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ON THE MAXIMUM EXTENSION OF CERTAIN *t*-TOUGH SETS OF MESH DERIVED ARCHITECTURES

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Abstract

Extended mesh and enhanced mesh architectures are highly versatile interconnection networks. The vulnerability of these architectures provides scope for its utility in various applications such as data communication, intruder locating, data security etc. In this paper we investigate the toughness and maximum extension of certain t-tough sets of the extended mesh architecture and the enhanced mesh architecture.

1. Introduction

Interconnection network utilization have become the basis of every other economic activity elaborating the need for its efficient performance. In the event of negative exploitation of certain nodes of the network, if these nodes have a small degree or if the remaining segments are well-connected, then the network can very well perform without much loss in efficiency. If these nodes have relatively large degree they can even disconnect the network. Hence the efficiency of the network relies on its remaining segments or unexploited segments. Therefore, a hierarchical grouping process which assesses the network performance as well as relays a risk propagation predicting system thus issuing an early warning to nodes which could be influenced by those negatively exploited is necessary. In this paper we establish a graph theory based network vulnerability hierarchical grouping process, namely, toughness and maximum extension of a t-tough set of a

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graph and investigate the same for extended mesh architecture and enhanced mesh architecture.

1.1 Preliminaries and Literature Survey

Toughness is a measure for the vulnerability of a graph. It estimates how closely the vertices of the graph are associated with each other. Toughness of graphs was introduced by Chvátal [3] to study the hamiltonicity property of graphs. Formally, the toughness of a graph G is defined as follows:

Definition 1. Toughness [3] of a graph G is defined as the real number $\tau > 0$ such that it is the minimum of the ratio of the number of vertices in the cutset S to the number of components in $G \setminus S$ taken over all possible cutsets S of G.

$$\tau = \min \frac{|S|}{\omega(G \setminus S)}, \forall S \subset V \tag{1}$$

Suppose S is a cutset of a connected graph G, then $\omega(G \setminus S) \geq 2$ and $|S| \geq \kappa$. Therefore, the relationship between the connectivity and toughness of a graph is described in the following theorem:

Theorem 1 [3]**.** *If G is not complete, then* $\tau \leq \frac{\kappa}{2}$.

Plummer [10] studied the relationship between toughness and n-extendibility of a graph G in view of classifying graphs with large toughness as well as graphs with toughness less than 1. Douglas et al. [1] determined the time complexity of recognizing a t-tough graph to be NP-hard. Woeginger [4, 6] studied the toughness of split graphs and determined a polynomial time algorithm to generate the same. Brouwer [2] studied the relationship between toughness and spectrum of a graph. Goddard [11] investigated the toughness of cubic graphs. Later, Cynthia et al. [8] investigated the toughness of cyclic split graphs and generalised prism graphs. Xiaofeng Gu [12] derived a tight lower bound for the toughness of (n, d, λ)-graphs moving closer to the toughness conjecture of Brouwer.

We have introduced the concept of extension of a t-tough set of a graph and hence the maximum extension of the same.

Definition 2. A t-tough set [9] of a connected graph G, denoted as S_t, is defined as a cutset $S \subset V(G)$ which satisfies the following equation:

$$t = \frac{|S|}{\omega(G \setminus S)},\ t \geq \tau$$

Definition 3. For a connected graph G, a t'-tough set $S_{t'}$ is called an extension [9] of a t-tough set S_t if whenever $t' \leq t$, $S \subseteq S'$.

(i) If $t' = t$, then $S_{t'}$ is called a weak extension of S_t.

(ii) If $t' < t$, then $S_{t'}$ is called a strong extension of S_t.

Definition 4. A t_m-tough set S_{t_m} is called a maximum extension of a t-tough set S_t if there does not exist a t'-tough set $S_{t'}$ such that $t_0 \leq t_m$ and $S_{t_0} \supset S_{t_m}$.

Definition 3 and definition 4 help to identify a series of supersets of S_t, $S \subset S_{t_1} \subset S_{t_2} \subset \ldots \subset S_{t_m}$ such that the toughness value decreases simultaneously (i.e.), $t \leq t_1 \leq t_2 \leq \ldots \leq t_m$. The set S_t can be identified as the set of negatively exploited nodes of the network and therefore depending on the stature of $G \setminus S_t$ the performance efficacy of G can be assessed. In view of predicting the risk propagation of S, the supersets $S_{t_1}, S_{t_2}, \ldots, S_{t_m}$ fall under the risk category and so an early threat warning can be issued to each of the supersets hierarchically.

2. Toughness and Maximum Extension of Some t-Tough Sets of the Extended Mesh Graph $EX(n, n)$, $n \geq 2$

The extended mesh graph [7] $EX(n, n)$ of dimension $n \times n$ is derived from a mesh graph $M(n, n)$ of dimension $n \times n$ by adding two intersecting edges to each of the bounded face of the mesh graph. The vertex set of $EX(n, n)$ is denoted as follows:

$$V(EX(n, n)) = \{v_{ij},\ 1 \leq i,\ j \leq n\}$$

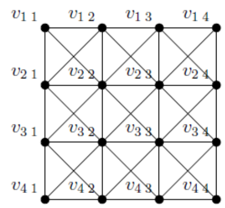

Figure 1. The Extended Mesh Graph $EX(4, 4)$.

Remark 1. The minimum toughness of the extended mesh graph $EX(3, 3)$ is 1.25 (i.e.),

$$\tau(EX(3, 3)) = 1.25$$

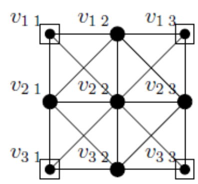

Figure 2. τ-Tough Set S_τ and the Components of $EX(3, 3) - S_\tau$.

We derive the toughness of $EX(n, n)$, $n \geq 3$ following which we will investigate the maximum extension of certain t-tough sets and infer the vulnerability of the graph.

Theorem 2. *The minimum toughness of the extended mesh graph* $EX(n, n)$, $n \geq 3$ *is* 1.5.

Proof. By the construction of $EX(n, n)$, $\delta(EX(n, n)) = 3$, which implies $\kappa \leq 3$. Hence, $1 \leq \tau(EX(n, n)) \leq 1.5$. We prove that this bound is sharp by contradiction. Consider the following cutsets of $EX(n, n)$.

$$S^1 = \{v_{1\,2}, v_{2\,1}, v_{2\,2}\} \tag{2}$$

$$S^2 = \{v_{1(n-1)}, v_{2\,n}, v_{2(n-1)}\} \tag{3}$$

$$S^3 = \{v_{(n-1)1}, v_{(n-1)2}, v_{n\,2}\} \tag{4}$$

$$S^4 = \{v_{(n-1)n}, v_{(n-1)(n-1)}, v_{n(n-1)}\} \tag{5}$$

Then,

$$\frac{|S^1|}{\omega(EX(n, n) - S^1)} = \frac{|S^2|}{\omega(EX(n, n) - S^2)} = \frac{|S^3|}{\omega(EX(n, n) - S^3)}$$

$$= \frac{|S^4|}{\omega(EX(n, n) - S^4)} = \frac{3}{2} \tag{6}$$

Suppose S^5 is a cutset of $EX(n, n)$ such that

$$\frac{|S^5|}{\omega(EX(n, n) - S^5)} < \frac{3}{2} \tag{7}$$

Then S^5 is classified as follows:

Type 1. S^5 consists of vertices of degree 5 and 8. Without loss of generality, the smallest cutset of this type is

$$S^5 = \{v_{12}, v_{13}, v_{21}, v_{23}, v_{31}, v_{32}\} \tag{8}$$

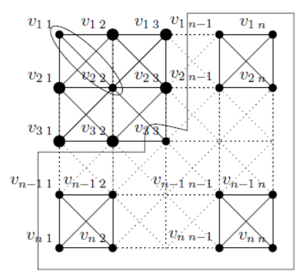

Figure 3. A Type 1 Cutset S^5 and the Components of $EX(n, n) - S^5$.

It follows from figure 3 that

$$\omega(EX(n, n) \setminus S^5) = 2 \tag{9}$$

and

$$\frac{| S^5 |}{\omega(EX(n,\, n) \setminus S^5)} = 3 > 1.5 \tag{10}$$

Hence, a contradiction to $\tau(EX(n,\, n)) < 1.5,\ n \geq 3$.

Type 2. S^5 consists of vertices of degree 3, 5 and 8. Without loss of generality, the smallest cutset of this type is

$$S^5 = \{v_{1\,1},\, v_{1\,3},\, v_{2\,3},\, v_{3\,1},\, v_{3\,2},\, v_{3\,3}\} \tag{11}$$

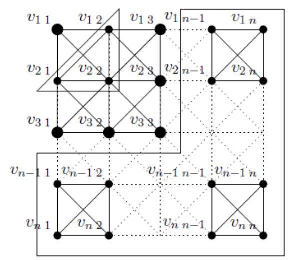

Figure 4. A Type 2 Cutset S^5 and the Components of $EX(n,\, n) - S^5$.

It follows from figure 4 that

$$\omega(EX(n,\, n) \setminus S^5) = 2 \tag{12}$$

and

$$\frac{| S^5 |}{\omega(EX(n,\, n) \setminus S^5)} = 3 > 1.5 \tag{13}$$

Hence, a contradiction to $\tau(EX(n,\, n)) < 1.5,\ n \geq 3$.

Therefore, $\tau(EX(n,\, n)) = 1.5,\ n \geq 3$

In the following theorem we investigate if it is possible to find a t-tough set $S_t,\ t > \tau$, of $EX(n,\, n)$ such that S_τ is its maximum extension.

Theorem 3. *Let* S_τ *be a* τ*-tough set of the extended mesh graph* $EX(n,\, n),\ n > 3$. *Then* S_τ *is not an extension of any* t*-tough* $S_t,\ t \geq \tau$, *of* $EX(n,\, n)$.

Proof. Consider the following τ-tough sets of $EX(n, n)$, $n > 3$.

$$S_\tau^1 = \{v_{1\,2}, v_{2\,1}, v_{2\,2}\} \tag{14}$$

$$S_\tau^2 = \{v_{1(n-1)}, v_{2\,n}, v_{2(n-1)}\} \tag{15}$$

$$S_\tau^3 = \{v_{(n-1)1}, v_{(n-1)2}, v_{n\,2}\} \tag{16}$$

$$S_\tau^4 = \{v_{(n-1)n}, v_{(n-1)(n-1)}, v_{n(n-1)}\} \tag{17}$$

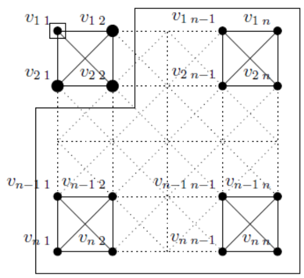

Figure 5. τ-Tough Set S_τ^1 and the Components of $EX(n, n) - S_\tau^1$.

If there exists a 1.5-tough set, say S_τ^5 of $EX(n, n)$ other than the tough sets S_τ^1, S_τ^2, S_τ^3 and S_τ^4 then S_τ^5 is classified as follows:

Case 1. Without loss of generality, let S_τ^5 be a weak extension of S_τ^1. Therefore,

$$|S_\tau^5| > |S_\tau^1| \tag{18}$$

and

$$\omega(EX(n, n) - S_\tau^5) > \omega(EX(n, n) - S_\tau^1) \tag{19}$$

But

$$\frac{|S_\tau^5|}{\omega(EX(n, n) - S_\tau^5)} \le \frac{|S_\tau^1|}{\omega(EX(n, n) - S_\tau^1)} = \frac{3}{2}. \tag{20}$$

Equations (18) to (20) imply that the cardinality of S_τ^5 and the number of components in $\omega(EX(n,\, n) - S_\tau^5)$ increases in the proportion 3.2. Without loss of generality, the smallest S_τ^5 is as follows:

$$S_\tau^5 = S_\tau^1 \cup \{v_{2\,3},\, v_{1\,4},\, v_{2\,4}\} \tag{21}$$

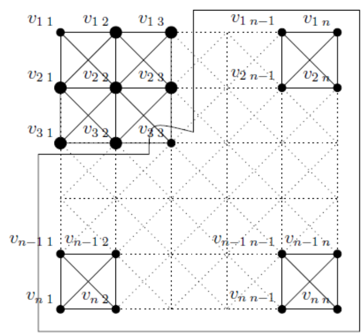

Figure 6. τ-Tough Set S_τ^5 and the Components of $EX(n,\, n) - S_\tau^5$.

It follows that

$$\frac{|\, S_\tau^5 \,|}{\omega(EX(n,\, n) \setminus S_\tau^5)} = 2 > 1.5 \tag{22}$$

Hence, a contradiction to the fact that S_τ^5 is a weak extension of S_τ^1. Therefore, S_τ^1, S_τ^2, S_τ^3 and S_τ^4 are the only τ-tough sets of $EX(n,\, n)$.

Case 2. Without loss of generality, let S_τ^5 be a strong extension of any of the tough sets S_τ^1. Then, by the degree properties of $EX(n,\, n)$ it is clearly not possible to find a tough set S such that $|\, S \,| < 3$ as S_τ^1, S_τ^2, S_τ^3 and S_τ^4 are the only τ-tough sets of $EX(n,\, n)$.

Hence, the proof. □

3. Maximum Extension of Some t-Tough Sets of the Enhanced Mesh Graph $EX(n,\, n)$, $n \geq 2$

The enhanced mesh graph [7] of dimension $n \times n$ is constructing from the extended mesh graph $EX(n, n)$ by adding a vertex to every pair of intersecting edges. The vertex set of the enhanced mesh graph is as follows:

$$V(EN(n, n)) = V(EX(n, n)) \cup \{v_k^l \mid 1 \le k \le n-1, 1 \le l \le n-1\} \tag{23}$$

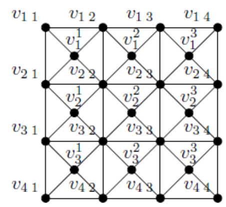

Figure 7. The Enhanced Mesh Graph $EN(4, 4)$.

Theorem 4. *The minimum toughness of the enhanced mesh graph* $EN(n, n)$, $n > 2$ *is* $\frac{n^2-4}{(n-1)^2}$ (*i.e.*),

$$\tau(EN(n, n)) = \frac{n^2-4}{(n-1)^2}, \; n > 2$$

Also, the τ*-tough set is*

$$S_\tau = \left\{\bigcup_{i=1}^{n} v_{i\,2}\right\} \cup \left\{\bigcup_{i=2}^{n-1} v_{i\,1}\right\} \cup \left\{\bigcup_{i=1}^{n}\bigcup_{j=3}^{n-1} v_{i\,j}\right\}$$

Proof. By the construction of $EN(n, n)$, $\delta(EN(n, n)) = 3$ implies $\kappa(EN(n, n)) \le 3$, we have $\tau(EN(n, n)) \le 1.5$. In this proof we will reduce the bound to $\frac{n^2-4}{(n-1)^2}$ and show that this bound is sharp for any n.

One two three four five six seven eight nine tan

Without loss of generality consider the following 1.5-tough set.

$$S_{1.5} = \{v_{1\,2}, v_{2\,1}, v_{2\,2}\} \tag{24}$$

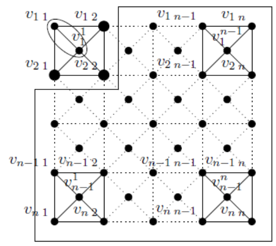

Figure 8. Tough set $S_{1.5} = \{v_{1\,2}, v_{21}, v_{2\,2}\}$ and the Components of $EN(n, n) - S_{1.5}$.

Let $S_t = S_{1.5}$. Now, we include more vertices to S_t and check if this set would increase the components in $EN(n, n) - S_t$ (i.e.), S_t would potentially decrease the value of t. Let

$$S_t = \{v_{1\,2}, v_{21}, v_{2\,2}\} \cup \{v_{3\,2}, v_{4\,2}, v_{5\,2}, \ldots, v_{n\,2}\} \tag{25}$$

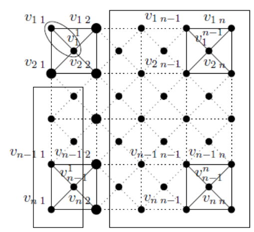

Figure 9. Illustration for Equation (25) and the Components of $EN(n, n) - S_t$.

Clearly, $t = \frac{n+1}{3} > 1.5$ though the number of components in $EN(n, n) - S_t$ have increased. Hence, we include more vertices to S_t in view of reducing t by increasing the components in $EN(n, n) - S_t$. Let

$$S_t = \{v_{1\,2}, v_{21}, v_{2\,2}\} \cup \{v_{3\,2}, v_{4\,2}, v_{5\,2}, \ldots, v_{n\,2}\} \cup \{v_{31}, v_{41}, v_{51}, \ldots, v_{n-11}\} \tag{26}$$

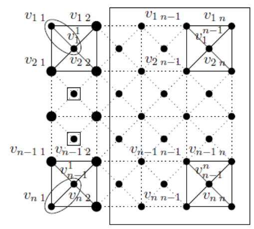

Figure 10. Illustration for Equation (26) and the Components of $EN(n, n) - S_t$.

Then, $t = 2 - \frac{2}{n}$. As n increases, $\frac{2}{n}$ approaches 0. Hence, $t > 1.5$. Therefore, let

$$S_t = \{v_{12}, v_{21}, v_{22}\} \cup \{v_{32}, v_{42}, v_{52}, \ldots, v_{n2}\} \cup \{v_{31}, v_{41}, v_{51}, \ldots, v_{n-11}\}$$
$$\cup \{v_{13}, v_{23}, v_{33}, \ldots, v_{n3}\} \tag{27}$$

In this case, $t = \frac{3n-2}{2n-1}$. Clearly, as n increases, $\frac{3n-2}{2n-1}$ approaches 1.5. Hence, $\tau \leq 1.5$ is sharp for this set. We proceed further to obtain a tough set which yields toughness less than 1.5. Hence, let

$$S_t = \{v_{12}, v_{21}, v_{22}\} \cup \{v_{32}, v_{42}, v_{52}, \ldots, v_{n2}\} \cup \{v_{31}, v_{41}, v_{51}, \ldots, v_{n-11}\}$$
$$\cup \{v_{13}, v_{23}, v_{33}, \ldots, v_{n3}\} \cup \{v_{14}, v_{24}, v_{34}, \ldots, v_{n4}\} \tag{28}$$

In this case S_t yields a toughness of $t = \frac{4n-2}{3n-2}$. It is easy to realize that as n increases, $\frac{4n-2}{3n-2}$ approaches a value less than 1.5. Hence, the bound for minimum toughness has reduced to $\frac{4n-2}{3n-2}$ (i.e.), $\tau \leq \frac{4n-2}{3n-2}$. Following the same strategy, it is possible to obtain the following tough set for some $k < n$ such that $t = \frac{kn-2}{(k-1)n-k+2}$ and $\tau \leq \frac{kn-2}{(k-1)n-k+2}$.

$$S_t = \{v_{12}, v_{21}, v_{22}\} \cup \{v_{32}, v_{42}, v_{52}, \ldots, v_{n2}\} \cup \{v_{31}, v_{41}, v_{51}, \ldots, v_{n-11}\}$$

$$\cup \{v_{1\,3}, v_{2\,3}, v_{3\,3}, \ldots, v_{n\,3}\} \cup \{v_{1\,4}, v_{2\,4}, v_{3\,4}, \ldots, v_{n\,4}\}$$

$$\cup \ldots \cup \{v_{1\,k}, v_{2\,k}, v_{3\,k}, \ldots, v_{n\,k}\} \tag{29}$$

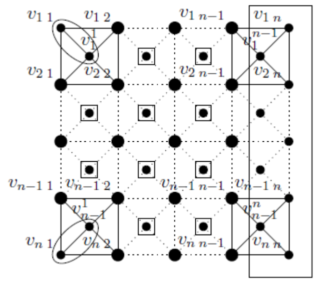

Figure 11. Illustration for Equation (29) and the Components of $EN(n, n) - S_t$.

Further, it is possible to find the following tough set by adding more vertices to S_t.

$$S_t = \{v_{1\,2}, v_{21}, v_{2\,2}\} \cup \{v_{3\,2}, v_{4\,2}, v_{5\,2}, \ldots, v_{n\,2}\} \cup \{v_{31}, v_{41}, v_{51}, \ldots, v_{n-11}\}$$

$$\cup \{v_{1\,3}, v_{2\,3}, v_{3\,3}, \ldots, v_{n\,3}\} \cup \{v_{1\,4}, v_{2\,4}, v_{3\,4}, \ldots, v_{n\,4}\} \cup \ldots \cup \{v_{1\,k}, v_{2\,k}, v_{3\,k}, \ldots, v_{n\,k}\}$$

$$\cup \ldots \cup \{v_{2\,n}, v_{3\,n}, v_{4\,n}, \ldots, v_{n-1\,n}\} \tag{30}$$

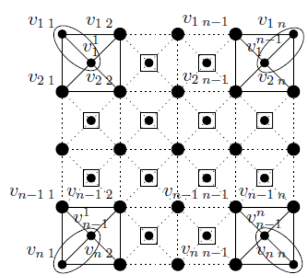

Figure 12. Illustration for Equation (30) and the Components of $EN(n, n) - S_t$.

It is easy to resolve that $\tau \leq \dfrac{n^2 - 4}{(n-1)^2}$. All that remains to prove is that

$\tau \geq \frac{n^2 - 4}{(n-1)^2}$. As n increases, τ approaches 1. Moreover, since $EN(n, n)$ is hamiltonian $\tau(EN(n, n)) \geq 1$. Hence, $\tau(EN(n, n)) = \frac{n^2 - 4}{(n-1)^2}$. □

Theorem 5. *Let S_τ be a τ-tough set of the enhanced mesh graph $EN(n, n), n \geq 3$. Then, S_τ is the maximum extension of the $\frac{n}{2}$-tough sets given by*

$$S^1_{\frac{n}{2}} = \left\{\bigcup_{i=1}^{n} v_{i\,2}\right\} \tag{31}$$

$$S^2_{\frac{n}{2}} = \bigcup_{i=1}^{n} \{v_{2\,i}\} \tag{32}$$

$$S^3_{\frac{n}{2}} = \bigcup_{i=1}^{n} \{v_{i\,n-1}\} \tag{33}$$

$$S^4_{\frac{n}{2}} = \bigcup_{i=1}^{n} \{v_{n-1\,i}\} \tag{34}$$

Proof. Without loss of generality consider the tough set $S^1_{\frac{n}{2}}$.

Let $S_{t_1} = S^1_{\frac{n}{2}}$. Consider the following tough sets and the corresponding toughness:

$$S_{t_1} = \{v_{1\,2}, v_{2\,2}, v_{3\,2}, \ldots, v_{n\,2}\}, t_1 = \frac{n}{2} \tag{35}$$

$$S_{t_2} = S_{t_1} \cup \{v_{2\,1}, v_{3\,1}, v_{4\,1}, \ldots, v_{n-1\,1}\}, t_2 = 2 - \frac{n}{2} \tag{36}$$

$$S_{t_3} = S_{t_2} \cup \{v_{1\,3}, v_{2\,3}, v_{3\,3}, \ldots, v_{n3}\}, t_3 = \frac{3n-2}{2n-1} \tag{37}$$

$$\vdots$$

$$S_{t_k} = S_{t_{k-1}} \cup \{v_{1k}, v_{2k}, v_{3k}, \ldots, v_{nk}\},\ t_k = \frac{kn-2}{(k-1)n-k+2} \tag{38}$$

$$\vdots$$

$$S_{t_{n-1}} = S_{n-2} \cup \{v_{1\,n-1}, v_{2\,n-1}, v_{3\,n-1}, \ldots, v_{n\,n-1}\},\ t_{n-1} = \frac{n^2-n-2}{n^2-3n+3} \tag{39}$$

$$S_{t_n} = S_{n-1} \cup \{v_{2n}, v_{3n}, v_{4n}, \ldots, v_{n-1n}\},\ t_n = \frac{n^2-4}{(n-1)^2} \tag{40}$$

Equations (35) to (40) imply that $S_{t_1} \subset S_{t_2} \subset S_{t_3} \subset \ldots \subset S_{t_k} \subset S_{t_{n-1}} \subset S_{t_n}$ and numerically $t_1 > t_2 > t_3 > \ldots > t_k > \ldots > t_{n-1} > t_n$. By definition, S_{t_n} is an extension of S_{t_1}. Since $t_n = \tau$, S_τ is the maximum extension of $S_{t_1} = S^1_{\frac{n}{2}}$.

By the degree properties of tough sets (31), (32), (33) and (34), these tough sets are isomorphic to each other. Hence, they yield the same toughness. Moreover, each of them is a subset of $S_{t_n} = S_\tau$. Since, S_r is the maximum extension of S^1_2, it is the maximum extension of S^2_2, S^3_2 and S^4_2. □

4. Conclusion

In this paper, we have extensively studied the toughness and maximum extension of certain t-tough sets of the extended and enhanced mesh graphs to their respective τ-tough sets. We infer that the extended mesh graph $EN(n, n)$ is less vulnerable since it is not possible to find a t-tough set of this graph such that its τ- tough sets are its maximum extension. Though certain t-tough sets of the enhanced mesh graph $EN(n, n)$ have maximum extension to its τ-tough sets, we infer that this graph is less vulnerable due to the relative largeness of its τ-tough set. Finally, we infer that the extended mesh graph is less vulnerable in comparison to the enhanced mesh graph.

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