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SOMEWHAT CONTINUITY OF κ-OPERATION ON TOPOLOGICAL SPACES

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Abstract

In this paper, we introduce the notion of somewhat (κ, κ')- continuity under the operation κ. Here κ and κ' are mapping from gs-open sets of (X, τ) and (y, σ) respectively to the power set, $P(X)$. Thus its properties and characterizations are studied with κ-dense and κ-equivalent defined.

1. Introduction

In 1979, Kasahara introduced the concepts of operation in topological spaces and operation-closed graph of a function. Several known characterization of compact spaces, H-closed spaces and nearly compact spaces are unified by generalizing the notion of compactness with the help of a certain operation of a topology τ into the power set $P(x)$, by choosing some special mappings $\gamma : \tau \rightarrow p(X)$ such as γ the identity mapping, the closure operation or the interior closure operation. In 1983, Jankvoic introduced and

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studied the concept of operation-closures of a subset, operation closed sets in a topological space and several related topics using the concept of α-closed sets and the α-closed graphs. Here in this paper the author has introduced a new concept called somewhat continuity of κ-operation where κ is the operation from gs-open sets to power set $p(X)$.

3. Preliminaries

Definition 3.1. Let (X, τ) be a topological space. A subset A of a space (X, τ) is called generalized semi closed gs-closed) set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ).

Definition 3.2. Let (X, τ) be a topological space. A subset A of a space (X, τ) is called generalized semi open (gs-open) set if $X \setminus A$ is gs-closed. The collection of all gs-open sets is denoted by $GSO(X, \tau)$. Clearly $\tau \subseteq GSO(X, \tau)$.

Definition 3.3. Let (X, τ) be a topological space. An operation $\gamma : \tau \to P(X)$ is a mapping from τ into the power set of X such that $V \subseteq V^{\gamma}$ for each $V \in \tau$, where V^{γ} denotes the value of γ at V.

Definition 3.4. A subset A of a space (X, τ) will be called a γ-open set of X, τ if for each $x \in A$, there exists an open set U such that $x \in U$ and $U^{\gamma} \subset A$. τ will denote the set of all γ-open sets. Clearly we have $\tau \supset \tau_{\gamma}$.

Definition 3.5 [7]**.** A subset B of (X, τ) is said to be γ-closed in (X, τ) if $X \setminus B$ is γ-open in (X, τ).

Definition 3.6 [7]**.** A point $x \in X$ is in the γ-closure of a set $A \subseteq X$ if $U^{\gamma} \cap A \neq \phi$ for each open set U of x. The γ closure of a set A is denoted by $Cl_{\gamma}(A)$.

Definition 3.7 [7]**.** An operation $\gamma : \tau \to P(X)$ is a mapping from τ into the power set $P(X)$.

$$\tau_{\gamma} - Cl(A) = \bigcap \{F : A \subseteq F, X \setminus F \in \tau_{\gamma}\}.$$

Where τ_γ denotes the set of all γ-open sets in (X, τ).

Definition 3.8 [4]**.** Let (X, τ) be a topological space. A mapping $\kappa : GSO(X, \tau) \to P(X)$ from the family of generalized semi open sets $GSO(X, \tau)$ to the power set of $P(X)$ such that $V \subseteq V^\kappa$ for every $V \in GSO(X, \tau)$ where V^κ denotes the value of V under the operation κ.

Definition 3.9 [4]**.** A subset A of a space (X, τ) will be called a κ-open set of (X, τ) if for each $x \in A$, there exists a gs-open neighbourhood U of x and $U^\kappa \subseteq A$.

Definition 3.10 [4]**.** A κ-operation $\kappa : GSO(X, \tau) \to P(X)$ is called regular κ operation given $x \in X$ and for each pair of gs-open neighbourhoods A and B of x, there exists a gs-open neighbourhood C of x such that $A^\kappa \cap B^\kappa \supseteq C^\kappa$.

Definition 3.11 [4]**.** A topological space (X, τ) is called κ-regular if for given $x \in X$ and each gs-open neighbourhood U of x, there exists a gs-open neighbourhood V of x such that $V^\kappa \subseteq U$.

Definition 3.12 [4]**.** A subset A of a topological space (X, τ) is called κ-closed whenever $X - A$ is κ-open.

Definition 3.13 [4]**.** Let κ be an operation on $GSO(X, \tau)$. A point $x \in X$ is said to be a κ-closure point of the set A if $U^\kappa \cap A \neq \phi$ for each gs-open neighbourhood U of x. $gs\,Cl_\kappa(A) = \{x \in X / U^\kappa \cap A \neq \phi, \forall U, gs\text{-open neighbourhood of } x\}$.

Definition 3.14 [4]**.** Let κ be an operation on $GSO(X, \tau)$. Then $gs_\kappa Cl(A)$ is defined as the intersection of all κ-closed sets containing $gs_\kappa Cl(A) = \bigcap \{F \subseteq X / A \subseteq F \text{ and } X/F \in \kappa 0(X, \tau)\}$.

Definition 3.15 [4]**.** An operation κ on $GSO(X, \tau)$ is said to be open κ operation if for every gs-open neighbourhood U of $x \in X$, there exists a κ-open set V such that $x \in V$ and $V \subset U^\kappa$.

Definition 3.16. A subset A of (X, τ) is said to be κ-g-closed if $gscl_{\kappa}(A) \subseteq U$ whenever $A \subseteq U$ and U is κ-open in (X, τ).

4. Somewhat (κ, κ')- Continuity

Definition 4.1. A function f is said to be somewhat (κ, κ')- continuity if for κ-open set V of (Y, σ) and $f^{-1}(V) \neq \emptyset$, there exists a non empty κ-g-open set U in (X, τ) such that $U \subseteq f^{-1}(V)$.

Example 4.2. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$, $\sigma = \{X, \emptyset, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ such that $f(a) = b$, $f(b) = a$ and $f = (c) = a$. Here κ-g-open sets are $X, \emptyset, \{a\}, \{b\}, \{c\}$ and κ'-open sets are $X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}$. Then f is somewhat (k, k')- continuity.

Example 4.3. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$, $\sigma = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ such that $f(a) = c$, $f(b) = b$ and $f(c) = c$. Here κ-g-open sets are $X, \emptyset, \{a\}, \{a, b\}, \{a, c\}$ and κ-open sets are $X, \emptyset, \{a, b\}$. Here the function is not f is somewhat (k, k')- continuity. Since for $f^{-1}(a, b) = \{b\}$ there is no κ-g-open set contained in $\{b\}$.

Remark 4.4. Composition of two somewhat (κ, κ')- continuous functions need not be somewhat (κ, κ')- continuous in general and is shown in the following example.

Theorem 4.5. *If f is somewhat (κ, κ')- continuous and g is gs-(κ, κ') continuous then $g \circ f$ is somewhat (κ, κ')- continuous.*

Proof. Let V be κ-open set, then $g^{-1}(V)$ is κ-open set in Y. Since g is gs-(κ, κ') continuous, now f is somewhat (κ, κ')- continuous. Thus $f^{-1}(g^{-1}(V))$ will contain a non empty κ-g-open set U. That is $\subseteq f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$. Therefore $g \circ f$ is somewhat (κ, κ')- continuous.

Corollary 4.6. *If f is somewhat (κ, κ')- continuous and g is super*

(κ, κ')-*continuous then* $g \circ f$ *is somewhat* (κ, κ')-*continuous.*

Proof. Let V is κ-open in Z. Since is g is super (κ, κ')-continuous, $g^{-1}(V) = \{A^{\kappa'}$ for same $A \in GSO(Y, \sigma)\}$. Since $A^{\kappa'}$ is κ-open, $g^{-1}(V)$ is κ-open (from lemma that every U^{κ} is κ-open). Since f is somewhat (κ, κ')-continuous, $f^{-1}(g^{-1}(V))$ contain a κ-open set U in X such that $U \subseteq f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$. Therefore $g \circ f$ is somewhat (κ, κ')-continuous.

Definition 4.7. A subset $A \subseteq X$ is called κ-dense in (X, τ) if $X = gs_{\kappa}cl(A)$.

Lemma 4.8. *A subset A of* (X, τ) *is* κ-*dense in* (X, τ) *if there is no proper* κ-*closed set C in* (X, τ) *such that* $A \subseteq C \subseteq X$.

Proof. Suppose there is a proper κ-closed set in X such that $A \subsetneq C \subsetneq X \ldots 1$.

Since A is κ-dense, $X = gs_{\kappa}cl(A)$.

$gs_{\kappa}cl(A) = \bigcap\{M \mid A \subseteq M$ and M is κ-closed in $X\} \ldots 2$

1 and 2 implies C is one set in the intersection of 2. Implies $X = C$.

Therefore C is not proper subset satisfying condition that $A \subseteq C \subseteq X$.

Theorem 4.9. *For a surjective function f the following statements are equivalent.*

(a) *f is somewhat* (κ, κ')-*continuous*

(b) *if C is* κ-*closed subset of* (Y, σ) *such that* $f^{-1}(C) \neq \emptyset$ *there exists a proper* κ-*g-closed subset D of* (X, τ) *such that* $f^{-1}(C) \subseteq D$.

(c) *If A is a* κ-*dense subset of* (X, τ) *then* $f(A)$ *is a dense subset of* (Y, σ).

Proof. $(a) \Rightarrow (b)$ Given f is somewhat (κ, κ')-continuous function and C is a κ'-closed subset of (Y, σ) then $Y - C$ is a κ'-open subset such that

$f^{-1}(C) \neq \emptyset$. Since f is somewhat (κ, κ')-continuous there is a proper κ-g-open set of (X, τ) such that $U \subseteq f^{-1}Y - C = X - f^{-1}(C) \neq \emptyset$. That is $f^{-1}C \subseteq X - U = D$ say, where D is κ-g-closed set.

(b) $\Rightarrow$ (c) Let $V \in_{\kappa} O(Y, \sigma)$ such that $f^{-1}V = \emptyset, Y - U = X - f^{-1}(U) \neq X$. By condition ($b$), there exists κ'-closed set D such that $f^{-1}(Y - U) \subset D$. Implies $X - f^{-1}(U) \subseteq D$. Thus $X - D \subseteq f^{-1}(U)$ where $X - D$ is κ-g-open. Hence f is somewhat (κ, κ')-continuous.

(b) $\Rightarrow$ (c) we have to prove $f(A)$ is dense in Y. Suppose $f(A)$ is not κ'-dense in Y, there exists a proper κ'-closed set C in Y such that $f(A) \subsetneq C \subsetneq Y$. Thus $f^{-1}(C) \neq X$. There exists a κ-g-closed set D such that $A \subseteq f^{-1}(C) \subseteq D \subseteq X$. Since A is dense there should not be any proper subset which is contained in X other than X. Hence a contradiction.

(c) $\Rightarrow$ (b) Suppose (b) is not true implies, for closed set C in Y such that $f^{-1}(C) \neq X$, there is no proper closed subset D in X such that $f^{-1}(C) \subseteq D$. This means $f^{-1}(C)$ is κ-dense in (X, τ). By (c) we get $f(f^{-1}(C))$ is κ'-dense. That is C is κ'-dense in Y. But C is a closed set in Y which is a contradiction.

Definition 4.10. Let X be a set with two topologies τ and σ. Then τ is said be equivalent to κ-provided if a non-empty subset $U \in \kappa O(X, \tau)$ then there exists a non-empty κ_{σ}-g-open set V such that $V \subseteq U$ and if for a non-empty subset $U \in \kappa O(Y, \sigma)$ then there exists a non-empty κ_{τ}-g-open set V such that $V \subseteq U$.

Theorem 4.11. *Let X be a set, τ and σ are κ-equivalent topologies on X. When f is identity then $f : (X, \tau) \to (X, \sigma)$ and $f^{-1} : (Y, \sigma) \to (X, \tau)$ are somewhat (κ, κ')-continuous. Conversely if the identity function f is somewhat (κ, κ')-continuous in both the directions, then τ and σ are κ-equivalent.*

Proof. Let f be identity and τ and σ are κ-equivalent. To prove: f and f^{-1} are somewhat (κ, κ')-continuous function. Let V be a κ-open set and $f^{-1}(V)$ since f is identity. By the definition of κ-equivalent there exists κ-open set W

such that $W \subseteq V = f^{-1}(V)$.

Proof of the Converse. Let f and f^{-1} be identity mapping and somewhat (κ, κ')-continuous function.

To Prove: τ and σ are κ-equivalent.

(i) Let f be somewhat (κ, κ')-continuous function. By the definition of somewhat (κ, κ')-continuity, for every κ-open set V of (Y, σ) and $f^{-1}(V) \neq \emptyset$, there exists a non empty κ-g-open set U in (X, τ) such that $U \subseteq f^{-1}(V) = V$ since it is an identity mapping.

(ii) Let f^{-1} be somewhat (κ, κ')-continuous function. By the definition of somewhat (κ, κ')-continuity, for every κ-open set V of (Y, σ) and $f^{-1}(V) \neq \emptyset$, there exists a non empty κ-g-open set U in (X, τ) such that $\mathrm{U} \subseteq f^{-1}(V) = V$ since it is an identity mapping.

Theorem 4.12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat (κ, κ')-continuous surjective function and τ^* be a topology for X which is κ-equivalent to τ. Then the function $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat (κ, κ')-continuous function.*

Proof. Le V be κ-open set of (Y, σ) such that $f^{-1}(V) \neq \emptyset$. Since $f : (X, \tau) \to (Y, \sigma)$ be a somewhat (κ, κ')-continuous there exists a κ-g-open set U in (X, τ) such that $\mathrm{U} \subseteq f^{-1}(V)$. To show $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat (κ, κ')-continuous function we have to prove that there exists a κ-g-open set W in (X, τ) such that $W \subseteq V = f^{-1}(V)$. Since τ^* and τ are equivalent, there exists a κ-open set W such that $W \subseteq U$ but $U \subseteq f^{-1}(V)$. Implies $W \subseteq V = f^{-1}(V)$. Therefore function $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat (κ, κ')-continuous function.

Theorem 4.13. $f : (X, \tau) \to (Y, \sigma)$ *be a somewhat* (κ, κ')*-continuous* *subjective function and* σ^* *be a topology for* Y *which is* κ*-equivalent to* σ. *Then the function* $f : (X, \tau) \to (Y, \sigma^*)$ *is somewhat* (κ, κ')*-continuous function*.

Proof. Let $V^* \in \sigma^*$ such that $f^{-1}(V) \neq \emptyset$ since σ^* is κ-equivalent to σ, there exists a non-empty κ-open set V in (Y, σ) such that $V \subseteq V^*$. This implies $\emptyset \subseteq f^{-1}(V) \subseteq f^{-1}(V^*)$. Since $f : (X, \tau) \to (Y, \sigma)$ is somewhat (κ, κ')-continuous function there exists a non empty κ-g-open set U in (X, τ) such that $U \subseteq f^{-1}(V)$. Then $U \subseteq f^{-1}(V^*)$, hence $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat (κ, κ')-continuous function.

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