Advances and Applications in Mathematical Sciences
Volume 21, Issue 6, April 2022, Pages 2989-3009

A STATISTICAL HYPOTHESES TECHNIQUES OF ONE WAY ANALYSIS OF VARIANCE USING TRIANGULAR FUZZY NUMBERS: A COMPARATIVE STUDY

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Abstract

The Analysis of Variance (ANOVA) is a statistical technique commonly used to compare several population means which are simultaneous. The classical ANOVA model's statistical analysis gives inference either accept or reject the null hypothesis at a level of significance. This paper proposed a comparative study on the statistical testing of ANOVA for one way with α-cut interval method, Exponential Triangular Fuzzy Numbers (ETFNs), and Logarithmic Triangular Fuzzy Numbers (LTFNs) using the Triangular Fuzzy Numbers (TFNs) based on the decision rule and illustrated with numerical examples.

1. Introduction

The Analysis of Variance (ANOVA) is one of the potent methods of statistical analysis. In order to measure the equality of means among several populations, an analysis of variance is used. The variability of the means of the different populations is tested. One-way ANOVA is a technique in which only one independent variable is considered at various stages, influencing the response variable. The values obtained from the experiment results are fuzzy

2020 Mathematics Subject Classification: 62F03, 62K86.
Keywords: Classical ANOVA, Fuzzy ANOVA, TFNs, α-Cut Interval Method, ETFNs, LTFNs, Decision Rule.
Received April 28, 2021; Accepted July 22, 2021

in many applied sciences, the theory of fuzzy sets must therefore be used to model and deal with these sciences' experimental results. Since the theory of fuzzy sets was introduced by Zadeh [1965] to the scientific community, this theory has been used in various science fields by many authors. Some recent works on ANOVA are discussed below in an exponential fuzzy nature. Dutta et al. [2011] suggested a counter-argument by showing that the α-cut approach is sufficiently general to deal with various forms of fuzzy arithmetic, including exponentiation, extraction of the nth root, and logarithm. In his first book on experimental design, Fisher [1934] demonstrated how valid conclusions could be drawn effectively from experiments with natural variations such as temperature, soil conditions and precipitation, that is, in the presence of variables of a nuisance. Gholamreza Hesamian [2016] has suggested that the one-way ANOVA should be applied to the case where the observed data is represented by closed intervals rather than real numbers. First, a notion of interval random variable is implemented in this approach. In particular, to examine hypotheses about the equality of interval means or assess the homogeneity of the assumption of interval variances, a normal distribution with interval parameters is implemented. Gou and Xu [2017] have examined the MADM problems, where the attribute values are expressed as interval numbers and the weight information is expressed as IFNs on the attributes. Jiryae et al. [2013] suggested that a different approach for one-way ANOVA is proposed when symmetric fuzzy random variables are all random variables. Both random errors and parameters are known as fuzzy numbers that are triangular or exponential. Khan et al. [2019] were proposed to deal with uncertainty in the form of a fuzzy picture set; the purpose of this study was to establish a logarithmic decision-making method. Kumar et al. [2010] have suggested the ranking formula for generalized exponential fuzzy numbers, and it has also been shown that the ranking function is not linear for generalized exponential fuzzy numbers. Li and Wei [2017] suggested the concept of a logarithmic operational law on IFNs in which the basis λ is a real number in the IFNs (0, 1). Then some of the operational law's properties are studied, and four intuitionistic methods of aggregation of fuzzy knowledge are proposed. Pak et al. [2014] proposed that when X and Y are statistically independent exponential random variables and the data obtained from both distributions are stated in the form of fuzzy numbers, the estimation of the stress-strength parameter R is considered. The method for ranking two exponential trapezoidal fuzzy numbers was described by Rezvani [2017]. For the ranking of exponential trapezoidal fuzzy numbers, a median value is suggested. The values proposed by Rezvani [2015] are calculated by using the probability density function corresponding to the membership functions of the given fuzzy number in order to determine the expected values and the correct ordering of the exponential trapezoidal fuzzy numbers is given. The method for the cardinal, median value, variance, and covariance of exponential fuzzy numbers with type function was proposed by Rezvani [2016]. Vahidi and Rezvani [2013] has suggested some new algebraic mathematics for positive fuzzy type numbers $(\overline{a}, \overline{\overline{a}}, \overline{\overline{\overline{a}}}, \overline{\overline{\overline{\overline{a}}}})$ and does not include the calculation of fuzzy algebraic number α-cuts. Direct mathematical expressions to calculate the exponential, square root, logarithms, inverse exponential, etc., of positive fuzzy form numbers $(\overline{a}, \overline{\overline{a}}, \overline{\overline{\overline{a}}}, \overline{\overline{\overline{\overline{a}}}})$ are obtained using the basic analytical principles of algebraic mathematics and Taylor series extension. Wu H.C. [2007] proposed that for the acceptance or rejection of null and alternative hypotheses with the concepts of pessimistic degree and optimistic degree by solving optimization problems, inaccurate data based on decision rules on the traditional ANOVA model would be used. Zhang et al. [2012] have proposed obtaining the crisp possibilistic mean preference pricing formula in the fuzzy double exponential jump-diffusion model by using the crisp possibilistic mean values of the fuzzy numbers. In this paper, a new approach to fuzzy one way ANOVA statistical analysis focused on arithmetic operations on fuzzy triangular numbers using the α-cut method with examples in fuzzy environments. This paper proposed a comparative study of statistical techniques to test the ANOVA method's hypothesis with fuzzy triangular data, as shown in the numerical examples below.

2. Preliminaries

2.1. Triangular Fuzzy Numbers (TFNs)

A TFNs $\widetilde{A}$ is a fuzzy number fully specified by triples (a, b, c) such that $a \leq b \leq c$ with membership function defined as

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0; & x \le a \\ \dfrac{x-a}{b-a}; & a \le x \le b \\ \dfrac{c-x}{b-c}; & b \le x \le c \\ 0; & x \ge c \end{cases}$$

where a is the lower level, b is the centre level and c is the upper level.

A TFNs can be represented as an interval number form as follows.

$$\widetilde{A} = [\{a + (b-a)\alpha\}^L; \{c - (b-c)\alpha\}^U]$$

The TFNs are represented in the graph is given below.

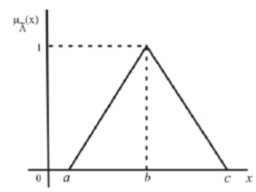

Figure 1. Triangular Fuzzy Numbers.

2.2. Exponential Triangular Fuzzy Numbers (ETFNs)

Let $\widetilde{X} = [a, b, c] > 0$ be a fuzzy number. Then $\alpha_A = [a + (b-a)\alpha, c - (c-b)\alpha]$ is the α-cut off the fuzzy numbers A. To calculate the exponential of the fuzzy number A, take exponential the α-cut of A using interval arithmetic.

$$\begin{aligned} \exp(\alpha_A) &= \exp([a + (b-a)\alpha, c - (c-b)\alpha]) \\ &= [\exp(a + (b-a)\alpha), \exp(c - (c-b)\alpha)]. \end{aligned} \tag{1}$$

To find the membership function $\mu_{\exp(x)}(x)$, equate to x both the first and second component in (1), which gives,

$$x = \exp(a + (b-a)\alpha) \text{ and } x = \exp(c - (c-b)\alpha).$$

Now, expressing in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (1) get $\alpha \in [0, 1]$ together with the domain of x.

$$\alpha = \frac{In(x) - a}{b-a}; \exp(a) \le x \le \exp(b) \text{ and } \alpha = \frac{c - In(x)}{c-b}; \exp(b) \le x \le \exp(c). \tag{2}$$

The ETFNs membership function is defined as

$$\mu_{\exp(x)}(x) = \begin{cases} \dfrac{In(x) - a}{b - a}; & \exp(a) \le a \le \exp(b) \\ \dfrac{c - In(x)}{c - b}; & \exp(b) \le x \le \exp(c) \end{cases}$$

2.3. Logarithmic Triangular Fuzzy Numbers (LTFNs)

Let $\widetilde{X} = [a, b, c] > 0$ be a fuzzy number. Then $\alpha_A = [a + (b - a)\alpha, c - (c - b)\alpha]$ is the α-cut off the fuzzy numbers A. To calculate the logarithm of the fuzzy number A, take the logarithm of the α-cut A using interval arithmetic.

$$\begin{aligned} In(\alpha_A) &= In([a + (b - a)\alpha, c - (c - b)\alpha]) \\ &= [In(a + (b - a)\alpha), In(c - (c - b)\alpha)]. \end{aligned} \tag{3}$$

To find the membership function $\mu_{In(x)}(x)$, we equate to x both the first and second component in (1), which gives,

$$x = In(a + (b - a)\alpha) \text{ and } x = In(c - (c - b)\alpha).$$

Now, expressing in terms of x and setting $\alpha = 0$ and $\alpha = 1$ in (3) get together with the domain x.

$$\alpha = \frac{\exp(x) - a}{b - a}; \ In(a) \le x \le In(b) \text{ and } \alpha = \frac{c - \exp(x)}{c - b}; \ In(b) \le x \le In(c). \tag{4}$$

The LTFNs membership function is defined as

$$\mu_{In(x)}(x) = \begin{cases} \dfrac{\exp(x) - a}{b - a}; & In(a) \le x \le In(b) \\ \dfrac{c - \exp(x)}{c - b}; & In(b) \le x \le In(c) \end{cases}$$

3. Classical ANOVA for One Way Model

The Classical ANOVA model can be found in any book on linear models, such as Cox's (2). This section gives a quick overview of traditional one-way ANOVA. It denotes r the number of factors under investigation, and the index $i\,(i = 1, 2, \ldots, r)$ denotes any one of these levels. The number of cases for the i^{th} factor level is denoted by, n_i, and the total number of cases in the study is denoted by n_i, where $n_i = \sum_{i=l}^{r} n_i$. The index j^{th} will be used to identify the given case or trial for a particular factor level. Therefore, we let y_{ij} denote the j^{th} observation on the dependent (response) variable for the i^{th} factor level. For instance, y_{ij} the employee's productivity in the i^{th} factor level of the j^{th} observations featuring the i^{th} type of shelf display. Since the number of cases or trials for the i^{th} factor level is denoted by, n_i, and $j\,(j = 1, 2, \ldots, n_i)$.

Now the ANOVA model can be stated as follows.

$$y_{ij} = \mu_i + e_{ij} \tag{5}$$

y_{ij} is the value of the response variable in the j^{th} trial for the i^{th} factor level, μ_i's are the factor level means which are regarded as parameters for testing hypotheses; e_{ij}'s are independent random variables having the normal distribution $N(0, \sigma^2) : i = 1, \ldots, r$ and $j = 1, \ldots, n_i$. Let y_{ij} 's are independent random variables having the normal distribution $N(0, \sigma^2)$. To test whether or not the factor level means μ_i are equal. The following testing hypotheses is considered as follows.

The Null Hypothesis (NH) is $H_0 : \mu_1 = \mu_2 = \ldots = \mu_r$ against the Alternative Hypothesis (AH) is $H_1 : \mu_1 \neq \mu_2 \neq \ldots \neq \mu_r$. To test whether or not the factor level means μ_i are equal. Then the sum of squares formula for the one-way ANOVA test is given below.

Total Sum of Squares (TSS) is

$$TSS = \sum_{i=1}^{r} \sum_{j=1}^{n_i} y_{ij}^2 - \frac{y^2}{n_t} \tag{6}$$

Treatment Sum of Squares (TrSS) is

$$TrSS = \sum_{i=1}^{r} \frac{y_i^2}{n_i} - \frac{y^2}{n_t} \tag{7}$$

and Error Sum of Squares (*ESS*) is

$$ESS = \sum_{i=1}^{r} \sum_{j=1}^{n_i} y_{ij}^2 - \sum_{i=1}^{r} \frac{y^2}{n_t} \tag{8}$$

Mean Sum of Squares Treatment (*MSTr*) and Mean Sum of Squares Error (*MSE*) are as follows

$$MSTr = \frac{TrSS}{(r-1)} \text{ and } MSE = \frac{ESS}{(n_t - r)} \tag{9}$$

Then, the *F*-Ratio's of Treatment and Error is

$$F = \frac{MSTr}{MSE} \text{ (or) } F = \frac{MSE}{MSTr} \tag{10}$$

Table 1. ANOVA Table for Classical One Way ANOVA.

SV	*df*	*SS*	*MSS*	*F-Ratio*
Treatment	$(r-1)$	*TrSS*	*MSTr*	$\frac{MSTr}{MSE}$ (or) $\frac{MSE}{MSTr}$
Error	$(n_t - r)$	*ESS*	*MSE*	-
Total	$(n_t - r)$	*TSS*	-	-

3.1. Decision Rule (DR)

If $F_{Tr} > F_T$, where F_{Tr} is the Calculated Value (CV) and F_T is the Tabulated Value (TV) of F with treatment *df* is $(r-1)$ $(n_t - r)$, 5% los, then accept the NH H_0, otherwise the AH H_1 is accepted. Similarly, if $F_E > F_r$.

3.2. Fuzzy One Way ANOVA Model

In the real world, genuine existence information may not always be as well defined. In this case, surveillance and recorded data are considered TFNs. The mathematical general linear model is as follows.

$$\tilde{y}_{ij} = \tilde{\mu}_i + e_{ij};\ i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, n_i \tag{11}$$

The interval fuzzy ANOVA models are the Lower Level Model (LLM) and Upper-Level Model (ULM), respectively, based on the above general linear model of crisp ANOVA. Because the statistical hypotheses and population parameters are clear, the linear model is designated as LLM, and the ULM is defined as: $(\tilde{y}_{ij})_l^{LL} = (\tilde{\mu}_i)_l^{LL} + (e_{ij})_l^{LL}$ and $(\tilde{y}_{ij})_u^{UL} = (\tilde{\mu}_i)_u^{UL} + (e_{ij})_u^{UL}$ in which $(\tilde{y}_{ij})_l^{LL}$ and $(\tilde{y}_{ij})_u^{UL}$ is the j^{th} observations of the i^{th} treatment; $(\tilde{\mu}_i)_l^{LL}$ and $(\tilde{\mu}_i)_u^{UL}$ is the general mean effect which is fixed; and $(e_{ij})_l^{LL}$ and $(e_{ij})_u^{UL}$ is the random error effect; $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, n_i$. Then, the analyze triangular fuzzy LLM and ULM using the crisp two ANOVA methods. Let 5% be the los and the LLM and ULM fuzzy statistical hypotheses is given by

The fuzzy NH is $\tilde{H}_0^{LL} : \tilde{\mu}_1^{LL} = \tilde{\mu}_2^{LL} = \ldots = \tilde{\mu}_r^{LL}$ against the fuzzy AH is $\tilde{H}_0^{LL} : \tilde{\mu}_1^{LL} \neq \tilde{\mu}_2^{LL} \neq \ldots \neq \tilde{\mu}_r^{LL}$.

The fuzzy NH is $\tilde{H}_0^{UL} : \tilde{\mu}_1^{UL} = \tilde{\mu}_2^{UL} = \ldots = \tilde{\mu}_r^{UL}$ against the fuzzy AH is $\tilde{H}_0^{UL} : \tilde{\mu}_1^{UL} \neq \tilde{\mu}_2^{UL} \neq \ldots \neq \tilde{\mu}_r^{UL}$.

For the triangular LLM and ULM from the NH of acceptance levels of TFNs. Using lower and upper levels model formulas are $a_{ij} + (b_{ij} - a_{ij})\alpha;\ c_{ij} - (c_{ij} - b_{ij})\alpha$ where $0 \leq i \leq r;\ 0 \leq j \leq n_i$. Then the sum of squares formula for triangular fuzzy LLM and ULM for fuzzy a one-way ANOVA test is given below:

Total Sum of Squares (TSS)

$$TSS_l^{LL} = \sum_{i=l}^{r}\sum_{j=1}^{n_i}[(\tilde{y}_{ij}^2)_l^{LL}] - \frac{[(\tilde{y}^2)_l^{LL}]}{n_t} \text{ and } TSS_u^{UL} = \sum_{i=l}^{r}\sum_{j=1}^{n_i}[(\tilde{y}_{ij}^2)_u^{UL}] - \frac{[(\tilde{y}^2)_u^{UL}]}{n_t} \tag{12}$$

where $$TSS_l^{LL} = \sum_{i=l}^{r}\sum_{j=1}^{n_i}[a_{ij} + (b_{ij} - a_{ij})\alpha]^2 - \frac{[(\tilde{y}^2)_l^{LL}]}{n_t}$$ and

$$TSS_u^{UL} = \sum_{i=l}^{r} \sum_{j=1}^{n_i} [c_{ij} - (c_{ij} - b_{ij})\alpha]^2 - \frac{[(\tilde{y}^2)_u^{UL}]}{n_t}$$

Treatment Sum of Squares (TrSS)

$$TSS_l^{LL} = \sum_{i=l}^{r} \frac{[(\tilde{y}^2)_l^{LL}]}{n_t} - \frac{[(\tilde{y}^2)_l^{LL}]}{n_t} \text{ and } TSS_u^{UL} = \sum_{i=l}^{r} \frac{[(\tilde{y}^2)_u^{UL}]}{n_t} - \frac{[(\tilde{y}^2)_u^{UL}]}{n_t} \quad (13)$$

where $$TSS_l^{LL} = \sum_{j=1}^{n_i} [a_{ij} + (b_{ij} - a_{ij})\alpha]^2;\ i = 1, 2, \ldots, r$$ and

$$TSS_l^{LL} = \sum_{j=1}^{n_i} [c_{ij} - (c_{ij} - b_{ij})\alpha]^2;\ i = 1, 2, \ldots, r$$

Error Sum of Squares (ESS)

$$ESS_l^{LL} = TSS_l^{LL} - TrSS_l^{LL} \text{ and } ESS_u^{UL} = TSS_u^{UL} - TrSS_u^{UL} \quad (14)$$

Mean Sum of Squares Treatment (MSTr)

$$MSTr_l^{LL} = \frac{TrSS_l^{LL}}{(r-1)} \text{ and } MSTr_u^{UL} = \frac{TrSS_u^{UL}}{(r-1)} \quad (15)$$

Mean Sum of Squares Error (MSE)

$$MSE_l^{LL} = \frac{ESS_l^{LL}}{(n_t - r)} \text{ and } MSE_u^{UL} = \frac{ESS_u^{UL}}{(n_t - r)} \quad (16)$$

$\tilde{F}$ – Ratio of test statistic is one way ANOVA model as follows:

$$(\tilde{F}_{Tr})_l^{LL} = \frac{MSTr_l^{LL}}{MSE_l^{LL}} \text{ and } (\tilde{F}_{Tr})_u^{UL} = \frac{MSTr_u^{UL}}{MSE_u^{UL}} \quad (17)$$

$$(\tilde{F}_E)_l^{LL} = \frac{MSE_l^{LL}}{MSTr_l^{LL}} \text{ and } (\tilde{F}_E)_u^{UL} = \frac{MSE_u^{UL}}{MSTr_u^{UL}} \quad (18)$$

Table 2. ANOVA Table for LLM of Fuzzy One Way ANOVA.

SV	*df*	*SS*	*MSS*	*F-Ratio*
Treatment	$(r-1)$	$TrSS$	$MSTr$	$\frac{MSTr}{MSE}$ (or) $\frac{MSE}{MSTr}$
Error	$(n_t - r)$	ESS	ESE	-
Total	$(n_t - 1)$	TSS	-	-

Since, using (3.1) DR of $(\widetilde{F}_{Tr})_l^{LL} > F_T$, where $(\widetilde{F}_{Tr})_l^{LL}$ is the CV and F_T is the TV of F with treatment *df* is $(r-1)\,(n_t - r)$. 5% los, then accept the NH H_0, otherwise the AH H_1 is accepted. Similarly, the calculate for error *df* for LLM.

Table 3. ANOVA Table for ULM of Fuzzy One Way ANOVA.

SV	*df*	*SS*	*MSS*	*F-Ratio*
Treatment	$(r-1)$	$TrSS_u^{UL}$	$MSTr_u^{UL}$	$\frac{MSTr_u^{UL}}{MSE_u^{UL}}$ (or) $\frac{MSE_u^{UL}}{MSTr_u^{UL}}$
Error	$(n_t - r)$	ESS_u^{UL}	MSE_u^{UL}	-
Total	$(n_t - 1)$	TSS_u^{UL}	-	-

Similarly, using (3.1) DR, the calculate for the treatment and error df for ULM.

Inspired by the fuzzy DR, suppose that if F TV of 5% Los, the NH of the LLM is accepted and the NH of the ULM is accepted for all $0 \le \alpha \le 1$ then, the NH of the ANOVA model is accepted.

Otherwise, the AH of the ANOVA model is accepted.

4. Applications

4.1. Example

A food company wished to test four different package designs for a new product. Ten stores with approximately equal sales volumes are selected as the experimental units. Package designs 1 and 4 are assigned to three stores each and package designs 2 and 3 are assigned to two stores each.

We cannot record the exact sales volume in a store due to unexpected situations, but we have fuzzy sales volumes. The fuzzy data are given below [8].

Table 4. One Way ANOVA for Triangular Fuzzy Numbers.

Package Design (i)	Store Observations (j)		
	1	2	3
1	9, 11, 13	14, 16, 18	-
2	11, 15, 19	10, 15, 20	11, 13, 15
3	15, 18, 21	14, 17, 20	17, 20, 23
4	15, 19, 23	21, 24, 27	-

To test whether or not the (fuzzy) mean sales are the same for the four designs. Let $\tilde{\mu}_i$ be the mean sales for the i^{th} design. The null hypothesis, $\tilde{H}_0 : \tilde{\mu}_1 = \tilde{\mu}_2 = \tilde{\mu}_3 = \tilde{\mu}_4$ and the alternative hypothesis, $\tilde{H}_1$: not all $\tilde{\mu}_i$'s are equal.

$\tilde{H}_0$: There is a relation between package designs and sales volumes.

$\tilde{H}_1$: There is no relation between the package designs and sales volumes.

Let us consider 4.1 example of table 4 one way ANOVA technique for LLM and ULM using α-cut interval method is given below.

Table 5. Fuzzy One Way ANOVA for TFNs using α-Cut Interval Method.

Package Design (i)	Store Observations (j)		
	1	2	3
1	$9 + 2\alpha, 13 - 2\alpha$	$14 + 2\alpha, 18 - 2\alpha$	-
2	$11 + 4\alpha, 19 - 4\alpha$	$10 + 5\alpha, 20 - 5\alpha$	$11 + 2\alpha, 15 - 2\alpha$
3	$15 + 3\alpha, 21 - 3\alpha$	$14 + 3\alpha, 20 - 3\alpha$	$17 + 3\alpha, 23 - 3\alpha$
4	$15 + 4\alpha, 23 - 4\alpha$	$21 + 3\alpha, 27 - 3\alpha$	-

In using sections 2.2, 2.3 and 3.2 the TFNs of LLM for Fuzzy One Way ANOVA table 4 values of Sum of Squares, Mean Sum of Squares, and F Ratio are computed.

$$TSS^{LL} = 8.9\alpha^2 - 3.4\alpha + 118.1;\ TrSS^{LL} = 3.7\alpha^2 + 5.3\alpha + 82.3;\ ESS^{LL} = 5.2\alpha^2 - 8.7\alpha + 35.8;\ MSTr^{LL} = 1.03\alpha^2 + 1.8\alpha + 27.4;\ MSE^{LL} = 0.9\alpha^2 - 1.5\alpha + 5.9;\ \tilde{F}^{LL} = \frac{1.03\alpha^2 + 1.8\alpha + 27.4}{0.9\alpha^2 - 1.5\alpha + 5.9}$$

Table 6. ANOVA Table for LLM of Fuzzy One Way ANOVA.

SV	*df*	*SS*	*MSS*	*F-Ratio*
Treatment	3	$3.7\alpha^2 + 5.3\alpha + 82.3$	$1.03\alpha^2 + 1.8\alpha + 27.4$	$\frac{1.03\alpha^2 + 1.8\alpha + 27.4}{0.9\alpha^2 - 1.5\alpha + 5.9}$
Error	6	$5.2\alpha^2 - 8.7\alpha + 35.8$	$0.9\alpha^2 - 1.5\alpha + 5.9$	-
Total	9	$8.9\alpha^2 - 3.4\alpha + 118.1$	-	-

If the DR using (3.1) in $\tilde{F}_{Tr} < F_T$, where $F_T = 4.76$ is the *F* TV of 5% los with (3, 6) df then, the NH $\tilde{H}_0$ is accepted. Therefore, there is a relation between package designs and sales volumes.

Similarly, the TFNs of ULM for Fuzzy One Way ANOVA table 4 values of Sum of Squares,

Mean Sum of Squares, and *F* Ratio are calculated in using sections 2.2, 2.3 and 3.2.

$$TSS^{UL} = 8.9\alpha^2 - 32.2\alpha + 146.9;\ TrSS^{UL} = 3.7\alpha^2 + 20.2\alpha + 107.7;\ ESS^{UL} = 5.2\alpha^2 - 12\alpha + 39.2;\ MSTr^{UL} = 1.2\alpha^2 + 6.7\alpha + 35.9;\ MSE^{UL} = 0.9\alpha^2 - 2\alpha + 6.5;\ \tilde{F}^{UL} = \frac{1.2\alpha^2 - 6.7\alpha + 35.9}{0.9\alpha^2 - 2\alpha + 6.5}$$

Table 7. ANOVA Table for ULM of Fuzzy One Way ANOVA.

SV	*df*	*SS*	*MSS*	*F-Ratio*
Treatment	3	$3.7\alpha^2 + 20.2\alpha + 107.7$	$1.2\alpha^2 + 6.7\alpha + 35.9$	$\frac{1.2\alpha^2 - 6.7\alpha + 35.9}{0.9\alpha^2 - 2\alpha + 6.5}$
Error	6	$5.2\alpha^2 - 12\alpha + 39.2$	$0.9\alpha^2 - 2\alpha + 6.5$	-
Total	9	$8.9\alpha^2 - 32.2\alpha + 146.9$	-	-

If the DR using (3.1) in $\widetilde{F}_{Tr} < F_T$, where $F_T = 4.76$ is the F TV of 5% los with (3, 6) df then, the null hypothesis $\widetilde{H}_0$ is rejected. Therefore, there is no relation between the package designs and sales volumes.

Consider the 4.1 example of table 4 using the one-way ANOVA method with ETFNs. The above relations 2.2, 2.3 and 3.2. calculation is shown below.

Table 8. Fuzzy One Way ANOVA using ETFNs.

Machines (i)	Sample Observations (j)		
	1	2	3
1	2.4	2.8	-
2	2.7	2.7	2.6
3	2.9	2.8	3.0
4	2.9	3.2	-

The ETFNs for Fuzzy One Way ANOVA table 4 values of Sum of Squares, Mean Sum of Squares, and F-Ratio are calculated in using sections 2.2, 2.3 and 3.2.

$$TSS = 0.44;\ TrSS = 0.29;\ ESS = 0.15;\ MSTr = 0.0967;\ MSE = 0.025;\ \widetilde{F} = 3.868$$

Table 9. ANOVA Table for Fuzzy One Way ANOVA ETFNs.

SV	*df*	*SS*	*MSS*	$\widetilde{F}$ *-Ratio*
Treatment	3	0.29	0.0967	3.868
Error	6	0.15	0.025	-
Total	9	0.44	-	-

If the DR using (3.1) in $\widetilde{F}_{Tr} < F_T$, where $F_T = 4.76$ is the F TV of 5% los with (3, 6) df then, the NH $\widetilde{H}_0$ is accepted. Therefore, there is a relation between package designs and sales volumes.

Consider the 4.1 example of table 4 using the one-way ANOVA method with ETFNs. The above relations (2) and (3.3) calculation is shown below.

Table 10. Fuzzy One Way ANOVA using LTFNs.

Machines (i)	Sample Observations (j)		
	1	2	3
1	1.04	1.2	-
2	1.17	1.17	1.11
3	1.25	1.23	1.3
4	1.27	1.38	-

The LTFNs for Fuzzy One Way ANOVA table 4 values of Sum of Squares, Mean Sum of Squares, and F Ratio are calculated in using sections 2.2, 2.3 and 3.2.

$$TSS = 0.09;\ TrSS = 0.07;\ ESS = 0.02;\ MSTr = 0.023;\ MSE = 0.003;\ \widetilde{F} = 7.67$$

Table 11. ANOVA Table for Fuzzy One Way ANOVA LTFNs.

SV	*df*	*SS*	*MSS*	$\widetilde{F}$ *-Ratio*
Treatment	3	0.07	0.023	7.67
Error	6	0.02	0.003	-
Total	9	0.09	-	-

If the DR using (3.1) in $\widetilde{F}_{Tr} < F_T$, where $F_T = 4.76$ is the F TV of 5% los with (3, 6) df then, the NH $\widetilde{H}_0$ is rejected. Therefore, there is no relation between the package designs and sales volumes.

4.2. Example

Four different machines are used to produce milk pouches of 100 ml each by a city diary. Before these pouches are dispatched for local distribution, the quality assurance manager selects two samples of pouches from machine one and machine four and three samples of pouches from machine two and machine three and determines the number of pouches that do not meet the specifications under the weights and measures act. We cannot record the exact number of pouches in a sample due to unexpected situations, but we have the fuzzy data for the number of pouches. The fuzzy data are given below [8].

Table 12. Fuzzy One Way ANOVA for TFNs.

Machines (i)	Sample Observations (j)		
	1	2	3
1	8, 9, 14	11, 14, 18	-
2	7, 11, 15	8, 10, 14	8, 11, 14
3	12, 15, 18	12, 14, 19	14, 17, 23
4	12, 15, 19	19, 21, 24	-

To test whether there is any significant difference in the performance of the machines. Let $\widetilde{\mu}_i$ be the mean number of non-specifications pouches for the i^{th} machines. In the null hypothesis $\widetilde{H}_0 : \widetilde{\mu}_1 = \widetilde{\mu}_2 = \widetilde{\mu}_3 = \widetilde{\mu}_4$ and the alternative hypothesis, $\widetilde{H}_1$: not all $\widetilde{\mu}_i$'s are equal.

$\widetilde{H}_0$: There is a significant difference in the performance of the machines.

$\widetilde{H}_1$: There is no significant difference in the performance of the machines.

Consider the LLM and ULM in Table 13 for one-way ANOVA using the α-cut interval method in Example 4.2.

Table 13. One Way ANOVA for TFNs using α-Cut Interval Method.

Package Design (i)	Store Observations (j)		
	1	2	3
1	$8+\alpha,\ 14-5\alpha$	$11+3\alpha,\ 18-4\alpha$	-
2	$7+4\alpha,\ 15-4\alpha$	$8+2\alpha,\ 14-4\alpha$	$8+3\alpha,\ 14-3\alpha$
3	$12+3\alpha,\ 18-3\alpha$	$12+2\alpha,\ 19-5\alpha$	$14+3\alpha,\ 23-6\alpha$
4	$12+3\alpha,\ 24-3\alpha$	$19+2\alpha,\ 24-3\alpha$	-

In using sections 2.2, 2.3 and 3.2 the TFNs of LLM for Fuzzy One Way ANOVA table 4 values of Sum of Squares, Mean Sum of Squares, and F Ratio are computed.

$$TSS^{LL} = 6.4\alpha^2 - 7.2\alpha + 118.9;\ TrSS^{LL} = 1.2\alpha^2 + 5.5\alpha + 86.6;\ ESS^{LL} = 5.2\alpha^2 - 1.7\alpha + 32.3;\ MSTr^{LL} = 0.4\alpha^2 + 1.8\alpha + 28.9;\ MSE^{LL} = 0.9\alpha^2 - 0.3\alpha + 5.38;\ \widetilde{F}^{UL} = \frac{0.4\alpha^2 - 1.8\alpha + 28.9}{0.9\alpha^2 - 0.3\alpha + 5.38}$$

Table 14. ANOVA Table for LLM of Fuzzy One Way ANOVA.

SV	*df*	*SS*	*MSS*	$\widetilde{F}$ *-Ratio*
Treatment	3	$1.2\alpha^2 - 5.5\alpha + 86.6$	$0.4\alpha^2 - 1.8\alpha + 28.0$	$\frac{0.4\alpha^2 - 1.8\alpha + 28.9}{0.9\alpha^2 - 0.3\alpha + 5.38}$
Error	6	$5.2\alpha^2 - 1.7\alpha + 32.3$	$0.9\alpha^2 - 0.3\alpha + 5.38$	-
Total	9	$6.4\alpha^2 - 7.2\alpha + 118.9$	-	-

If the DR using (3.1) in $\widetilde{F}_{Tr} < F_T$, where $F_T = 4.76$ is the F TV of 5% los with (3, 6) df then, the NH $\widetilde{H}_0$ is rejected. Therefore, there is no significant difference in the performance of the machines.

Similarly, the TFNs of ULM for Fuzzy One Way ANOVA table 4 values of Sum of Squares, Mean Sum of Squares, and F Ratio are calculated in using sections 2 and 3.

$$TSS^{LL} = 8.9\alpha^2 - 10.2\alpha + 119.6;\ TrSS^{LL} = 2.6\alpha^2 + 4.5\alpha + 84.4;\ ESS^{UL}$$

$$= 6.3\alpha^2 - 5.7\alpha + 35.2;\ MSTr^{LL} = 0.9\alpha^2 + 1.5\alpha + 28.1;\ MSE^{UL} = 1.1\alpha^2$$

$$- 0.9\alpha + 5.9;\ \widetilde{F}^{UL} = \frac{0.9\alpha^2 - 1.5\alpha + 28.1}{1.1\alpha^2 - 0.9\alpha + 5.9}$$

Table 15. ANOVA Table for ULM Fuzzy One Way ANOVA.

SV	*df*	*SS*	*MSS*	$\widetilde{F}$ *-Ratio*
Treatment	3	$2.6\alpha^2 - 4.5\alpha + 84.4$	$0.9\alpha^2 - 1.5\alpha + 28.1$	$\frac{0.9\alpha^2 - 1.5\alpha + 28.1}{1.1\alpha^2 - 0.9\alpha + 5.9}$
Error	6	$6.3\alpha^2 - 5.7\alpha + 35.2$	$1.1\alpha^2 - 0.9\alpha + 5.9$	-
Total	9	$8.9\alpha^2 - 10.2\alpha + 119.6$	-	-

If the DR using (3.1) in $\widetilde{F}_{Tr} < F_T$, where $F_T = 4.76$ is the F TV of 5% los with (3, 6) df then, the null hypothesis $\widetilde{H}_0$ is accepted. Therefore, there is no relation between the package designs and sales volumes.

In using sections 2.2, 2.3 and 3.2. calculate the ETFNs for Fuzzy One Way ANOVA table 4 in Example 4.2 values of Sum of Squares, Mean Sum of Squares, and F Ratio.

Table 16. Fuzzy One Way ANOVA using ETFNs.

Machines (i)	Store Observations (j)		
	1	2	3
1	2.2	2.6	-
2	2.4	2.3	2.4
3	2.7	2.6	2.8
4	2.7	3.0	-

The ETFNs for Fuzzy One Way ANOVA table 8 values of Sum of Squares, Mean Sum of Squares, and F Ratio are calculated in using sections 2.2, 2.3 and 3.2.

$$TSS = 0.541;\ TrSS = 0.389;\ ESS = 0.152;\ MSTr = 0.1297;\ MSE = 0.0253;\ \widetilde{F} = 5.13$$

Table 17. ANOVA Table for ETFNs of Fuzzy One Way ANOVA.

SV	*df*	*SS*	*MSS*	$\widetilde{F}$ *-Ratio*
Treatment	3	0.389	0.1297	5.13
Error	6	0.152	0.0253	-
Total	9	0.541	-	-

If the DR using (3.1) in $\widetilde{F}_{Tr} > F_T$, where $F_T = 4.76$ is the F TV of 5% los with (3, 6) df then, the NH $\widetilde{H}_0$ is rejected. Therefore, there is no significant difference in the performance of the machines.

In using sections 2.2, 2.3 and 3.2.calculate the ETFNs for Fuzzy One Way ANOVA table 4 in Example 4.2 values of Sum of Squares, Mean Sum of Squares, and F Ratio.

Table 18. Fuzzy One Way ANOVA using LTFNs.

Machines (i)	Store Observations (j)		
	1	2	3
1	0.95	1.14	-
2	1.04	1.0	1.04
3	1.17	1.14	1.23
4	1.17	1.32	-

The LTFNs for Fuzzy One Way ANOVA table 8 values of Sum of Squares, Mean Sum of Squares, and F Ratio are calculated in using sections 2.2, 2.3 and 3.2.

$$TSS = 0.114;\ TrSS = 0.079;\ ESS = 0.035;\ MSTr = 0.026;\ MSE = 0.005;\ \widetilde{F} = 5.2$$

Table 19. ANOVA Table for LTFNs of Fuzzy One Way ANOVA.

SV	df	SS	MSS	$\widetilde{F}$ -Ratio
Treatment	3	0.079	0.026	5.2
Error	6	0.035	0.005	-
Total	9	0.114	-	-

If the DR using (3.1) in Tr $\widetilde{F}_{Tr} < F_T$, where $F_T = 4.76$ is the F TV of 5% los with (3, 6) df then, the NH $\widetilde{H}_0$ is rejected. Therefore, there is no significant difference in the performance of the machines.

In examples 1 and 2, Table 20 compares the results of a comparative study of various methods.

Table 20. ANOVA Model for Decision Rule Obtained.

ANOVA Model	Decision Rule Obtained	
	Example 1	Example 2
α-cut Interval Method	The lower level $\widetilde{H}_0$ is accepted and the upper $\widetilde{H}_0$ is rejected	The lower level $\widetilde{H}_0$ is rejected and the level $\widetilde{H}_0$ is accepted
Exponential	$\widetilde{H}_0$ is rejected	$\widetilde{H}_0$ is rejected
Logarithmic TFNs	$\widetilde{H}_0$ is rejected	$\widetilde{H}_0$ is rejected

5. Conclusion

It can be challenging to record data in the real world, and experimental outcomes, such as fuzzy environments, cannot always be well-defined. Consequently, it would be beneficial to associate statistical methods for processing fuzzy data. This paper proposed a comparative study on the statistical method of the hypothesis testing of the single-factor ANOVA method with the triangular fuzzy data. Our proposed fuzzy test is superior to the classical test of significance. When comparing the various methods' results, the alpha cut interval method fits better than the Exponential TFNs, and Logarithmic TFNs approaches produce similar results. In future works, there is a plan in testing to smear the method of this paper to other experimental designs such as two factor ANOVA, multi-factor ANOVA model, etc., and where crisp more exactly than actual numbers.

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