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VAGUE $\hat{g}$ FUNCTIONS IN VAGUE TOPOLOGICAL SPACES

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Abstract

In this paper we introduced vague $\hat{g}$ continuous mappings, perfectly vague $\hat{g}$ continuous mappings, completely vague $\hat{g}$ continuous mappings, almost vague $\hat{g}$ continuous mappings and vague $\hat{g}$ irresolute mappings. We investigated some of their properties and also provided some characterization of the above mappings.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [13] in 1965. The theory of fuzzy topology was introduced by C. L. Chang [1] in 1967. The theory of vague sets was first initiated by Gau and Buehrer [2]. Norman Levine [3] initiated generalized closed (briefly g-closed) sets in 1970. M. K Veera Kumar [12] introduced $\hat{g}$-closed sets in topological spaces in 2000.

In this paper we introduced the notion of vague $\hat{g}$ continuous mappings, perfectly vague $\hat{g}$ continuous mappings, strongly vague $\hat{g}$ continuous

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mappings and vague $\hat{g}$ irresolute mappings and studied some of their properties and characterizations.

2. Preliminaries

Definition 2.1 [2]**.** A vague set $\mathbb{A}$ in the universe of discourse X is characterized by two membership functions given by:

1. A true membership function $T_{\mathbb{A}} : X \to [0, 1]$ and

2. A false membership function $F_{\mathbb{A}} : X \to [0, 1]$,

where $T_A(x)$ is lower bound on the grade of membership of x derived from the "evidence for x", $F_{\mathbb{A}}(x)$ is a lower bound on the negation of x derived from the "evidence against x" and $T_{\mathbb{A}}(x) + F_{\mathbb{A}}(x) \leq 1$. Thus the grade of membership of x in the vague set $\mathbb{A}$ is bounded by a subinterval $[T_{\mathbb{A}}(x), 1 - F_{\mathbb{A}}(x)]$ of $[0, 1]$. This indicates that if the actual grade of membership $\mu(x)$, then $T_{\mathbb{A}}(x) \leq \mu(x) \leq F_{\mathbb{A}}(x)$. The vague set $\mathbb{A}$ is written as, $\mathbb{A} = \{\langle x, [T_{\mathbb{A}}(x), 1 - F_{\mathbb{A}}(x)]\rangle / x \in X\}$ where the interval $[T_{\mathbb{A}}(x), 1 - F_{\mathbb{A}}(x)]$ is called the "vague value of x" in $\mathbb{A}$ and is denoted by $V_{\mathbb{A}}(x)$.

Definition 2.2 [2]**.** Let $\mathbb{A}$ ad $\mathbb{B}$ be vague sets of the form $\mathbb{A} = \{\langle x, [T_{\mathbb{A}}(x), 1 - F_{\mathbb{A}}(x)]\rangle / x \in X\}$ and $\mathbb{B} = \{\langle x, [T_B(x), 1 - F_B(x)]\rangle / x \in X\}$ Then

(a) $\mathbb{A} \subseteq \mathbb{B}$ if and only if $T_{\mathbb{A}}(x) \leq T_{\mathbb{B}}(x)$ and $1 - F_{\mathbb{A}}(x) \leq 1 - F_{\mathbb{B}}(x)$ for all $x \in X$

(b) $\mathbb{A} = \mathbb{B}$ if and only if $\mathbb{A} \subseteq \mathbb{B}$ and $\mathbb{B} \subseteq \mathbb{A}$

(c) $\mathbb{A}^C = \{\langle x, F_A(x), 1 - T_A(x) / x \in X\rangle\}$

(d) $\mathbb{A} \cap \mathbb{B} = \{\langle x, \min(T_{\mathbb{A}}(x), T_{\mathbb{B}}(x)), \min(1 - F_{\mathbb{A}}(x), 1 - F_{\mathbb{B}}(x))\rangle / x \in X\}$

(e) $\mathbb{A} \cup \mathbb{B} = \{\langle x, \max(T_{\mathbb{A}}(x), T_{\mathbb{B}}(x)), \max(1 - F_{\mathbb{A}}(x), 1 - F_{\mathbb{B}}(x))\rangle / x \in X\}$

For the sake of simplicity, we shall use the notation $\mathbb{A} = \langle x, T_{\mathbb{A}}, 1 - F_{\mathbb{A}} \rangle$ instead of

$$\mathbb{A} = \{\langle x, T_{\mathbb{A}}(x), 1 - F_{\mathbb{A}}(x) \rangle / x \in X\}.$$

Vague Topological Space

Definition 2.3 [5]**.** A vague topology (VT in short) on X is a family $\mathfrak{T}$ of vague sets (VS in short) in X satisfying the following axioms.

(a) $0, 1 \in \mathfrak{T}$

(b) $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \mathfrak{T}$

(c) $\cup G_i \in \tau$ for any family $\{G_i / i \in J\} \subseteq \mathfrak{T}$

In this case the pair $(X, \mathfrak{T})$ is called a vague topological space (VTS in short) and any VS in $\mathfrak{T}$ is known as a vague open set (VOS in short) in X.

The complement $\mathbb{A}^C$ of a VOS in a VTS (X, τ) is called a vague closed set (VCS in short) in X.

Definition 2.4 [5]. Let (X, τ) be a VTS and $\mathbb{A} = \langle x, T_{\mathbb{A}}, 1 - F_{\mathbb{A}} \rangle$ be a VS in X. Then the vague interior and a vague closure are defined by

$V\operatorname{int}(\mathbb{A}) = \cup\{G \setminus G$ is an VOS in X and $G \subseteq \mathbb{A}\}$ and $Vcl(A) = \cup\{K \setminus K$ is an VCS in X and $\mathbb{A} \subseteq K\}$

Note that for any Vague Set $\mathbb{A}$ in (X, τ), we have $Vcl(\mathbb{A}^C) = (V\operatorname{int}(\mathbb{A}))^C$ and $V\operatorname{int}(\mathbb{A}^C) = (Vcl(\mathbb{A}))^C$.

Definition 2.5 [5]**.** A Vague set $\mathbb{A}$ of (X, τ) is said to be

(a) a vague semi closed set (VSCS in short) if $V\operatorname{int}(Vcl(\mathbb{A})) \subseteq \mathbb{A}$.

(b) a vague semi open set (VSOS in short) if $\mathbb{A} \subseteq Vcl(V\operatorname{int}(\mathbb{A}))$.

Definition 2.6 [7]**.** A vague set $\mathbb{A}$ of (X, τ) is said to be a vague $\hat{g}$-closed sets ($V\hat{G}CS$ in short) if $Vcl(\mathbb{A}) \subseteq U$ whenever $\mathbb{A} \subseteq U$ and U is vague semi open set in X.

Every VCS is a $V\hat{G}CS$ and every $V\hat{G}CS$ is VGSCS, VGCS, VRGCS, VGPCS, VαGCS, VGαCS but the separate converses may not be true in general. The family of all $V\hat{G}CSs$ of a VTS (X, τ) is denoted by $V\hat{G}C$ (X).

Definition 2.7. Let *f* be a mapping from a VTS (X, τ) into a VTS (Y, σ). Then *f* is said to be a

(i) [6] Vague continuous mapping (V continuous mapping for short) if $f^{-1}(\mathbb{B}) \in \mathrm{VO}(X)$ for each VOS $\mathbb{B} \in Y$.

(ii) [6] Vague generalized semi continuous mapping (VGS continuous mapping for short) if $f^{-1}(\mathbb{B}) \in \mathrm{VGSO}(X)$ for each VOS $\mathbb{B} \in Y$.

(iii) [6] Vague generalized continuous mapping (VG continuous mapping for short) if $f^{-1}(\mathbb{B}) \in \mathrm{VGO}(X)$ for each VOS $\mathbb{B} \in Y$.

(iv) Vague regular generalized continuous mapping (VRG continuous mapping for short) if $f^{-1}(\mathbb{B}) \in \mathrm{VRGO}(X)$ for each VOS $\mathbb{B} \in Y$.

(v) [6] Vague generalized pre continuous mapping (VGP continuous mapping for short) if $f^{-1}(\mathbb{B}) \in \mathrm{VGPO}(X)$ for each VOS $\mathbb{B} \in Y$.

(vi) [6] Vague generalized α continuous mapping (VGα continuous mapping for short) if $f^{-1}(\mathbb{B}) \in \mathrm{VG\alpha O}(X)$ for each VOS $\mathbb{B} \in Y$.

(vii) [5] Vague α generalized continuous mapping (VαG continuous mapping for short) if $f^{-1}(\mathbb{B}) \in \mathrm{V\alpha GO}(X)$ for each VOS $\mathbb{B} \in Y$.

(viii) [5] Vague irresolute mapping (V irresolute mapping for short) if $f^{-1}(\mathbb{B}) \in \mathrm{VO}(X)$ for each VOS $\mathbb{B} \in Y$.

(ix) Completely vague continuous mapping ($\mathcal{C}$V continuous mapping for short) if $f^{-1}(\mathbb{B})$ is VRO(X) for each VOS $\mathbb{B} \in Y$.

(x) Perfectly vague continuous mapping ($\mathcal{P}$V continuous mapping for short) if $f^{-1}(\mathbb{B})$ is both VO(X) and VC(X) for each VOS $\mathbb{B} \in Y$.

(xi) Almost vague continuous mapping ($\mathcal{A}$V continuous mapping for short) if $f^{-1}(\mathbb{B})$ is VO(X) for each VROS $\mathbb{B} \in Y$.

3. Vague $\hat{g}$ Continuous Mappings

In this section we introduce vague $\hat{g}$ continuous mapping and investigate some of its properties.

Definition 3.1. A mapping $f : (X, \tau) \to (Y, \sigma)$ is called vague $\hat{g}$ continuous (V$\hat{g}$ continuous for short) mapping if $f^{-1}(\mathbb{V})$ is a V$\hat{g}$CS in (X, τ) for every VCS in (Y, σ).

Example 3.2. Let $X = \{a, b\}, Y = \{u, v\}$ and $\tau = \{0, G_1, 1\}$ and $\sigma = \{0, G_2, 1\}$ are VTs on X and Y respectively.

$G_1 = \{\langle x, [0.4, 0.8], [0.3, 0.6] \rangle\}, G_2 = \{\langle x, [0.5, 0.9], [0.4, 0.6] \rangle\}$. Define a mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$. Since the inverse image of a vague closed set $\mathbb{A} = \{\langle y, [0.1, 0.5], [0.4, 0.6] \rangle\}$ in (Y, σ) is a V$\hat{g}$CS in (X, τ). Hence f is a vague $\hat{g}$ continuous mapping.

Theorem 3.3. *Every vague continuous mapping is a V$\hat{g}$ continuous mapping but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a vague continuous mapping. Let $\mathbb{V}$ be a VCS in Y. Then $f^{-1}(\mathbb{V})$ is a V$\hat{g}$CS in X. Since every VCS is a V$\hat{g}$CS, $f^{-1}(\mathbb{V})$ is a V$\hat{g}$CS. Hence f is $V\hat{g}$ continuous mapping.

Example 3.4. Let $X = \{a, b\}, Y = \{u, v\}$ and $\tau = \{0, G_1, 1\}$ and $\sigma = \{0, G_2, 1\}$ are VTs on X and Y respectively.

$G_1 = \{\langle x, [0.4, 0.8], [0.3, 0.6] \rangle\}, G_2 = \{\langle x, [0.5, 0.9], [0.4, 0.6] \rangle\}$. Define a mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$. Since the inverse image of a vague closed set $\mathbb{A} = \{\langle y, [0.1, 0.5], [0.4, 0.6] \rangle\}$ in (Y, σ) is a V$\hat{g}$CS in (X, τ) but $f^{-1}(\mathbb{A})$ is not vague closed in (X, τ). Hence f is a vague $\hat{g}$ continuous mapping but not vague continuous mapping.

Theorem 3.5. *Every vague $\hat{g}$ continuous mapping is a vague gs continuous mapping but not conversely.*

Proof. Let $f:(X, \tau) \to (Y, \sigma)$ be a vague $\hat{g}$ continuous mapping. Let $\mathbb{V}$ be a VCS in *Y*. Then $f^{-1}(\mathbb{V})$ is a $\text{V}\hat{g}\text{CS}$ in X. Since every $\text{V}\hat{g}\text{CS}$ is a VGSCS, $f^{-1}(\mathbb{V})$ is a VGSCS. Hence *f* is VGS continuous mapping.

Example 3.6. Let $X = \{a, b\}, Y = \{u, v\}$ and $\tau = \{0, G_1, 1\}$ and $\sigma = \{0, G_2, 1\}$ are VTs on *X* and *Y* respectively.

$G_1 = \{\langle x, [0.2, 0.5], [0.1, 0.4]\rangle\}, G_2 = \{\langle x, [0.2, 0.7], [0.3, 0.6]\rangle\}$. Define a mapping $f:(X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$. Since the inverse image of a vague closed set $\mathbb{A} = \{\langle y, [0.2, 0.4], [0.1, 0.3]\rangle\}$ in (Y, σ) is a VGSCS in (X, τ) but $f^{-1}(\mathbb{A})$ is not vague $\hat{g}$ closed in (X, τ). Hence *f* is a vague GS continuous mapping but not vague $\hat{g}$ continuous mapping.

Theorem 3.7. *Every vague* $\hat{g}$ *continuous mapping is a vague g continuous mapping but not conversely*.

Proof. Let $f:(X, \tau) \to (Y, \sigma)$ be a vague $\hat{g}$ continuous mapping. Let $\mathbb{V}$ be a VCS in *Y*. Then $f^{-1}(\mathbb{V})$ is a $\text{V}\hat{g}\text{CS}$ in X. Since every $\text{V}\hat{g}\text{CS}$ is a VGCS, $f^{-1}(\mathbb{V})$ is a VGCS. Hence *f* is VG continuous mapping.

Example 3.8. Let $X = \{a, b\}, Y = \{u, v\}$ and $\tau = \{0, G_1, 1\}$ and $\sigma = \{0, G_2, 1\}$ are VTs on *X* and *Y* respectively.

$G_1 = \{\langle x, [0.3, 0.4], [0.2, 0.5]\rangle\}, G_2 = \{\langle x, [0.2, 0.6], [0.3, 0.6]\rangle\}$. Define a mapping $f:(X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$. Since the inverse image of a vague closed set $\mathbb{A} = \{\langle y, [0.2, 0.4], [0.1, 0.3]\rangle\}$ in (Y, σ) is a VGCS in (X, τ) but $f^{-1}(\mathbb{A})$ is not vague $\hat{g}$ closed in (X, τ). Hence *f* is a vague *g* continuous mapping but not vague $\hat{g}$ continuous mapping.

Theorem 3.9. *Every vague* $\hat{g}$ *continuous mapping is a vague rg continuous mapping but not conversely*.

Proof. Let $f:(X, \tau) \to (Y, \sigma)$ be a vague $\hat{g}$ continuous mapping. Let $\mathbb{V}$

be a VCS in Y. Then $f^{-1}(\mathbb{V})$ is a $\text{V}\hat{g}\text{CS}$ in X. Since every $\text{V}\hat{g}\text{CS}$ is a VRGCS, $f^{-1}(\mathbb{V})$ is a VRGCS. Hence f is VRG continuous mapping.

Example 3.10. Let $X = \{a, b\}$, $Y = \{u, v\}$ and $\tau = \{0, G_1, 1\}$ and $\sigma = \{0, G_2, 1\}$ are VTs on X and Y respectively.

$G_1 = \{\langle x, [0.5, 0.9], [0.4, 0.8]\rangle\}$, $G_2 = \{\langle x, [0.5, 0.8], [0.4, 0.7]\rangle\}$. Define a mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$ and f is a vague RG continuous mapping but not vague $\hat{g}$ continuous mapping.

Theorem 3.11. *Every vague $\hat{g}$ continuous mapping is a vague gp continuous mapping but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a vague $\hat{g}$ continuous mapping. Let $\mathbb{V}$ be a VCS in Y. Then $f^{-1}(\mathbb{V})$ is a $\text{V}\hat{g}\text{CS}$ in X. Since every $\text{V}\hat{g}\text{CS}$ is a VGPCS, $f^{-1}(\mathbb{V})$ is a VGPCS. Hence f is VGP continuous mapping.

Example 3.12. Let $X = \{a, b\}$, $Y = \{u, v\}$ and $\tau = \{0, G_1, 1\}$ and $\sigma = \{0, G_2, 1\}$ are VTs on X and Y respectively.

$G_1 = \{\langle x, [0.5, 0.9], [0.4, 0.8]\rangle\}$, $G_2 = \{\langle x, [0.4, 0.8], [0.4, 0.7]\rangle\}$. Define a mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and v.

Since the inverse image of a vague closed set $\mathbb{A} = \{\langle y, [0.2, 0.6], [0.3, 0.6]\rangle\}$ in (Y, σ) is a VGPCS in (X, τ) but $f^{-1}(\mathbb{A})$ is not vague $\hat{g}$ closed in (X, τ). Hence f is a vague GP continuous mapping but not vague $\hat{g}$ continuous mapping.

Theorem 3.13. *Every vague $\hat{g}$ continuous mapping is a vague gsp continuous mapping but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a vague $\hat{g}$ continuous mapping. Let V be a VCS in Y. Then $f^{-1}(\mathbb{V})$ is a $\text{V}\hat{g}\text{CS}$ in X. Since every $\text{V}\hat{g}\text{CS}$ is a *VGSPCS*, $f^{-1}(\mathbb{V})$ is a VGSPCS. Hence f is VGSP continuous mapping.

Example 3.14. Let $X = \{a, b\}, Y\{u, v\}$ and $\tau = \{0, G_1, 1\}$ and $\sigma = \{0, G_2, 1\}$ are VTs on X and Y respectively.

$G_1\{\langle x, [0.5, 0.8], [0.4, 0.7]\rangle\}, G_2\{\langle x, [0.4, 0.8], [0.4, 0.7]\rangle\}$. Define a mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$ and f is a vague GSP continuous mapping but not vague $\hat{g}$ continuous mapping.

Theorem 3.15. *Every vague $\hat{g}$ continuous mapping is a vague αg continuous mapping but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a vague $\hat{g}$ continuous mapping. Let $\mathbb{V}$ be a VCS in Y. Then $f^{-1}(\mathbb{V})$ is a $V\hat{g}CS$ in X. Since every $V\hat{g}CS$ is a VαGCS, $f^{-1}(\mathbb{V})$ is a VαGCS. Hence f is VαG continuous mapping.

Example 3.16. Let $X = \{a, b\}, Y\{u, v\}$ and $\tau = \{0, G_1, 1\}$ and $\sigma = \{0, G_2, 1\}$ are VTs on X and Y respectively.

$G_1\{\langle x, [0.5, 0.8], [0.4, 0.7]\rangle\}, G_2\{\langle x, [0.4, 0.8], [0.4, 0.7]\rangle\}$. Define a mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$. Since the inverse image of a vague closed set $\mathbb{A} = \{\langle y, [0.2, 0.4], [0.3, 0.6]\rangle\}$ in (Y, σ) is a VαGCS in (X, τ) but $f^{-1}(\mathbb{A})$ is not vague $\hat{g}$ closed in (X, τ). Hence f is a vague αG continuous mapping but not vague $\hat{g}$ continuous mapping.

The relation between various types of vague $\hat{g}$ continuity is given in the figure 4.1. In that figure ‘cts’ means continuous.

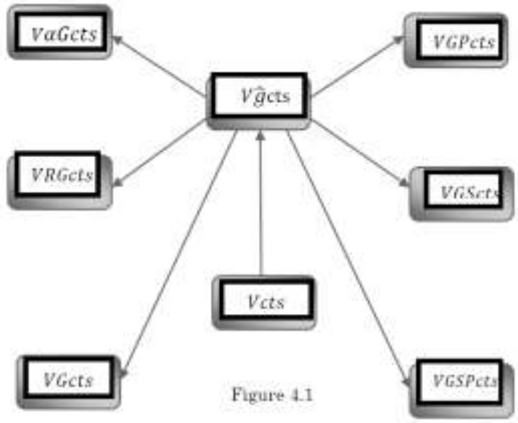

Figure 4.1

The reverse implications are not true in general in the above figure.

Theorem 3.17. *The following statements are equivalent for a function* $f : (X, \tau) \to (Y, \sigma)$

(i) *f is vague* $\hat{g}$ *continuous.*

(ii) *For every vague open set V of Y,* $f^{-1}(\mathbb{V})$ *is vague* $\hat{g}$ *open set in X.*

Proof. (i) $\Rightarrow$ (ii) Let V be vague open subset of Y and let $x \in f^{-1}(\mathbb{V})$ be any arbitrary point. Since $f(x) \in \mathbb{V}$ by (i), there exist vague $\hat{g}$ open set in Ux in X, containing x such that arbitrary union of vague $\hat{g}$ open sets is vague $\hat{g}$ open, $f^{-1}(\mathbb{V})$ is vague $\hat{g}$ open in X.

(ii) $\Rightarrow$ (i) it is obvious.

Theorem 3.18. *If* $f : (X, \tau) \to (Y, \sigma)$ *is vague* $\hat{g}$ *continuous then for each vague point* $c(\alpha, \beta)$ *of X and each vague open set* $\mathbb{V}$ *of Y such that* $f(c(\alpha, \beta)) \subseteq \mathbb{V}$ *there exist a vague* $\hat{g}$ *open set U of X such that* $c(\alpha, \beta) \subseteq U$ *and* $f(U) \in \mathbb{V}$.

Proof. Let $c(\alpha, \beta)$ be a vague point of X and $\mathbb{V}$ be a vague open set of Y such that $f(c(\alpha, \beta)) \subseteq \mathbb{V}$. Put $U = f^{-1}(\mathbb{V})$. Then by hypothesis U is a vague $\hat{g}$ open set of X such that $c(\alpha, \beta) \subseteq U$ and $f(U) = f(f^{-1}(\mathbb{V})) \subseteq \mathbb{V}$.

Theorem 3.19. *Let* $f:(X, \tau) \rightarrow (Y, \sigma)$ *is a vague* $\hat{g}$ *continuous then for each vague point* $c(\alpha, \beta)$ *of X and each vague open set* $\mathbb{V}$ *of Y such that* $f(c(\alpha, \beta))_{Vq}\mathbb{V}$, *there exists a vague* $\hat{g}$ *open set U of X such that* $c(\alpha, \beta)_{Vq}U$ *and* $f(U) \subseteq \mathbb{V}$.

Proof. Let $c(\alpha, \beta)$ be a vague point if X and $\mathbb{V}$ be a vague open set of Y such that $f(c(\alpha, \beta))_{Vq}\mathbb{V}$. Put $U = f^{-1}(\mathbb{V})$. Then by hypothesis U is a vague $\hat{g}$ open set of X such that $c(\alpha, \beta)_{Vq}U$ and $f(U) = f(f^{-1}(\mathbb{V})) \subseteq \mathbb{V}$.

Theorem 3.20. *If* $f:(X, \tau) \rightarrow (Y, \sigma)$ *is vague* $\hat{g}$ *continuous then* $f(V\hat{g} - cl(A)) \subset Vcl(f(\mathbb{A}))$ *for every vague subset* $\mathbb{A}$ *of X.*

Proof. Let $\mathbb{A} \subseteq X$. Then $Vcl(f(\mathbb{A}))$ is a vague closed in Y, since f is vague $\hat{g}$ continuous, $f^{-1}(Vcl(f(\mathbb{A})))$ is vague $\hat{g}$ closed in X and $\mathbb{A} \subseteq f^{-1}(f(A))$ $\subseteq f^{-1}(Vcl(f(\mathbb{A})))$. Therefore $V\hat{g} - cl(\mathbb{A}) \subseteq V\hat{g}cl(f^{-1}(Vcl(f(\mathbb{A}))) = f^{-1}(Vcl(f(\mathbb{A})))$. Hence $f(V\hat{g} - cl(A)) \subset Vcl(f(\mathbb{A}))$ for every vague subset $\mathbb{A}$ of X.

Theorem 3.21. *Let* (X, τ) *and* (Y, σ) *be any two VTS. Let* $f:(X, \tau) \rightarrow (Y, \sigma)$ *be a vague* $\hat{g}$ *continuous mapping. Then for every vague set A in Y,* $V\hat{g}cl(f^{-1}(\mathbb{A})) \subseteq f^{-1}(Vcl(\mathbb{A}))$.

Proof. Let $\mathbb{A}$ be a vague set in (Y, σ). Let $\mathbb{B} = f^{-1}(\mathbb{A})$. Then $f(B) = f(f^{-1}(\mathbb{A})) \subseteq \mathbb{A}$. Then by the theorem 3.20 $f(V\hat{g} - cl(f^{-1}(\mathbb{A})))$ $\subseteq Vcl(f(f^{-1}(\mathbb{A})))$. Thus $V\hat{g}cl(f^{-1}(\mathbb{A})) \subseteq f^{-1}(Vcl(\mathbb{A}))$.

Theorem 3.22. *Let* $f:(X, \tau) \rightarrow (Y, \sigma)$ *be a mapping from a VTS X into a VTS Y. Then the following conditions are equivalent, if X is a* $V\hat{g}T_{1/2}$ *space*:

(i) *f is a* $V\hat{g}$ *continuous mapping*.

(ii) *If* $\mathbb{B}$ *is a VOS in Y then* $f^{-1}(\mathbb{B})$ *is a* $V\hat{g}OS$ *in X.*

(iii) $f^{-1}(V \operatorname{int} \mathbb{B}) \subseteq Vcl(V \operatorname{int}(Vcl(f^{-1}(\mathbb{B}))))$ *for every vague set* $\mathbb{B}$ *in* Y.

Proof. (i) $\Rightarrow$ (ii) is obviously true by the theorem.

(ii) $\Rightarrow$ (iii) Let $\mathbb{B}$ be any vague set in Y. Then $\operatorname{int}(\mathbb{B})$ is a vague open set in Y. Then $f^{-1}(V \operatorname{int} \mathbb{B})$ is a V$\hat{g}$OS in X. Since X is a $V\hat{g}T_{1/2}$ space, $f^{-1}(V \operatorname{int} \mathbb{B})$ is a V$\hat{g}$OS in X. Therefore $f^{-1}(V \operatorname{int}(\mathbb{B})) \subseteq Vcl(V \operatorname{int}(Vcl(f^{-1}(V \operatorname{int}(\mathbb{B})))) \subseteq Vcl(V \operatorname{int}(Vcl(f^{-1}(\mathbb{B})))$.

(iii) $\Rightarrow$ (i) Let $\mathbb{B}$ be a VCS in Y. Then its complement $\mathbb{B}^C$ is a VOS in Y, then $\operatorname{int} \mathbb{B}^C = \mathbb{B}^C$ (say $\mathbb{B}^C = \mathbb{A}$). By hypothesis $f^{-1}(\operatorname{int} A) \subseteq cl(\operatorname{int}(cl(f^{-1}(\mathbb{A}))) \Rightarrow f^{-1}(A) \subseteq cl(\operatorname{int}(cl(f^{-1}(\mathbb{A})))$. Hence $f^{-1}(\mathbb{A})$ is a VOS in X. Since every VOS is a V$\hat{g}$OS, hence $f^{-1}(\mathbb{A})$ is a V$\hat{g}$OS in X. Thus $f^{-1}(\mathbb{B})$ is a V$\hat{g}$CS in X, since $f^{-1}(\mathbb{A}) = f^{-1}(\mathbb{B}^C)$. Hence f is a $V\hat{g}$ continuous mapping.

Theorem 3.23. *Let* $f : (X, \tau) \to (Y, \sigma)$ be a $V\hat{g}$ *continuous mapping and* $g : (Y, \sigma) \to (Z, \gamma)$ *is a V continuous mapping, then* $g \circ f : (X, \tau) \to (Z, \gamma)$ *is a* $V\hat{g}$ *continuous mapping.*

Proof. Let $\mathbb{V}$ be a VCS in Z. Then $g^{-1}(\mathbb{V})$ is a VCS in Y, by hypothesis. Since f is a $V\hat{g}$ continuous mapping, $f^{-1}(g^{-1}(\mathbb{V}))$ is a V$\hat{g}$CS in X. Hence $g \circ f$ is a $V\hat{g}$ continuous mapping.

Theorem 3.24. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a* $v\hat{g}$ *continuous mapping and* $g : (Y, \sigma) \to (Z, \gamma)$ *be a VG continuous mapping, then* $g \circ f : (X, \tau) \to (Z, \gamma)$ *is a* $v\hat{g}$ *continuous mapping.*

Proof. Let $\mathbb{V}$ be a VCS in Z. Then $g^{-1}(\mathbb{V})$ is a VCS in Y. Since f is a $V\hat{g}$ continuous mapping, $f^{-1}(g^{-1}(\mathbb{V}))$ is a V$\hat{g}$CS in X. Hence $g \circ f$ is a $V\hat{g}$ continuous mapping.

4. Vague $\hat{g}$ Irresolute Continuous Mappings

In this section we have introduced vague $\hat{g}$ irresolute mappings and studied some of their properties.

Definition 4.1. A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be a vague $V\hat{g}$ irresolute ($V\hat{G}$ irresolute for short) mapping if $f^{-1}(\mathbb{V})$ is a $V\hat{g}$ CS in (X, τ) for every $\mathrm{V}\hat{g}\mathrm{CS}$ $\mathbb{V}$ of (Y, σ).

Theorem 4.2. *If $f : (X, \tau) \to (Y, \sigma)$ is a $V\hat{g}$ irresolute mapping, then f is a $V\hat{g}$ continuous mapping but not conversely.*

Proof. Let *f* be a $V\hat{g}$ irresolute mapping. Let $\mathbb{V}$ be any VCS in *Y*. Then $\mathbb{V}$ is a $\mathrm{V}\hat{g}\mathrm{CS}$ and by hypothesis $f^{-1}(\mathbb{V})$ is a $V\hat{g}$ CS in *X*. Hence *f* is a $V\hat{g}$ continuous mapping.

Example 4.3. Let $X = \{a, b\}$, $Y = \{u, v\}$ and $\tau = \{0, G_1, 1\}$ and $\sigma = \{0, G_2, 1\}$ are VTs on *X* and *Y* respectively.

$G_1\{\langle x, [0.5, 0.6], [0.4, 0.5]\rangle\}$, $G_2\{\langle x, [0.3, 0.5], [0.5, 0.5]\rangle\}$. Define a mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$. Then f is a $V\hat{g}$ continuous mapping but not a $v\hat{g}$ irresolute mapping.

Theorem 4.4. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \gamma)$ be a $V\hat{g}$ irresolute mapping. Then $g \circ f : (X, \tau) \to (Z, \gamma)$ is a $V\hat{g}$ irresolute mapping.*

Proof. Let $\mathbb{V}$ be a $\mathrm{V}\hat{g}\mathrm{CS}$ in *Z*. Then $g^{-1}(\mathbb{V})$ is a $\mathrm{V}\hat{g}\mathrm{CS}$ in *Y*, since *g* is a $V\hat{g}$ irresolute. Now by hypothesis $f^{-1}(g^{-1}(\mathbb{V}))$ is a $\mathrm{V}\hat{g}\mathrm{CS}$ in *X*. Hence $g \circ f$ is a $V\hat{g}$ irresolute mapping.

Theorem 4.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a $V\hat{g}$ irresolute mapping and $g : (Y, \sigma) \to (Z, \gamma)$ be a $V\hat{g}$ continuous mapping, then $g \circ f : (X, \tau) \to (Z, \gamma)$ is a $V\hat{g}$ continuous mapping.*

Proof. Let $\mathbb{V}$ be a VCS in Z. Then $g^{-1}(\mathbb{V})$ is a $V\hat{g}CS$ in Y. Since f is a $V\hat{g}$ irresolute mapping, $f^{-1}(g^{-1}(\mathbb{V}))$ is a $V\hat{g}CS$ in X. Hence $g \circ f$ is a $V\hat{g}$ continuous mapping.

5. Perfectly Vague $\hat{g}$ Continuous Mappings, Completely Vague $\hat{g}$ Continuous Mappings and Almost Vague $\hat{g}$ Continuous Mappings

In this section we have introduced perfectly vague $\hat{g}$ continuous mappings, completely vague $\hat{g}$ continuous mappings and almost vague $\hat{g}$ continuous mappings. Also investigated some of its properties.

Definition 5.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be perfectly vague $\hat{g}$ continuous ($\mathcal{P}V\hat{g}$ continuous mapping for short) if $f^{-1}(\mathbb{B})$ is both VO(X) and VC(X) for each V$\hat{g}$CS $\mathbb{B} \in Y$.

Definition 5.2. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be completely vague $\hat{g}$ continuous ($\mathcal{C}V\hat{g}$ continuous mapping for short) if $f^{-1}(\mathbb{B})$ is VRC(X) for each V$\hat{g}$C set $\mathbb{B} \in Y$.

Definition 5.3. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost vague $\hat{g}$ continuous ($\mathcal{A}V\hat{g}$ continuous mapping for short) if $f^{-1}(\mathbb{B})$ is V$\hat{g}$C(X) for each VRC set $\mathbb{B} \in Y$.

Theorem 5.4. *Every $\mathcal{P}V\hat{g}$ continuous mapping is a vague continuous mapping but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\mathcal{P}V\hat{g}$ continuous mapping. Let $\mathbb{A}$ be a VCS in Y. Since every VCS is a $V\hat{g}CS$, $\mathbb{A}$ is a V$\hat{g}$CS in Y. Since f is a $\mathcal{P}V\hat{g}$ continuous mapping, $f^{-1}(\mathbb{A})$ is a vague clopen in X. Thus $f^{-1}(\mathbb{A})$ is a VCS in X. Hence f is a V continuous mapping.

Example 5.5. Let $X = \{a, b\}, Y\{u, v\}$ and $\tau = \{0, G_1, 1\}$ and $\sigma = \{0, G_2, 1\}$ are VTs on X and Y respectively.

$G_1\{\langle x, [0.4, 0.5], [0.5, 0.6]\rangle\}$, $G_2\{\langle x, [0.4, 0.5], [0.5, 0.6]\rangle\}$. Define a mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$. Then f is a V continuous mapping but not a $\mathcal{P}V\hat{g}$ continuous mapping.

Theorem 5.6. *Every $\mathcal{P}V\hat{g}$ continuous mapping is a vague $\hat{g}$ continuous mapping but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\mathcal{P}V\hat{g}$ continuous mapping. Let $\mathbb{A}$ be a VCS in Y. Since every VCS is a $V\hat{g}CS$, $\mathbb{A}$ is a V$\hat{g}$CS in Y. Since f is a $\mathcal{P}V\hat{g}$ continuous mapping, $f^{-1}(\mathbb{A})$ is a vague clopen in X. Thus $f^{-1}(\mathbb{A})$ is a VCS in X. Since every VCS is a $V\hat{g}CS$, $f^{-1}(\mathbb{A})$ is a V$\hat{g}$CS. Hence f is a $V\hat{g}$ continuous mapping.

Example 5.7. Let $X = \{a, b\}, Y\{u, v\}$ and $\tau = \{0, G_1, 1\}$ and $\sigma = \{0, G_2, 1\}$ are VTs on X and Y respectively.

$G_1\{\langle x, [0.4, 0.5], [0.5, 0.6]\rangle\}$, $G_2\{\langle x, [0.6, 0.7], [0.3, 0.4]\rangle\}$. Define a mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$. Then f is a $V\hat{g}$ continuous mapping but not a $\mathcal{P}V\hat{g}$ continuous mapping.

Theorem 5.8. *Every $\mathcal{P}V\hat{g}$ continuous mapping is an almost vague $\hat{g}$ continuous mapping but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\mathcal{P}V\hat{g}$ continuous mapping. Let $\mathbb{A}$ be a VRCS in Y. Since every VRCS is a V$\hat{g}$CS, $\mathbb{A}$ is a V$\hat{g}$CS in Y. Since f is a $\mathcal{P}V\hat{g}$ continuous mapping, $f^{-1}(\mathbb{A})$ is a vague clopen in X. Thus $f^{-1}(\mathbb{A})$ is a VCS in X. Since every VCS is a $V\hat{g}CS$, $f^{-1}(\mathbb{A})$ is a V$\hat{g}$CS. Hence f is a $\mathcal{A}V\hat{g}$ continuous mapping.

Example 5.9. Let $X = \{a, b\}, Y\{u, v\}$ and $\tau = \{0, G_1, 1\}$ and $\sigma = \{0, G_2, 1\}$ are VTs on X and Y respectively.

$G_1\{\langle x, [0.4, 0.5], [0.5, 0.6]\rangle\}$, $G_2\{\langle x, [0.2, 0.3], [0.7, 0.8]\rangle\}$. Define a mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$. Then f is a $\mathcal{A}V\hat{g}$ continuous mapping but not a $\mathcal{P}V\hat{g}$ continuous mapping.

Theorem 5.10. *Every $\mathcal{P}V\hat{g}$ continuous mapping is a $\mathcal{C}V\hat{g}$ continuous mapping.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\mathcal{P}V\hat{g}$ continuous mapping. Let $\mathbb{A}$ be a $V\hat{g}CS$ in Y. Since f is a $PV\hat{g}$ continuous mapping, $f^{-1}(\mathbb{A})$ is a vague clopen in X. Thus $f^{-1}(\mathbb{A})$ is a VCS in X. Since every VCS is a $V\hat{g}CS$, $f^{-1}(\mathbb{A})$ is a $V\hat{g}CS$. Hence f is a $\mathcal{A}V\hat{g}$ continuous mapping. Therefore $Vcl(f^{-1}(\mathbb{A}))$ $= f^{-1}(\mathbb{A})$ and $\text{int}(f^{-1}(\mathbb{A})) = f^{-1}(\mathbb{A})$. Now $Vcl(V\,\text{int}(f^{-1}(\mathbb{A}))) = Vcl(Vf^{-1}(\mathbb{A}))$ $= f^{-1}(\mathbb{A})$. Therefore $f^{-1}(\mathbb{A})$ is VRCS in X. Hence f is a $\mathcal{C}V\hat{g}$ continuous mapping.

Theorem 5.11. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is a $\mathcal{P}V\hat{g}$ continuous mapping if and only if the inverse image of each* $\text{V}\hat{g}\text{OS}$ *in Y is a vague clopen in X.*

Proof. Necessity. Let a mapping $f : (X, \tau) \to (Y, \sigma)$ be $\mathcal{P}V\hat{g}$ continuous mapping. Let $\mathbb{A}$ be a $\text{V}\hat{g}\text{OS}$ in Y. Then $\mathbb{A}^C$ is a $\text{V}\hat{g}\text{CS}$ in Y. Since f is a $\mathcal{P}V\hat{g}$ continuous mapping, $f^{-1}(\mathbb{A}^C)$ is vague clopen in X. As $f^{-1}(\mathbb{A}^C) = (f^{-1}(\mathbb{A}))^C$. Thus $(f^{-1}(\mathbb{A}))$ is vague clopen in X.

Sufficiency. Let $\mathbb{B}$ be a $V\hat{g}CS$ in Y. Then $\mathbb{B}^C$ is a $\text{V}\hat{g}\text{OS}$ in Y. By hypothesis, $(f^{-1}(\mathbb{B}^C))$ is vague clopen in X. Thus $(f^{-1}(\mathbb{B}))$ is vague clopen in X, as $f^{-1}(\mathbb{B}^C) = (f^{-1}(\mathbb{B}))^C$. Therefore f is a $\mathcal{P}V\hat{g}$ continuous mapping.

Theorem 5.12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a vague continuous mapping and $g : (Y, \sigma) \to (Z, \gamma)$ is $\mathcal{P}V\hat{g}$ continuous mapping, then $g \circ f : (X, \tau) \to (Z, \gamma)$ is a $\mathcal{P}V\hat{g}$ continuous mapping.*

Proof. Let $\mathbb{A}$ be a $V\hat{g}CS$ in Z. Since g is a $\mathcal{P}V\hat{g}$ continuous mapping, $g^{-1}(\mathbb{A})$ is a vague clopen in Y. Since f is a vague continuous mapping,

$f^{-1}(g^{-1}(\mathbb{A}))$ is a VCS in X, as well as VOS in X. Hence $g \circ f$ is a $\mathcal{P}V\hat{g}$ continuous mapping.

Theorem 5.13. *The composition of two* $\mathcal{P}V\hat{g}$ *continuous mapping is a* $\mathcal{P}V\hat{g}$ *continuous mapping*.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \gamma)$ be any two $\mathcal{P}V\hat{g}$ continuous mapping. Let $\mathbb{A}$ be a $V\hat{g}CS$ in Z. By hypothesis, $g(\mathbb{A})$ is vague clopen in Y. Since every VCS is a $V\hat{g}CS$, $g^{-1}(\mathbb{A})$ is a $V\hat{g}CS$ in Y. Since f is a $\mathcal{P}V\hat{g}$ continuous mapping $f^{-1}(g^{-1}(\mathbb{A})) = (g \circ f)^{-1}(\mathbb{A})$ is vague clopen in X. Hence $g \circ f$ is a $\mathcal{P}V\hat{g}$ continuous mapping.

Theorem 5.14. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a* $\mathcal{P}V\hat{g}$ *continuous mapping and* $g : (Y, \sigma) \to (Z, \gamma)$ *is a* $V\hat{g}$ *irresolute mapping, then* $g \circ f : (X, \tau) \to (Z, \gamma)$ *is a* $\mathcal{P}V\hat{g}$ *continuous mapping*.

Proof. Let $\mathbb{A}$ be a $V\hat{g}CS$ in Z. By hypothesis, $g^{-1}(\mathbb{A})$ is a $V\hat{g}CS$ in Y. Since f is a $\mathcal{P}V\hat{g}$ continuous mapping, $f^{-1}(g^{-1}(\mathbb{A})) = (g \circ f)^{-1}(\mathbb{A})$ is vague clopen in X. Hence $g \circ f$ is a $\mathcal{P}V\hat{g}$ continuous mapping.

Theorem 5.15. *If* $f : (X, \tau) \to (Y, \sigma)$ $\mathcal{P}V\hat{g}$ *continuous mapping and* $\mathbb{A}$ *is a subset of X, then* $g : \mathbb{A} \to (Y, \sigma)$ $\mathcal{P}V\hat{g}$ *continuous mapping*.

Proof. Let V be any vague open subset of Y. Then $f^{-1}(\mathbb{V})$ is vague clopen in X. Then $f^{-1}(\mathbb{V}) \cap \mathbb{A} = g^{-1}(\mathbb{V})$ is vague clopen in $\mathbb{A}$. Hence g is $\mathcal{P}V\hat{g}$ continuous mapping.

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