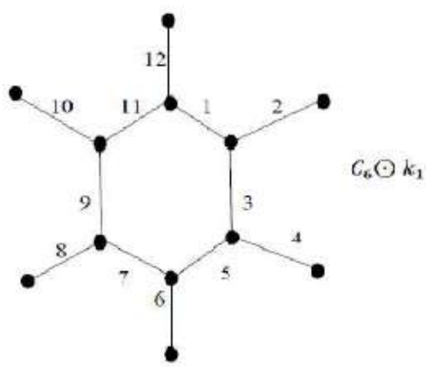


Figure 2a.

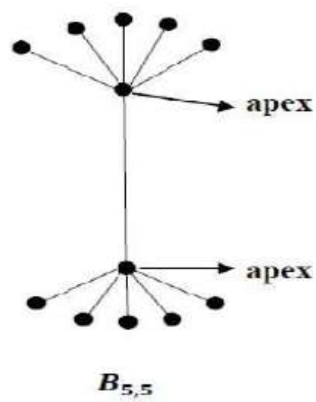


Figure 2b.

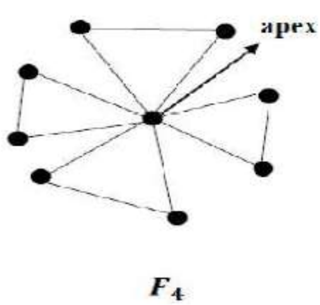


Figure 2c.

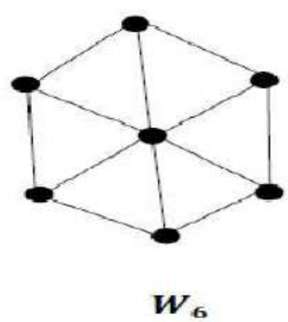


Figure 2d.

Definition 2.7. Let G be a graph. For a vertex $u \in V(G)$, the neighborhood of u is defined as the set of all adjacent vertices of u in G and it is denoted by $N(u)$.

If u and v are any two vertices of G, then $\mu(uv)$ is defined as

$$\mu(uv) = \begin{cases} 1 & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

3. Some Properties of $\mathcal{AL}(G)$

In this section, we obtain the formula for the order and size of adjacent line graph $\mathcal{AL}(G)$ and establish some properties of $\mathcal{AL}(G)$. Let $x(uv)$ and $y(vw)$ be two adjacent edges in G and v be the common end vertex of x and y. Then xy is a vertex in $\mathcal{AL}(G)$.

Theorem 3.1. *If G is a connected graph with n vertices, then*

i. $|V(\mathcal{AL}(G))| = \sum_{i=1}^{n} \binom{d_i}{2}$, *where d_i is the degree of the i^{th} vertex.*

ii. *For a vertex $xy \in V(\mathcal{AL}(G))$,*

$$\deg(xy) = \binom{d(u)}{2} + \binom{d(v)}{2} + \binom{d(w)}{2} - 1 + \sum_{t \in M - \{u, v, w\}} \left(\binom{d(t)}{2} - \binom{d(t)-s}{2} \right)$$

where $M = N(u) \cup N(v) \cup N(w)$ and $s = \mu(ut) + \mu(vt) + \mu(wt)$

Proof. Let G be a simple connected graph with $|V(G)| = n$. We know that adjacent edges have a common vertex and the number of edges incident on a vertex is the degree of that vertex. Therefore the number of vertices in $\mathcal{AL}(G)$ corresponding to a vertex in G is equal to $\binom{d}{2}$, where d is the degree of that vertex. Hence the total number of vertices in the adjacent line graph of G is

$$|V(\mathcal{AL}(G))| = \sum_{i=1}^{n} \binom{d_i}{2}.$$

Let x and y be the adjacent edges in G, then $xy \in V(\mathcal{AL}(G))$. Let v be the common end vertex of x and y and u and w are other end vertices of x and y

respectively. Since the degree of the vertex v contributes $\binom{d(v)}{2}$, vertices to $\mathcal{AL}(G)$ and xy is one among them, $\binom{d(v)}{2} - 1$ vertices are adjacent to xy. Also the degree of the vertex u and the degree of the vertex w together contribute $\binom{d(u)}{2} + \binom{d(w)}{2}$ vertices to $\mathcal{AL}(G)$ and these vertices are also adjacent to xy. Finally, the degree of the vertices in the set of vertices in $M - \{u, v, w\}$ also contribute some vertices adjacent to xy. If $t \in M - \{u, v, w\}$ and t is adjacent to any one of u, v and w, then $\binom{d(t)}{2} - \binom{d(t)-1}{2}$ vertices are adjacent to xy, because the degree of the vertex t contributes $\binom{d(t)}{2}$ vertices to $\mathcal{AL}(G)$ out of which $\binom{d(t)-1}{2}$ edges are not adjacent to xy. Similarly, if t is adjacent to any two of them, then $\binom{d(t)}{2} - \binom{d(t)-2}{2}$ vertices are adjacent to xy and if t is adjacent to all the three then $\binom{d(t)}{2} - \binom{d(t)-3}{2}$ vertices are adjacent to xy. Thus the total number of vertices adjacent to xy is

$$\binom{d(u)}{2} + \binom{d(v)}{2} + \binom{d(w)}{2} - 1 + \sum_{t \in M - \{u, v, w\}} \left(\binom{d(t)}{2} - \binom{d(t)-s}{2} \right)$$

Hence

$$\deg(xy) = \binom{d(u)}{2} + \binom{d(v)}{2} + \binom{d(w)}{2} - 1 + \sum_{t \in M - \{u, v, w\}} \left(\binom{d(t)}{2} - \binom{d(t)-s}{2} \right)$$

We apply the above theorem to find the orders and sizes of adjacent line graphs of crown graph $C_n \odot K_1$, friendship graph F_n, bistar graph $B_{n,n}$, wheel graph W_n, complete graph K_n and complete bipartite graph $K_{m,n}$.

Theorem 3.2. *The order and size of the crown graph, friendship graph, bistar graph and the wheel graph are*

i. $|V(\mathcal{AL}(C_n \odot K_1))| = 3n$ *and* $|E(\mathcal{AL}(C_n \odot K_1))| = 15n$

ii. $|V(\mathcal{AL}(F_n))| = n(2n+1)$ *and* $|E(\mathcal{AL}(F_n))| = \dfrac{n(2n+1)(2n^2+n-1)}{2}$

iii. $|V(\mathcal{AL}(B_{n,n}))| = n(n+1)$ *and* $|E(\mathcal{AL}(B_{n,n}))| = \frac{n}{4}(n^3 + 6n^2 - n - 2)$

iv. $|V(\mathcal{AL}(W_n))| = \frac{n(n+5)}{2}$ *and* $|E(\mathcal{AL}(W_n))| = \frac{n}{8}(n^3 + 6n^2 + 31n + 26)$

Proof.

i. The Crown graph $C_n \odot K_1$ has n vertices of degree 3 and n pendant vertices. Thus the order of $\mathcal{AL}(C_n \odot K_1)$ is $n\binom{3}{2} = 3n$.

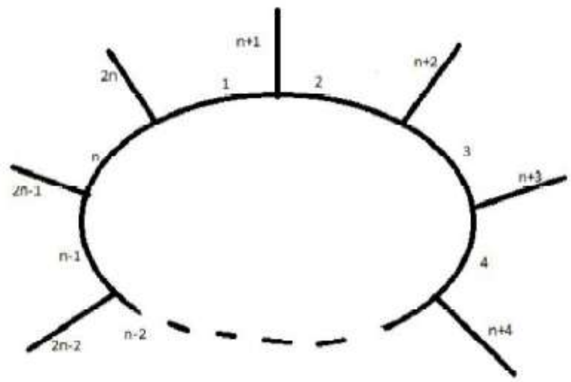


Figure 3.

The vertices of $\mathcal{AL}(C_n \odot K_1)$ are of two types: type I is formed by the adjacent edges on the cycle and type II is formed by an edge on the cycle along with adjacent pendant edge.

There are n vertices of type I and $2n$ vertices of type II in $\mathcal{AL}(C_n \odot K_1)$.

The degree of type I vertex is $\binom{3}{2} + \binom{3}{2} + \binom{3}{2} - 1 + 2 + 2 = 12$.

The degree of type II vertex is $\binom{3}{2} + \binom{3}{2} + 0 - 1 + 2 + 2 = 9$.

Therefore the sum of the degrees of all the vertices is $12n + 9(2n) = 30n$. Hence the size of the graph $\mathcal{AL}(C_n \odot K_1)$. is $15n$.

ii. The Friendship graph F_n has one vertex of degree $2n$ and $2n$ vertices of degree 2.

Therefore the number of vertices in $\mathcal{AL}(F_n)$ is $\binom{2n}{2} + 2n\binom{2}{2} = n(2n+1)$.

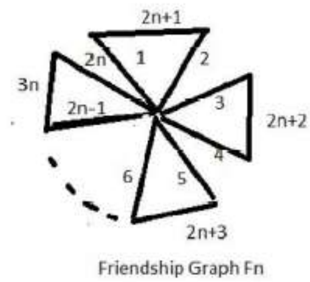


Figure 4.

The degree of the vertex $(1, 2)$ in $\mathcal{AL}(F_n)$ is

$$\binom{2}{2} + \binom{2n}{2} + \binom{2}{2} - 1 + \sum_{1}^{2n-2} 1 = 2n^2 + n - 1.$$

The degree of the vertex $(1, 2n+1)$ in $\mathcal{AL}(F_n)$ is

$$\binom{2n}{2} + \binom{2}{2} + \binom{2}{2} - 1 + \sum_{1}^{2n-2} 1 = 2n^2 + n - 1.$$

Since all the vertices are of degree $2n^2 + n - 1$, the sum of the degrees of all the vertices of $\mathcal{AL}(F_n)$ is $n(2n+1)(2n^2 + n - 1)$.

Hence the size of $\mathcal{AL}(F_n) = \dfrac{n(2n+1)(2n^2 + n - 1)}{2}$.

iii. The bistar graph $B_{n,n}$ contains two vertices of degree $n+1$ and $2n$ vertices of degree 1.

So the two vertices of degree $n+1$ contributes $2\binom{n+1}{2}$ vertices to $\mathcal{AL}(B_{n,n})$.

Hence the order of the graph $\mathcal{AL}(B_{n,n})$ is $n(n+1)$.

Let us label the edges of $B_{n,n}$ by $1, 2, 3, \ldots, 2n, 2n+1$ as in figure.

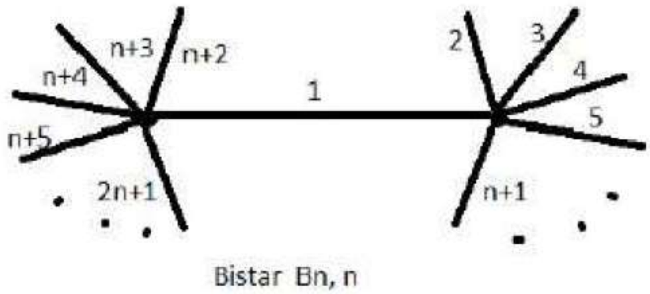


Figure 5.

The degree of vertex (1, 2) in $\mathcal{AL}(B_{n,n})$ is $\binom{n+1}{2} + 0 + \binom{n+1}{2} - 1 + 0 = n(n+1) - 1.$

The degree of vertex (2, 3) in $\mathcal{AL}(B_{n,n})$ is

$$\binom{n+1}{2} + 0 + 0 - 1 + \left(\binom{n+1}{2} - \binom{n}{2}\right) = \frac{n^2 + 3n - 2}{2}.$$

There are $2n$ vertices of type (1, 2) and $n(n-1)$ vertices of type (2, 3).

Hence the size of $\mathcal{AL}(B_{n,n})$ is $\frac{n}{4}(n^3 + 6n^2 - n - 2).$

iv. The Wheel graph W_n has one vertex of degree n and n vertices of degree 3. Therefore the order of adjacent line graph of wheel graph is $|V(\mathcal{AL}(W_n))| = \frac{n(n+5)}{2}.$

Let us label the edges of W_n by 1, 2, 3, ..., $2n$ as in figure.

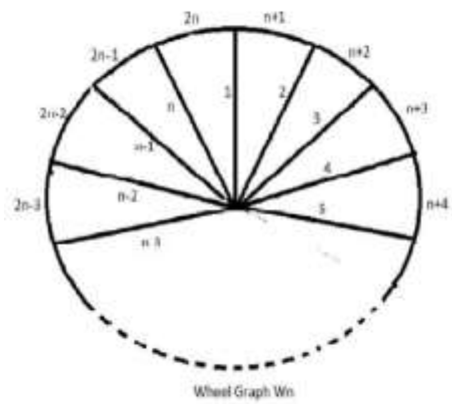


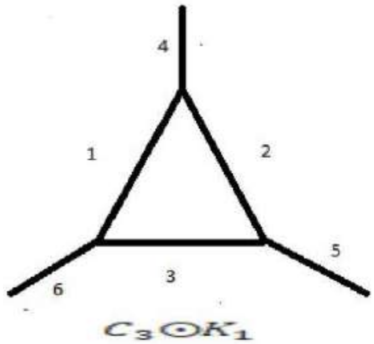
Figure 6.

There are five types of vertices in $\mathcal{AL}(W_n)$ formed by the adjacent edges of W_n. There are n vertices of each type $(n+1, n+2)$, $(1, 2)$, and $(1, 3)$, $2n$ vertices of type $(1, n+1)$ and remaining $\frac{n(n-5)}{2}$ vertices of type $(1, 4)$. So using the above theorem the sum of the degrees of all the vertices in $\mathcal{AL}(W_n)$ is $\frac{n}{4}(n^3 + 6n^2 + 31n + 26)$ (after simplification). Hence $|E(\mathcal{AL}(W_n))| = \frac{n}{8}(n^3 + 6n^2 + 31n + 26)$.

Example 3.3.

1. The order and size of $\mathcal{AL}(C_3 \odot K_1)$ are, respectively, 9 and 36, which is a complete graph K_9.

2. The order and size of $\mathcal{AL}(C_4 \odot K_1)$ are, respectively, 12 and 58.



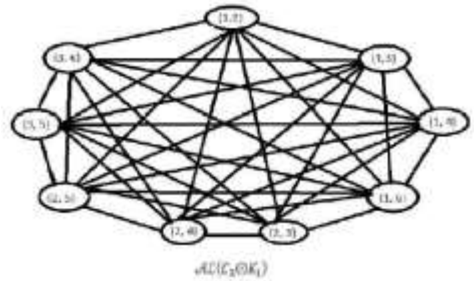


Figure 7a. **Figure 7b.**

It might be expected that the adjacent line graphs are so dense, but the below theorem suggest that they are not as dense as one might expect. For example, $\mathcal{AL}(K_n)$ has order $0(n^3)$ and size $0(n^5)$.

Theorem 3.4. *The size of the graphs* $\mathcal{AL}(K_n)$ *and* $\mathcal{AL}(K_{m,n})$ *are, respectively*,

$$\frac{n(n-1)(n-2)}{8}(9n^2 - 45n + 58) \text{ and}$$

$$\frac{mn}{4}(m^3 + n^3 + 9mn(m+n) - 11(m^2 + n^2 - 40mn + 30(m+n) - 18)$$

Proof. The order of $\mathcal{AL}(K_n)$ is $\frac{n(n-1)(n-2)}{2}$, as there are n vertices, each of degree $n-1$ in K_n.

The degree of a vertex $xy \in E(\mathcal{AL}(K_n))$ can be found, using the Theorem 3.1, as:

$$\binom{n-1}{2} + \binom{n-1}{2} + \binom{n-1}{2} - 1 + (n-3)\left(\binom{n-1}{2} - \binom{n-4}{2}\right) = \frac{1}{2}(9n^2 - 45n + 58)$$

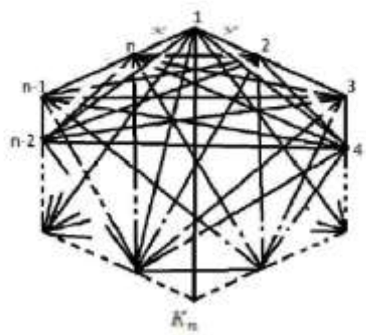


Figure 8.

As there are $\frac{n(n-1)(n-2)}{2}$ such vertices, the sum of the degrees of all the vertices is

$$\frac{n(n-1)(n-2)}{4}(9n^2 - 45n + 58).$$

Hence the size of $\mathcal{AL}(K_n)$ is $\frac{n(n-1)(n-2)}{4}(9n^2 - 45n + 58).$

Now consider the graph $K_{m,n}$, it has m vertices of degree n and n vertices of degree m, so the order of $\mathcal{AL}(K_{m,n})$ is $m\binom{n}{2} + n\binom{m}{2}$ $= \frac{mn}{2}(m + n - 2).$

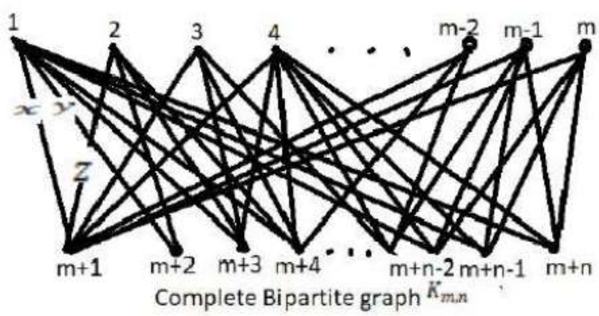


Figure 9.

There are two types of vertices in $\mathcal{AL}(K_{m,n})$ formed by the adjacent edges of $K_{m,n}$.

The degree of type I (e.g., xy) is

$$\binom{m}{2}+\binom{n}{2}+\binom{m}{2}-1+(n-2)\left(\binom{m}{2}-\binom{m-1}{2}\right)+(m-1)\left(\binom{n}{2}-\binom{n-2}{2}\right)$$

$$= 2m^2+n^2+7mn-13m-8n+9$$

Therefore sum of all degrees of type I vertices

$$= \frac{mn(n-1)}{2}(2m^2+n^2+7mn-13m-8n+9)$$

The degree of type II (e.g., xz) is

$$\binom{n}{2}+\binom{m}{2}+\binom{n}{2}-1+(m-2)\left(\binom{n}{2}-\binom{n-1}{2}\right)+(n-1)\left(\binom{m}{2}-\binom{m-2}{2}\right)$$

$$= 2n^2+m^2+7mn-13n-8m+9$$

Therefore sum of all degrees of type II vertices

$$= \frac{nm(n-1)}{2}(2n^2+m^2+7mn-13n-8m+9)$$

Hence the size of $\mathcal{AL}(K_{m,n})$ is

$$\frac{mn}{4}(m^3+n^3+9mn(m+n)-11(m^2+n^2-40mn+30(m+n)-18)).$$

Remark 3.5. From Figure 1d, it is observed that $C_3 \cong \mathcal{AL}(C_3)$. So naturally one question may arise that, For what types of graphs G, G and $\mathcal{AL}(G)$ are isomorphic? The following theorem gives answer to this question.

Theorem 3.6. *Let G be a simple connected graph with at least 3 vertices. Then $\mathcal{AL}(G)$ is isomorphic to G if and only if $G \cong C_3$.*

Proof. Let us assume that $\mathcal{AL}(G) \cong G$. Let $|V(G)| = n$, $|E(G)| = m$ and $d = (d_1, d_2, \ldots, d_n)$ be the degree sequence of G. Then $|V(\mathcal{AL}(G))| = \sum_{i=1}^{n}\binom{d_i}{2}$ and $\sum_{i=1}^{n} d_i = 2m$.

Since $\mathcal{AL}(G) \cong G$, we have $\sum_{i=1}^{n}\binom{d_i}{2} = n$.

That is, $d_1(d_1-1)+d_2(d_2-1)+d_3(d_3-1)+\ldots+d_n(d_n-1) = 2n$.

For a connected graph, this is possible if and only if $d_i = 2$ for each $i = 1, 2, \ldots, n$.

So, we have $m = n$. Hence G must be the cycle C_n. But $|E(\mathcal{AL}(C_n))| > n$, if $n \geq 4$. Thus $|E(\mathcal{AL}(C_n))| = n$, if and only if $n = 3$. That is, if $\mathcal{AL}(G) \cong G$ then $G \cong C_3$.

Converse is trivial.

Remark 3.7. From Figure 1c and Figure1d, it is observed that $\mathcal{AL}(K_{1,3}) \cong \mathcal{AL}(C_3)$. Also the graphs $\mathcal{AL}(K_{1,4}) \cong \mathcal{AL}(Z_2)$. So there are graphs G_1 and G_2 such that $G_1 \ncong G_2$ but $\mathcal{AL}(G_1) \cong \mathcal{AL}(G_2)$.

With this observation we pose the following problem:

Find all graphs G_1 and G_2 such that $G_1 \ncong G_2$ but $\mathcal{AL}(G_1) \cong \mathcal{AL}(G_2)$.

Example 3.8. The adjacent line graphs of the following graphs G_1 and G_2 and G_3 and G_4 are isomorphic.

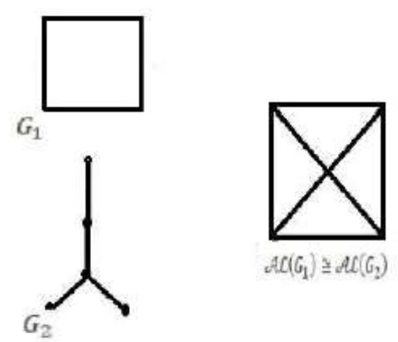


Figure 10a.

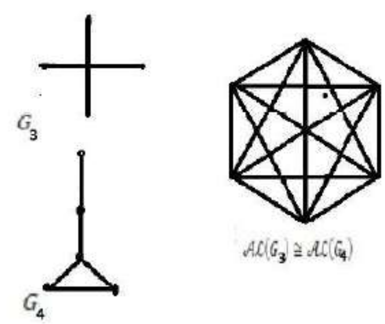


Fig. 10b.

Theorem 3.9. *Let G be a simple connected graph with at least* 3 *vertices. Then* $\mathcal{AL}(G)$ *is a tree if and only if G is either* P_3 or P_4.

Proof. Assume that G is a graph other than P_3 or P_4. Then, it is clear that $\mathcal{AL}(G)$ contains at least one cycle. Therefore $\mathcal{AL}(G)$ is not a tree. Thus if $\mathcal{AL}(G)$ is a tree then G must be either P_3 or P_4.

Conversely, assume that G is P_3 or P_4. We know that $\mathcal{AL}(P_3)$ is a trivial graph and $\mathcal{AL}(P_4)$ is P_2. Hence $\mathcal{AL}(G)$ is a tree if G is P_3 or P_4.

Graphs can be isomorphic to a subgraph of their derived graphs. So finding all graphs G such that G is isomorphic to a subgraph of $\mathcal{AL}(G)$ is one such problem. The following proposition shows that the graphs C_n and K_n are subgraphs of their adjacent line graphs.

Proposition 3.10.

1. $C_n \subseteq \mathcal{AL}(C_n)$, $n \geq 3$
2. $K_n \subseteq \mathcal{AL}(K_n)$, $n \geq 3$

Proof. 1. Let the edges of C_n be labelled as $1, 2, 3, \ldots, n$. Then the vertices of $\mathcal{AL}(C_n)$ are $12, 23, 34, \ldots, (n-1)n, n1$. These n vertices form of a cycle of length n in $\mathcal{AL}(C_n)$ as each vertex is adjacent to the succeeding vertex and also the vertex $n1$ is adjacent to the vertex 12. Thus $C_n \subseteq \mathcal{AL}(C_n)$, $n \geq 3$.

2. We know that adjacent line graph of a complete graph K_n is a complete graph of order $n\left(\frac{n-1}{2}\right)$ i.e., $\mathcal{AL}(K_n) = K_{n\left(\frac{n-1}{2}\right)}$.

Thus $K_n \subseteq \mathcal{AL}(K_n)$.

Definition 3.11. A connected graph G is said to be pancyclic if it contains all the cycles of length from 3 to the order of the graph.

Theorem 3.12. *If C_n, $n \geq 3$ is a cycle, then $\mathcal{AL}(C_n)$ is pancyclic.*

Proof. We know that, if $n = 3, 4$ or 5, then $\mathcal{AL}(C_3)$, $\mathcal{AL}(C_4)$ and $\mathcal{AL}(C_5)$ are complete graphs and hence pancyclic.

Consider $n \geq 6$. The adjacent line graph of a cycle of length $n \geq 5$ is a 4-regular graph on n vertices.

Let the edges of C_n be labelled as $1, 2, 3, \ldots, n$. Then the vertices of $\mathcal{AL}(C_n)$ are $12, 23, 34, \ldots, (n-1)n, n1$ Then a cycle of length 3 in $\mathcal{AL}(C_n)$ is

12, 34, 23, 12, a cycle of length 4 in $\mathcal{AL}(C_n)$ is 12, 34, 45, 23, 12 and a cycle of length 5 in $\mathcal{AL}(C_n)$ is 12, 34, 56, 45, 23, 12. In general, the cycle 12, 34, 45, 56, …, $n(n+1)$, $(n-1)n$, $(n-3)(n-2)$, …, 23, 12 defines a cycle of odd length in $\mathcal{AL}(C_n)$. The cycle 12, 34, 45, …, $n(n+1)$, $(n-2)(n-1)$, $(n-3)(n-2)$, …, 23, 12 defines a cycle of even length in $\mathcal{AL}(C_n)$. Hence $\mathcal{AL}(C_n)$ is pancyclic.

Definition 3.13. Let G be a simple connected graph. The distance between any two vertices u and v in G is equal to the length of a shortest path joining u and v and is denoted by $d(u, v)$. The diameter of G, denoted by $diam(G)$ is the maximum distance between any two pair of vertices of G.

Theorem 3.14. *Let G be a connected graph with $n \geq 3$. Then $diam(\mathcal{AL}(G)) = 1$, if $G \cong K_n$ or $G \cong K_{1,n}$.*

Proof. If $G \cong K_n$, then $\mathcal{AL}(K_n) = K_{n\binom{n-1}{2}}$ and $diam(\mathcal{AL}(K_n)) = diam(K_{n\binom{n-1}{2}}) = 1$.

If $G \cong K_{1,n}$, then $\mathcal{AL}(K_{1,n}) = K_{\binom{n}{2}}$ and $diam(\mathcal{AL}(K_{1,n})) = diam(K_{\binom{n}{2}}) = 1$.

Remark 3.15. Converse of the above theorem is not true. That is, if $diam(\mathcal{AL}(G)) = 1$ then G may not be isomorphic to K_n or $K_{1,n}$. For example, consider the graph G_4 in Example 3.8.

For this graph the diameter is 3 but $diam(\mathcal{AL}(G_4)) = 1$.

Theorem 3.16. *Let G be a connected graph with $n \geq 3$. If $diam(G) \leq 3$. then $diam(\mathcal{AL}(G)) \leq 2$.*

Proof. Let the diameter of the graph G be at most 3. Then each vertex of G is at a distance of at most 3 from every other vertices. Hence by the definition of adjacent line graph, each vertex of $\mathcal{AL}(G)$ is at a distance of at most 2 from every other vertices. Thus the diameter of $\mathcal{AL}(G)$ is at most 2.

Remark 3.17. If $diam(\mathcal{AL}(G)) \leq 2$ then $diam(G) \nleq 3$. For example, consider the graph P_5, the path on five vertices. We know that $\mathcal{AL}(P_5) = K_3$. Therefore $diam(\mathcal{AL}(P_5)) = diam(K_3) = 1$, but $diam(P_5) = 4$.

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