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SOME INTEGRAL TRANSFORMS OF MARICHEV-SAIGO-MAEDA FRACTIONAL DIFFERENTIAL OPERATORS INVOLVING A PRODUCT OF SPECIAL FUNCTIONS

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Abstract

In literature, a lot of remarkable fractional differential operators whose integrand include various special functions have been given. In this article, first we shall set up two theorems which give Marichev-Saigo-Maeda fractional differential operators involving a product of general polynomial, multivariate Mittage-Leffler function and modified I-function. Next, we shall set up some transforms like Beta transform, Laplace transform and Verma transforms of these Marichev-Saigo-Maeda fractional differential operators.

1. Introduction and Preliminaries

In this paper, first we are introducing modified I-function of two variables, which is a generalization of modified H-function given in 1979 by Prasad and Prasad [8] and I-function given in 2012 by Shantha Kumari et al. [3], in the following style:

$$I[z_1,\, z_2] = I^{m,\, n: m_1,\, n_1 : m_2,\, n_2 : m_3,\, n_3}_{p,\, q: p_1,\, q_1 : p_2,\, q_2 : p_3,\, q_3}$$

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$$\left[\begin{matrix} z_1 \\ z_1 \end{matrix}\middle| \begin{matrix} (a_j; \alpha_j, A_j; \xi_j)_{1,p}; (c_j; \gamma_j, C_j, \xi'_j)_{1,p_1} : (e_j, E_j; U_j)_{1,p_2}; (g_j, G_j, P_j)_{1,p_3} \\ (b_j; \beta_j, B_j; \eta_j)_{1,q}; (d_j; \delta_j, D_j, \eta'_j)_{1,q_1} : (f_j, F_j; V_j)_{1,q_2}; (h_j, H_j, Q_j)_{1,q_3} \end{matrix}\right]$$

$$= \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \psi(s_1, s_2) \theta_1(s_1) \theta_2(s_2) z_1^{s_1} z_2^{s_2} ds_1 ds_2 \tag{1.1}$$

where

$$\psi(s_1, s_2) = \frac{\prod_{j=1}^{m} \Gamma^{\eta_j}(b_j - \beta_j s_1 - B_j s_2) \prod_{j=1}^{n} \Gamma^{\xi_j}(1 - a_j + \alpha_j s_1 + A_j s_2)}{\prod_{j=1}^{q} \Gamma^{\eta_j}(1 - b_j - \beta_j s_1 - B_j s_2) \prod_{j=n+1}^{p} \Gamma^{\xi_j}(a_j + \alpha_j s_1 + A_j s_2)}$$

$$\times \frac{\prod_{j=1}^{m_1} \Gamma^{\eta'_j}(d_j - \delta_j s_1 - D_j s_2) \prod_{j=1}^{n_1} \Gamma^{\xi'_j}(1 - c_j + \gamma_j s_1 + C_j s_2)}{\prod_{j=1}^{q_1} \Gamma^{\eta'_j}(1 - d_j - \delta_j s_1 - D_j s_2) \prod_{j=n+1}^{p_1} \Gamma^{\xi'_j}(c_j + \gamma_j s_1 + C_j s_2)} \tag{1.2}$$

$$\theta_1(s_1) = \frac{\prod_{j=1}^{m_2} \Gamma^{U_j}(f_j - F_j s_1) \prod_{j=1}^{n_2} \Gamma^{V_j}(1 - e_j + E_j s_1)}{\prod_{j=1}^{q_2} \Gamma^{U_j}(1 - f_j - F_j s_1) \prod_{j=n_2+1}^{p_2} \Gamma^{V_j}(e_j + E_j s_1)} \tag{1.3}$$

$$\theta_2(s_2) = \frac{\prod_{j=1}^{m_3} \Gamma^{P_j}(h_j - H_j s_2) \prod_{j=1}^{n_3} \Gamma^{Q_j}(1 - g_j + G_j s_2)}{\prod_{j=m_3+1}^{q_3} \Gamma^{P_j}(1 - h_j - H_j s_2) \prod_{j=n_3+1}^{p_3} \Gamma^{Q_j}(g_j + G_j s_2)} \tag{1.4}$$

Here, the variables z_1 and z_2 are non-zero real or complex numbers and an empty product is interpreted as unity. $m, n, m_1, n_1, m_2, n_2, m_3, n_3, p, q, p_1, q_1, p_2, q_2, p_3, q_3,$ are all non-negative integers such that $0 \le n \le p, 0 \le m \le q, 0 \le n_1 \le p_1, 0 \le m_1 \le q_1, 0 \le n_2 \le p_2, 0 \le m_2 \le q_2,$ $0 \le n_3 \le p_3, 0 \le m_3 \le q_3$ and $\alpha_j, \beta_j, \gamma_j, \delta_j, A_j, B_j, C_j, D_j, E_j, F_j, G_j, H_j,$ $\xi_j, \eta_j, \xi'_j, \eta'_j, U_j, V_j, P_j, Q_j$ are all positive real numbers. $a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j$ are all complex numbers. The integration path L_1 in the complex s_1 plane runs from $\sigma_1 - i\infty$ to $\sigma_1 + i\infty$ such that all the poles of $\Gamma^{\eta_j}(b_j - \beta_j s_1 - B_j s_2)$ for $j = 1, \ldots, m, \Gamma^{\eta_j}(d_j - \delta_j s_1 - D_j s_2)$ for

$j = 1, \ldots, m_1$ and $\Gamma^{U_j}(f_j - F_j s_1)$ for $j = 1, \ldots, m_2$ lie to the right of L_1 while all the poles of $\Gamma^{\xi_j}(1 - a_j - \alpha_j s_1 - A_j s_2)$ for $j = 1, \ldots, n$, $\Gamma^{\xi'_j}(1 - c_j - \gamma_j s_1 - C_j s_2)$ for $j = 1, \ldots, n_1$ and $\Gamma^{V_j}(1 - e_j - E_j s_1)$ for $j = 1, \ldots, n_2$ lie to the left of L_1. The integration path L_2 in the complex s_2 plane runs from $\sigma_2 - i\infty$ to $\sigma_2 - i\infty$ such that all the poles of $\Gamma^{\eta_j}(b_j - \beta_j s_1 - B_j s_2)$ for $j = 1, \ldots, m$, $\Gamma^{\eta_j}(1 - c_j - \gamma_j s_1 - C_j s_2)$ for $j = 1, \ldots, n_1$ and $\Gamma^{P_j}(h_j - H_j s_2)$ $j = 1, \ldots, m_3$ lie to the right of L_2 while all the poles of $\Gamma^{\xi_j}(1 - a_j - \alpha_j s_1 - A_j s_2)$ for $j = 1, \ldots, n$, $\Gamma^{\eta'_j}(d_j - \delta_j s_1 - D_j s_2)$ for $j = 1, \ldots, m_1$ and $\Gamma^{Q_j}(1 - g_j - G_j s_2)$ for $j = 1, \ldots, n_3$ lie to the left of L_2. When the exponents of various gamma functions in equations (2), (3) and (4) are not integers, the poles of gamma functions in numerator are converted to branch points and branch cuts can be chosen that the paths of integration can be distorted for each of contours as long as that there is no coincidence of poles.

Using the results of Braaksma [2] and Rathie [1], it is easy to show that the function $I[z_1, z_2]$ defined by (1) is an analytic function of z_1 and z_2 if

$$V_1 = \sum_{j=1}^{p} \xi_j \alpha_j + \sum_{j=1}^{p_1} \xi'_j \gamma_j + \sum_{j=1}^{p_2} U_j E_j - \sum_{j=1}^{q} \eta_j \beta_j - \sum_{j=1}^{q_1} \eta'_j \delta_j - \sum_{j=1}^{q_2} V_j F_j < 0 \quad (1.5)$$

$$V_2 = \sum_{j=1}^{p} \xi_j A_j + \sum_{j=1}^{q_1} \eta'_j D_j + \sum_{j=1}^{p_3} P_j G_j - \sum_{j=1}^{q} \eta_j \beta_j - \sum_{j=1}^{p_1} \xi'_j C_j - \sum_{j=1}^{q_3} Q_j H_j < 0 \quad (1.6)$$

Similarly following Shantha Kumari et al. [3] and Prasad and Prasad [8], it is easy to show that the double integral defined in (1) converges absolutely if

$| \arg z_1 | < \frac{1}{2} \pi \Omega_1$ and $| \arg z_2 | < \frac{1}{2} \pi \Omega_2$ where

$$\Omega_1 = \sum_{j=1}^{n}\xi_j\alpha_j + \sum_{j=n+1}^{p}\xi_j\alpha_j + \sum_{j=1}^{m}\eta_j\beta_j - \sum_{j=1}^{q}\eta_j\beta_j - \sum_{j=1}^{n_1}\xi'_j\gamma_j - \sum_{j=1}^{p_1}\xi'_j\gamma_j + \sum_{j=1}^{m_1}\eta'_j\delta_j$$

$$- \sum_{m_1+1}^{q_1}\eta'_j\delta_j + \sum_{j=1}^{n_2}U_jE_j - \sum_{j=n_2+1}^{p_2}U_jE_j + \sum_{j=1}^{m_2}V_jF_j - \sum_{j=m_2+1}^{q_2}V_jF_j > 0 \quad (1.7)$$

and

$$\Omega_2 = \sum_{j=1}^{n}\xi_jA_j + \sum_{j=n+1}^{p}\xi_jA_j + \sum_{j=1}^{m}\eta_jB_j - \sum_{j=1}^{q}\eta_jB_j - \sum_{j=1}^{n_2}\xi'_jC_j - \sum_{j=1}^{p_2}\xi'_jC_j + \sum_{j=1}^{m_2}\eta'_jL_j$$

$$- \sum_{m_1+1}^{q_2}\eta'_jD_j + \sum_{j=1}^{n_3}P_jG_j - \sum_{j=n_3+1}^{p_3}P_jG_j + \sum_{j=1}^{m_3}Q_jH_j - \sum_{j=m_3+1}^{q_3}Q_jH_j > 0 \quad (1.8)$$

Now the asymptotic behavior may be establish in the following convenient form, see Braaksma [2].

$$I(z_1, z_2) = 0(|z_1|^{\alpha_1}, |z_2|^{\alpha_2}), \max(|z_1|^{\alpha_1}, |z_2|^{\alpha_2}) \to 0$$

$$I(z_1, z_2) = 0(|z_1|^{\beta_1}, |z_2|^{\beta_2}), \min(|z_1|^{\beta_1}, |z_2|^{\beta_2}) \to 0,$$

where

$$\alpha_1 = \min_{1\le j\le m_2} \Re\left[V_j\left(\frac{f_j}{F_j}\right)\right] \text{ and } \alpha_2 = \min_{1\le j\le m_3} \Re\left[Q_j\left(\frac{h_j}{H_j}\right)\right]$$

$$\beta_1 = \max_{1\le j\le n_2} \Re\left[U_j\left(\frac{e_j-1}{E_j}\right)\right] \text{ and } \beta_2 = \max_{1\le j\le n_3} \Re\left[P_j\left(\frac{g_j-1}{G_j}\right)\right]$$

For simplicity, we shall use the following notations;

$$U = m_1, n_1 : m_2, n_2; m_3, n_3 \quad (1.9)$$

$$V = p_1, q_1 : p_2, q_2; p_3, q_3 \quad (1.10)$$

$$A_1 = (a_j; \alpha_j, A_j; \xi_j)_{1, p} \quad (1.11)$$

$$A_2 = (c_j; \gamma_j, C_j; \xi'_j)_{1, p_1} \quad (1.12)$$

$$A_3 = (e_j, E_j; U_j)_{1, p_2} \quad (1.13)$$

$$A_4 = (g_j; G_j; P_j)_{1, p_3} \tag{1.14}$$

$$B_1 = (b_j; \beta_j, B_j; \eta_j)_{1, q} \tag{1.15}$$

$$B_2 = (d_j; \delta_j, D_j; \eta'_j)_{1, q_1} \tag{1.16}$$

$$B_3 = (f_j, F_j, V_j; \eta_j)_{1, q_3} \tag{1.17}$$

$$B_4 = (h_j, H_j, Q_j)_{1, q_3} \tag{1.18}$$

Marichev-Saigo-Maeda integral operators, including the Saigo operators and involving the Appell's function $F_3(\cdot)$ as the kernel, given by Marichev, (see [5], [9]) are defined in the following way:

Definition. Let $\alpha, \alpha', \beta, \beta', \eta \in C$ and $x > 0$, then for $\Re(\eta) > 0$ we have (see [5], [9])

$$(I_{0,x}^{\alpha, \alpha', \beta, \beta', \eta} f)(x) = \frac{x^{-\alpha}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} t^{-\alpha'} F_3\left(\alpha, \alpha', \beta, \beta'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t) dt \tag{1.19}$$

and

$$(I_{x,\infty}^{\alpha, \alpha', \beta, \beta', \eta} f)(x) = \frac{x^{-\alpha'}}{\Gamma(\eta)} \int_x^\infty (t-x)^{\eta-1} t^{-\alpha} F_3\left(\alpha, \alpha', \beta, \beta'; \eta; 1 - \frac{x}{t}, 1 - \frac{t}{x}\right) f(t) dt \tag{1.20}$$

The corresponding Marichev-Saigo-Maeda fractional differential operators are given as follows:

Definition. Let $\alpha, \alpha', \beta, \beta', \eta \in C$ and $x > 0$, then for $\Re(\eta) > 0$ we have (see [5], [9])

$$(D_{0,x}^{\alpha, \alpha', \beta, \beta, \eta} f)(x) = (I_{0,x}^{-\alpha', -\alpha, -\beta', -\beta, -\eta} f)(x) = \left(\frac{d}{dx}\right)^m (I_{0,x}^{-\alpha', -\alpha, -\beta'+m, -\beta, -\eta+m} f)(x)$$

$$= \frac{1}{\Gamma(m-\eta)} \left(\frac{d}{dx}\right)^m (x)^{\alpha'} \int_0^x (x-t)^{m-\eta-1} t^{\alpha}$$

$$\times F_3\left(-\alpha', -\alpha, m-\beta', -\beta; m-\eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)f(t)dt \tag{1.21}$$

and

$$(D_{x,\infty}^{\alpha,\alpha',\beta,\beta,\eta}f)(x) = (I_{x,\infty}^{-\alpha',-\alpha,-\beta',-\beta,-\eta}f)(x) = \left(-\frac{d}{dx}\right)^m (I_{0,x}^{-\alpha',-\alpha,-\beta'+m,-\beta,-\eta+m}f)(x)$$

$$= \frac{1}{\Gamma(m-\eta)}\left(-\frac{d}{dx}\right)^m (x)^{\alpha}\int_x^{\infty}(t-x)^{m-\eta-1}t^{\alpha'}$$

$$\times F_3\left(-\alpha', -\alpha, -\beta', m-\beta; m-\eta; 1-\frac{x}{t}, 1-\frac{t}{x}\right)f(t)dt \tag{1.22}$$

where $m = [\Re(\eta)]+1$ and $[x]$ denotes the greatest integer function.

Now, we are giving the following lemma (see [9]) which gives the power function formulas for the above discussed factional differential operators. These power function formulas are required for our present study.

Lemma. *Let* $\alpha, \alpha', \beta, \beta', \eta \in C$ *and* $x > 0$. *Then the following formulas hold* (*see* [9])

(a) *Let* $\Re(n) > 0$ *and* $\Re(\rho) > \max\{0, \Re(\eta-\alpha-\alpha'-\beta'), \Re(\beta-\alpha)\}$, *then*

$$(D_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}t^{\rho-1})(x) = \frac{\Gamma(\rho)\Gamma(\rho-\eta+\alpha+\alpha'+\beta')\Gamma(\rho-\beta+\alpha)}{\Gamma(\rho-\eta+\alpha+\alpha')\Gamma(\rho-\eta+\alpha+\beta')\Gamma(\rho-\beta)}x^{\rho-\eta+\alpha+\alpha'-1} \tag{1.23}$$

(b) *Let* $\Re(n) > 0$ *and* $\Re(\rho) < 1 + \min\{\Re(\beta'), \Re(\eta-\alpha-\alpha'), \Re(\eta-\alpha'-\beta)\}$, *then*

$$(D_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}t^{\rho-1})(x)$$

$$= \frac{\Gamma(1+\beta'-\rho)\Gamma(1-\alpha-\alpha'+\eta-\rho)\Gamma(1-\alpha'-\beta+\eta-\rho)}{\Gamma(1-\rho)\Gamma(1-\alpha-\alpha'-\beta+\eta-\rho)\Gamma(1-\alpha'+\beta'-\rho)}x^{\rho-\eta+\alpha+\alpha'-1} \tag{1.24}$$

In 1903, the function $E_{\alpha}(y)$ is defined by Mittage-Leffler [6] in the following style:

$$E_{\alpha}(y) = \sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\alpha k+1)} \text{ where } \alpha, y \in C, \Re(\alpha) > 0 \tag{1.25}$$

In 1905, the function $E_\alpha(y)$ is generalized by Wiman [15] and gave the function $E_{\alpha,\beta}(y)$ in the following style:

$$E_{\alpha,\beta}(y)=\sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\alpha k+\beta)} \text{ where } \alpha,\ \beta,\ y\in C,\ \Re(\alpha)>0,\ \Re(\beta)>0 \tag{1.26}$$

In 1971, the function $E_{\alpha,\beta}(y)$ is generalized by Prabhakar [7] and gave the function $E_{\alpha,\beta}^{\lambda}(y)$ in the following style:

$$E_{\alpha,\beta}^{\lambda}(y)=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{\Gamma(\alpha k+\beta)}\frac{y^k}{k!} \tag{1.27}$$

where $\alpha,\ \beta,\ \lambda,\in C,\ \Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(\lambda)>0$

Saxena et al. [11] gave the multivariate analogue of multivariable Mittage-Leffler function in the following style:

$$E_{\mu_i,c}^{\lambda_i}(y_1,\ \dots,\ y_l)=E_{\mu_1,\dots,\mu_l,c}^{\lambda_1,\dots,\lambda_l}(y_1,\ \dots,\ y_l)$$

$$=\sum_{k_1,\dots,k_l=0}^{\infty}\frac{(\lambda_1)_{k_1},\ \dots,\ (\lambda_l)_{k_l}}{\Gamma(c+\sum_{i=1}^{l}\mu_i k_i)}\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_l^{k_l}}{k_l!} \tag{1.28}$$

where $c,\ \lambda_i,\ \mu_i\in C,\ \Re(\mu_i)>0,\ \forall\, i=1,\ 2,\ \dots,\ l$

For convenience, let $F_{\mu_i,c}^{\lambda_i}=\dfrac{(\lambda_1)_{k_1},\ \dots,\ (\lambda_l)_{k_l}}{\Gamma(c+\sum_{i=1}^{l}\mu_i k_i)}$

Srivastava [14] defined the general class of polynomials in the following style:

$$S_N^M(y)=\sum_{k=0}^{[N/M]}\frac{(-N)_{Mk}}{k!}A_{N,k}y^k \quad N=0,\ 1,\ 2,\ \dots \tag{1.29}$$

where the coefficients $A_{N,k}(N,\ k\geq 0)$ are arbitrary constants, real or complex, $(\lambda)_N$ is the pochhammer symbol and M is an arbitrary positive integer.

2. Main Results

Fractional differentiation of the product of general polynomial, multivariate Mittage-Leffler function and modified I-function.

Here, we shall state and prove two theorems for Marichev-Saigo-Maeda fractional differential operators pertaining general polynomial, multivariate Mittage-Leffler function and modified I-function of two variables.

Theorem 2.1. *Let* $x > 0, \alpha, \alpha', \beta, \beta', \eta, \rho, \delta, \upsilon \in C, \Re(\eta) > 0, \mu, \mu', \lambda, m_i \in R^+$ *for* $i = 1, 2, \ldots, l$ *and* $|\arg z_j| < \frac{1}{2}\Omega_j \pi (j = 1, 2)$ *with* Ω_j *same as in the equations* (1.7) *and* (1.8). *Further more*

$$\mu \min_{1 \le j \le m_2} \Re\left(-V_j \frac{f_j}{F_j}\right) + \mu' \min_{1 \le j \le m_3} \Re\left(-Q_j \frac{h_j}{H_j}\right) < \Re(\rho)$$

$$+ \min\{0, \Re(\beta - \alpha), \Re(\eta - \alpha - \alpha' - \beta')\}. \textit{ Then}$$

$$\{D_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}(t^{\rho-1} S_N^M(\sigma t^{\lambda}) E_{\mu_i,c}^{\lambda_i}(\sigma_1 t^{m_1}, \ldots, \sigma_l t^{m_l}) I_{p,q:V}^{m,n:U}[\delta t^{\mu}, \nu t^{\mu'}])\}(x)$$

$$= x^{\rho+\alpha+\alpha'-\eta-1} \sum_{k=0}^{[N/M]} \sum_{k_1,\ldots,k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i,c}^{\lambda_i}(\sigma x^{\lambda})^k \prod_{i=1}^{l} \frac{(\sigma_i x^{m_i})^{k_i}}{k_i!}$$

$$\times I_{p+3,q+3:V}^{m,n+3:U}\left[\begin{matrix} \delta x^{\mu} \\ \nu x^{\mu'} \end{matrix} \middle| \begin{matrix} J_1, J_2, J_3, A_1; A_2 : A_3; A_4 \\ B_1, K_1, K_2, K_3; B_2 : B_3; B_4 \end{matrix}\right] \tag{2.1}$$

where

$$J_1 = (1 - \rho - \lambda k - \sum_{i=1}^{l} m_i k_i; \mu, \mu'; 1) \tag{2.2}$$

$$J_2 = (1 - \rho - \lambda k - \sum_{i=1}^{l} m_i k_i + \beta - \alpha; \mu, \mu'; 1) \tag{2.3}$$

$$J_3 = (1 - \rho - \lambda k - \sum_{i=1}^{l} m_i k_i + \eta - \alpha - \alpha' - \beta'; \mu, \mu'; 1) \tag{2.4}$$

$$K_1 = (1 - \rho - \lambda k - \sum_{i=1}^{l} m_i k_i + \beta; \mu, \mu'; 1) \tag{2.5}$$

$$K_2 = (1 - \rho - \lambda k - \sum_{i=1}^{l} m_i k_i + \eta - \alpha - \alpha'; \mu, \mu'; 1) \tag{2.6}$$

$$K_3 = (1 - \rho - \lambda k - \sum_{i=1}^{l} m_i k_i + \eta - \alpha - \beta'; \mu, \mu'; 1) \tag{2.7}$$

Proof. To prove this theorem, using equations (1.1), (1.28) and (1.29) and interchanging the order of integration, which is permitted under the given absolute convergence conditions and denoting left hand side by Δ, we get

$$\Delta = \sum_{k=0}^{[N/M]} \sum_{k_1, \ldots, k_l = 0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \sigma^k \frac{\sigma_1^{k_1}}{k_1!} \cdots \frac{\sigma_l^{k_l}}{k_l!} \frac{1}{(2\pi i)^2}$$

$$\int_{L_1} \int_{L_2} \Phi(s_1, s_2) \theta_1(s_1) \theta_2(s_2)$$

$$\times \delta^{s_1} \nu^{s_2} (D_{0,x}^{\alpha, \alpha', \beta, \beta', \eta} t^{\rho + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2 - 1})(x) ds_1 ds_2 \tag{2.8}$$

now using the power function formula (1.23), we get

$$\Delta = x^{\rho + \alpha + \alpha' - \eta - 1} \sum_{k=0}^{[N/M]} \sum_{k_1, \ldots, k_l = 0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} (\sigma x^{\lambda})^k$$

$$\prod_{i=1}^{l} \frac{(\sigma_i x^{m_i})^{k_i}}{k_i!} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2)$$

$$\times \frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \beta + \mu s_1 + \mu' s_2)}$$

$$\frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \alpha' + \beta' + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \alpha' + \mu s_1 + \mu' s_2)}$$

$$\times \frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \beta + \alpha + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \beta' + \mu s_1 + \mu' s_2)}$$

$$\theta_1(s_1)\theta_2(s_2)(\delta x^{\mu})^{s_1}(\nu x^{\mu'})^{s_2} ds_1 ds_2 \tag{2.9}$$

Now explaining the equation (2.9) in terms of Mellin-Barnes contour integral with the help of equation (1.1), we get the right hand side of equation (2.1).

Theorem 2.2. *Let* $x > 0$, $\alpha, \alpha', \beta, \beta', \eta, \rho, \delta, \nu \in C$, $\Re(\eta) > 0$, $\mu, \mu', \lambda, m_i \in R^{+}$ *for* $i = 1, 2, \ldots, l$ *and* $|\arg z_j| < \frac{1}{2}\Omega_j \pi (j = 1, 2)$ *with* Ω_j *same as in the equations* (1.7) *and* (1.8). *Further more*

$$1 + \mu \min_{1 \le j \le m_2} \Re\left(U_j \frac{1 - e_j}{E_j}\right) + \mu' \min_{1 \le j \le m_3} \Re\left(P_j \frac{1 - g_j}{G_j}\right) > \Re(\rho)$$

$$- \min\{\Re(\eta - \alpha - \alpha' - m), \Re(-\alpha' - \beta + \eta), \Re(\beta')\}.$$

Then

$$\{D_{x,\infty}^{\alpha, \alpha', \beta, \beta', \eta}(t^{\rho - 1} S_N^M(\sigma t^{\lambda}) E_{\mu_i, c}^{\lambda_i}(\sigma_1 t^{m_1}, \ldots, \sigma_l t^{m_l}) I_{p, q:V}^{m, n:U}[\delta t^{-\mu}, \nu t^{-\mu'}])\}(x)$$

$$= x^{\rho + \alpha + \alpha' - \eta - 1} \sum_{k=0}^{[N/M]} \sum_{k_1, \ldots, k_l = 0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i}(\sigma x^{\lambda})^k \prod_{i=1}^{l} \frac{(\sigma_i x^{m_i})^{k_i}}{k_i!}$$

$$\times I_{p+3, q+3:V}^{m, n+3:U}\left[\begin{matrix} \delta x^{-\mu} \\ \nu x^{-\mu'} \end{matrix}\middle| \begin{matrix} J_4, J_5, J_6, A_1; A_2 : A_3; A_4 \\ B_1, K_4, K_5, K_6; B_2 : B_3; B_4 \end{matrix}\right] \tag{2.10}$$

where

$$J_4 = (\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \alpha'; \mu, \mu'; 1) \tag{2.11}$$

$$J_5 = (\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha' + \beta; \mu, \mu'; 1) \tag{2.12}$$

$$J_6 = (\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \beta'; \mu, \mu'; 1) \tag{2.13}$$

$$K_4 = (\rho + \lambda k + \sum_{i=1}^{l} m_i k_i;\ \mu,\ \mu';\ 1) \tag{2.14}$$

$$K_5 = (\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \alpha' + \beta;\ \mu,\ \mu';\ 1) \tag{2.15}$$

$$K_5 = (\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \beta' + \alpha';\ \mu,\ \mu';\ 1) \tag{2.16}$$

3. Integral Transforms

In this section we shall obtain some integral transforms like Beta transform, Laplace transform, Verma transform of the fractional derivative formulas obtained in the previous section.

3.1 Beta Transform

Definition. The beta transform of a function $f(z)$ is defined as (see [13])

$$B\{f(z) : s,\ p\} = \int_0^1 z^{s-1}(1-z)^{p-1} f(z)dz \tag{3.1.1}$$

Theorem 3.1.1. *Let* $x > 0,\ \alpha,\ \alpha',\ \beta,\ \beta',\ \eta,\ \rho,\ \delta,\ \nu \in C,\ \Re(\eta) > 0,\ \mu,\ \mu',\ \lambda,$ $m_i \in R^{+}$ *for* $i = 1, 2, \ldots, l$ *and* $|\arg z_j| < \frac{1}{2}\Omega_j \pi (j = 1, 2)$ *with* Ω_j *same as in the equations* (1.7) *and* (1.8). *Further more*

$$\mu \min_{1 \le j \le m_2} \Re\left(-V_j \frac{f_j}{F_j}\right) + \mu' \min_{1 \le j \le m_3} \Re\left(-Q_j \frac{h_j}{H_j}\right) < \Re(\rho) + \min\{0,\ \Re(\beta - \alpha),$$

$$\Re(\eta - \alpha - \alpha' - \beta')\}$$

Then

$$B\{D_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}(t^{\rho-1} S_N^M(zt^{\lambda}) E_{\mu_i, c}^{\lambda_i}(\sigma_1(zt)^{m_1},$$

$$\ldots,\ \sigma_l(zt)^{m_l}) I_{p,q:V}^{m,n:U}[\delta(zt)^{-\mu},\ \nu(zt)^{-\mu'}])(x) : s,\ p\}$$

$$= x^{\rho+\alpha+\alpha'-\eta-1}\Gamma(\rho) \sum_{k=0}^{[N/M]} \sum_{k_1,\ldots,k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i}(\sigma x^{\lambda})^k \prod_{i=1}^{l} \frac{(\sigma_i x^{m_i})^{k_i}}{k_i!}$$

$$\times I^{m, n+4:U}_{p+4, q+4:V}\left[\begin{matrix}\delta x^{\mu} \\ \nu x^{\mu'}\end{matrix}\left|\begin{matrix}J_1, J_2, J_3, J_7, A_1; A_2 : A_3; A_4 \\ B_1, K_1, K_2, K_3, K_7; B_2 : B_3; B_4\end{matrix}\right.\right] \tag{3.1.2}$$

where $J_1, J_2, J_3, K_1, K_2, K_3$ *are already given in the equations from* (2.2) *to* (2.7) *respectively and*

$$J_7 = \left(s + \lambda k + \sum_{i=1}^{l} m_i k_i; \mu, \mu'; 1\right) \tag{3.1.3}$$

$$K_7 = \left(s + p + \lambda k + \sum_{i=1}^{l} m_i k_i; \mu, \mu'; 1\right) \tag{3.1.4}$$

Proof. To prove this theorem, using the equation (3.1.1), we get

$$B\{D^{\alpha, \alpha', \beta, \beta', \eta}_{0, x}(t^{\rho-1} S^M_N(\sigma(zt)^{\lambda}) E^{\lambda_i}_{\mu_i, c}(\sigma_1(zt)^{m_1}, \ldots, \sigma_l(zt)^{m_l}) I[\delta(zt)^{\mu}, \nu(zt)^{\mu'}])(x) : s, p\}$$

$$= \int_0^1 z^{s-1}(1-z)^{p-1}\{D^{\alpha, \alpha', \beta, \beta', \eta}_{0, x}(t^{\rho-1} S^M_N(\sigma(zt)^{\lambda}) E^{\lambda_i}_{\mu_i, c}(\sigma_1(zt)^{m_1}, \ldots, \sigma_l(zt)^{m_l})$$

$$\times I[\delta(zt)^{\mu}, \nu(zt)^{\mu'}])(x)\}\, dz \tag{3.1.5}$$

Now using equations (1.1), (1.28) and (1.29) and interchanging the order of integration, which is permitted under the given absolute convergence conditions, we get

$$= \sum_{k=0}^{[N/M]} \sum_{k_1, \ldots, k_l = 0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N, k} F^{\lambda_i}_{\mu_i, c} \sigma^k \frac{\sigma_1^{k_1}}{k_1!} \cdots \frac{\sigma_l^{k_l}}{k_l!} \frac{1}{(2\pi i)^2}$$

$$\int_{L_1}\int_{L_2} \Phi(s_1, s_2)\theta_1(s_1)\theta_2(s_2) \times \delta^{s_1} \nu^{s_2}$$

$$\left[\int_0^1 z^{s+\lambda k+\sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2 - 1}(1-z)^{p-1} dz\right]$$

$$(D^{\alpha, \alpha', \beta, \beta', \eta}_{0, x} t^{\rho+\lambda k+\sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2 - 1})(x) ds_1 ds_2 \tag{3.1.6}$$

Now using the power function formula (1.23), we get

$$= x^{\rho+\alpha+\alpha'-\eta-1}\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_i,c}^{\lambda_i}(\sigma x^{\lambda})^k$$

$$\prod_{i=1}^{l}\frac{(\sigma_i x^{m_i})^{k_i}}{k_i!}\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(s_1, s_2)$$

$$\times\frac{\Gamma(\rho+\lambda k+\sum_{i=1}^{l}m_ik_i+\mu s_1+\mu' s_2)}{\Gamma(\rho+\lambda k+\sum_{i=1}^{l}m_ik_i-\beta+\mu s_1+\mu' s_2)}$$

$$\frac{\Gamma(\rho+\lambda k+\sum_{i=1}^{l}m_ik_i-\beta+\alpha+\mu s_1+\mu' s_2)}{\Gamma(\rho+\lambda k+\sum_{i=1}^{l}m_ik_i-\eta+\alpha+\alpha'+\mu s_1+\mu' s_2)}$$

$$\times\frac{\Gamma(\rho+\lambda k+\sum_{i=1}^{l}m_ik_i-\eta+\alpha+\alpha'+\beta'+\mu s_1+\mu' s_2)}{\Gamma(\rho+\lambda k+\sum_{i=1}^{l}m_ik_i-\eta+\alpha+\beta'+\mu s_1+\mu' s_2)}$$

$$\int_0^1 z^{s+\lambda k+\sum_{i=1}^{l}m_ik_i+\mu s_1+\mu' s_2-1}(1-z)^{p-1}dz$$

$$\times\theta_1(s_1)\theta_2(s_2)(\delta x^{\mu})^{s_1}(\nu x^{\mu'})^{s_2}ds_1ds_2 \tag{3.1.7}$$

Now computing the z-integral and putting in the above equation (3.1.7), we get

$$= x^{\rho+\alpha+\alpha'-\eta-1}\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_i,c}^{\lambda_i}(\sigma x^{\lambda})^k$$

$$\prod_{i=1}^{l}\frac{(\sigma_i x^{m_i})^{k_i}}{k_i!}\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(s_1, s_2)$$

$$\times\frac{\Gamma(\rho+\lambda k+\sum_{i=1}^{l}m_ik_i+\mu s_1+\mu' s_2)}{\Gamma(\rho+\lambda k+\sum_{i=1}^{l}m_ik_i-\beta+\mu s_1+\mu' s_2)}$$

$$\frac{\Gamma(\rho+\lambda k+\sum_{i=1}^{l} m_i k_i-\beta+\alpha+\mu s_1+\mu' s_2)}{\Gamma(\rho+\lambda k+\sum_{i=1}^{l} m_i k_i-\eta+\alpha+\alpha'+\mu s_1+\mu' s_2)}$$

$$\times\frac{\Gamma(\rho+\lambda k+\sum_{i=1}^{l} m_i k_i-\eta+\alpha+\alpha'+\beta'+\mu s_1+\mu' s_2)}{\Gamma(\rho+\lambda k+\sum_{i=1}^{l} m_i k_i-\eta+\alpha+\beta'+\mu s_1+\mu' s_2)}$$

$$\frac{\Gamma(s+\lambda k+\sum_{i=1}^{l} m_i k_i+\mu s_1+\mu' s_2)}{\Gamma(s+p+\lambda k+\sum_{i=1}^{l} m_i k_i+\mu s_1+\mu' s_2)}$$

$$\times\theta_1(s_1)\theta_2(s_2)(\delta x^{\mu})^{s_1}(\nu x^{\mu'})^{s_2}\,ds_1 ds_2 \tag{3.1.8}$$

Now explaining the above equation (3.1.8) with the help of equation (1.1) in terms of Mellin-Barnes contour integral, we get the required result (3.1.2).

Theorem 3.1.2. *Let* $x > 0$, $\alpha, \alpha', \beta, \beta', \eta, \rho, \delta, \nu \in C$, $\Re(\eta) > 0$, μ, μ', λ, $m_i \in R^+$ *for* $i = 1, 2, \ldots, l$ *and* $|\arg z_j| < \frac{1}{2}\Omega_j\pi (j = 1, 2)$ *with* Ω_j *same as in the equations* (1.7) *and* (1.8). *Further more*

$$1+\mu\min_{1\le j\le m_2}\Re\left(U_j\frac{1-e_j}{E_j}\right)+\mu'\min_{1\le j\le m_3}\Re\left(P_j\frac{1-g_j}{G_j}\right)>\Re(\rho)$$

$$-\min\{\Re(\eta-\alpha-\alpha'-m), \Re(-\alpha'-\beta+\eta), \Re(\beta')\}.$$

Then

$$B\{D_{x,\infty}^{\alpha,\alpha',\beta,\beta',\eta}(t^{\rho-1}S_N^M(\sigma(zt)^{\lambda})E_{\mu_i,c}^{\lambda_i}(\sigma_1(zt)^{m_1},\ldots,\sigma_l(zt)^{m_l})I[\delta(zt)^{\mu},\nu(zt)^{\mu'}])(x):s,\ p\}$$

$$=x^{\rho+\alpha+\alpha'-\eta-1}\Gamma(\rho)\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_i,c}^{\lambda_i}(\sigma x^{\lambda})^k\prod_{i=1}^{l}\frac{(\sigma_i x^{m_i})^{k_i}}{k_i!}$$

$$\times I_{p+4,q+4:V}^{m,n+4:U}\left[\begin{matrix}\delta x^{-\mu}\\ \nu x^{-\mu'}\end{matrix}\middle|\begin{matrix}J_4, J_5, J_6, J_7, A_1; A_2 : A_3; A_4\\ B_1, K_4, K_5, K_6, K_7; B_2 : B_3; B_4\end{matrix}\right] \tag{3.1.9}$$

where $J_4, J_5, J_6, J_7, K_4, K_5, K_6, K_7$ *are given above in the equations*

(2.11), (2.12), (2.13), (3.1.3), (2.14), (2.15), (2.16) *and* (3.14) *respectively*.

3.2 Laplace Transform

Definition. The Laplace transform of a function $f(z)$ is defined as (see [12])

$$L\{f(z)\} = \int_0^\infty e^{-qz} f(z) dz \tag{3.2.1}$$

Theorem 3.2.1. *Let* $x > 0$, $\alpha, \alpha', \beta, \beta', \eta, \rho, \delta, \nu \in C$, $\Re(\eta) > 0$, μ, μ', λ, $m_i \in R^+$ *for* $i = 1, 2 \ldots, l$ *and* $|\arg z_j| < \frac{1}{2}\Omega_j \pi (j = 1, 2)$ *with* Ω_j *same as in the equations* (1.7) *and* (1.8). *Further more*

$$\mu \min_{1 \le j \le m_2} \Re\left(-V_j \frac{f_j}{F_j}\right) + \mu' \min_{1 \le j \le m_3} \Re\left(-Q_j \frac{h_j}{H_j}\right) < \Re(\rho) + \min\{0, \Re(\beta - \alpha),$$

$$\Re(\eta - \alpha - \alpha' - \beta')\}$$

Then

$$L\{D_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}(t^{\rho-1} S_N^M(\sigma(zt)^\lambda) E_{\mu_i,c}^{\lambda_i}(\sigma_1(zt)^{m_1}, \ldots, \sigma_l(zt)^{m_l}) I[\delta(zt)^\mu, \nu(zt)^{\mu'}])(x)\}$$

$$= \frac{x^{\rho+\alpha+\alpha'-\eta-1}}{q} \sum_{k=0}^{[N/M]} \sum_{k_1,\ldots,k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i,c}^{\lambda_i} \left(\frac{\sigma x^\lambda}{q^\lambda}\right)^k \prod_{i=1}^{l} \frac{1}{k_i!} \left(\frac{\sigma_i x^{m_i}}{q^{m_i}}\right)^{k_i}$$

$$\times I_{p+4,q+3:V}^{m,n+4:U} \left[\begin{matrix} \delta(q^{-1}x)^\mu \\ \nu(q^{-1}x)^{\mu'} \end{matrix} \middle| \begin{matrix} J_1, J_2, J_3, J_8, A_1; A_2 : A_3; A_4 \\ B_1, K_1, K_2, K_3; B_2 : B_3; B_4 \end{matrix} \right] \tag{3.2.2}$$

where $J_1, J_2, J_3, K_1, K_2, K_3$ *are already given in the equations from* (2.2) *to* (2.7) *respectively and*

$$J_8 = (-\lambda k - \sum_{i=1}^{l} m_i k_i; \mu, \mu'; 1) \tag{3.2.3}$$

Proof. To prove this theorem, using equation (3.2.1), we get

$$L\{D_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}(t^{\rho-1} S_N^M(\sigma(zt)^\lambda) E_{\mu_i,c}^{\lambda_i}(\sigma_1(zt)^{m_1}, \ldots, \sigma_l(zt)^{m_l}) I[\delta(zt)^\mu, \nu(zt)^{\mu'}])(x)\}$$

$$= \int_0^\infty e^{-qz} \{D_{0,x}^{\alpha, \alpha', \beta, \beta', \eta} (t^{\rho-1} S_N^M (\sigma(zt)^\lambda) E_{\mu_i, c}^{\lambda_i} (\sigma_1 (zt)^{m_1}, \ldots, \sigma_l (zt)^{m_l})$$

$$\times [\delta(zt)^\mu, \nu(zt)^{\mu'}]) (x)\} dz \tag{3.2.4}$$

Now using equations (1.1), (1.28) and (1.29) and interchanging the order of integration, which is permitted under the given absolute convergence conditions, we get

$$= \sum_{k=0}^{[N/M]} \sum_{k_1, \ldots, k_l = 0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \sigma^k \frac{\sigma_1^{k_1}}{k_1!} \cdots \frac{\sigma_l^{k_l}}{k_l!} \frac{1}{(2\pi i)^2}$$

$$\int_{L_1} \int_{L_2} \Phi(s_1, s_2) \theta_1(s_1) \theta_2(s_1) \times \delta^{s_1} \nu^{s_2}$$

$$\left[\int_0^\infty e^{-qz} z^{\lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2} dz \right]$$

$$(D_{0,x}^{\alpha, \alpha', \beta, \beta', \eta} t^{\rho + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2 - 1}) (x) ds_1 ds_2 \tag{3.2.5}$$

Now using the power function formula (1.23), we get

$$= x^{\rho + \alpha + \alpha' - \eta - 1} \sum_{k=0}^{[N/M]} \sum_{k_1, \ldots, k_l = 0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} (\sigma x^\lambda)^k$$

$$\prod_{i=1}^{l} \frac{(\sigma_i x^{m_i})^{k_i}}{k_i!} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2)$$

$$\times \frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \beta + \mu s_1 + \mu' s_2)}$$

$$\frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \beta + \alpha + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \alpha' + \mu s_1 + \mu' s_2)}$$

$$\times \frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \alpha' + \beta' + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \beta' + \mu s_1 + \mu' s_2)}$$

$$\int_0^1 z^{s + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2 - 1} (1 - z)^{p-1} dz$$

$$\times \theta_1(s_1)\theta_2(s_2)(\delta x^{\mu})^{s_1}(\nu x^{\mu'})^{s_2} ds_1 ds_2 \tag{3.2.6}$$

Now computing the z-integral and putting in the above equation (3.2.6), we get

$$= x^{\rho+\alpha+\alpha'-\eta-1} \sum_{k=0}^{[N/M]} \sum_{k_1, \ldots, k_l = 0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} (\sigma x^{\lambda})^k$$

$$\prod_{i=1}^{l} \frac{(\sigma_i x^{m_i})^{k_i}}{k_i!} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2)$$

$$\times \frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \beta + \mu s_1 + \mu' s_2)}$$

$$\frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \beta + \alpha + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \alpha' + \mu s_1 + \mu' s_2)}$$

$$\times \frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \alpha' + \beta' + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \beta' + \mu s_1 + \mu' s_2)}$$

$$\frac{\Gamma(1 + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2)}{q^{\lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2 + 1}}$$

$$\times \theta_1(s_1)\theta_2(s_2)(\delta x^{\mu})^{s_1}(\nu x^{\mu'})^{s_2} ds_1 ds_2 \tag{3.2.7}$$

Now explaining the above equation (3.2.7) with the help of equation (1.1) in terms of Mellin-Barnes contour integral, we get the required result (3.2.2).

Theorem 3.2.2. *Let* $x > 0, \alpha, \alpha', \beta, \beta', \eta, \rho, \delta, \nu \in C, \Re(\eta) > 0, \mu, \mu', \lambda,$ $m_i \in R^+$ *for* $i = 1, 2 \ldots, l$ *and* $|\arg z_j| < \frac{1}{2}\Omega_j \pi (j = 1, 2)$ *with* Ω_j *same as in the equations* (1.7) *and* (1.8). *Further more*

$$1 + \mu \min_{1 \le j \le m_2} \Re\left(U_j \frac{1 - e_j}{E_j}\right) + \mu' \min_{1 \le j \le m_3} \Re\left(P_j \frac{1 - g_j}{G_j}\right) > \Re(\rho)$$

$$- \min\{\Re(\eta - \alpha - \alpha' - m), \Re(-\alpha' - \beta + \eta), \Re(\beta')\}.$$

Then

$$\{D_{x,\infty}^{\alpha,\alpha',\beta,\beta',\eta}(t^{\rho-1} S_N^M(\sigma(zt)^\lambda) E_{\mu_i,c}^{\lambda_i}(\sigma_1(zt)^{m_1}, \ldots, \sigma_l(zt)^{m_l}) I_{p,q:V}^{m,n:U}[\delta(zt)^{-\mu}, \nu(zt)^{-\mu'}])(x)\}$$

$$= \frac{x^{\rho+\alpha+\alpha'-\eta-1}}{q} \sum_{k=0}^{[N/M]} \sum_{k_1,\ldots,k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i,c}^{\lambda_i}\left(\frac{\sigma x^\lambda}{q^\lambda}\right)^k \prod_{i=1}^{l} \frac{1}{k_i!}\left(\frac{\sigma_i x^{m_i}}{q^{m_i}}\right)^{k_i}$$

$$\times I_{p+4,q+3:V}^{m,n+4:U}\left[\begin{matrix}\delta(qx)^{-\mu} \\ \nu(qx)^{-\mu'}\end{matrix}\middle| \begin{matrix} J_4, J_5, J_6, J_8, A_1; A_2 : A_3; A_4 \\ B_1, K_4, K_5, K_6; B_2 : B_3; B_4\end{matrix}\right] \quad (3.2.8)$$

where $J_4, J_5, J_6, J_8, K_4, K_5, K_6$ *are given above in the equations* (2.11), (2.12), (2.13), (3.2.3), (2.14), (2.15) *and* (2.16) *respectively*.

3.3 Verma Transform

Definition. The Verma transform of a function $f(z)$ is defined as (see [4])

$$V\{f(z)\} = \int_0^\infty (sz)^{q-1} e^{-\frac{1}{2}sz} W_{\kappa,\tau}(sz) f(z) dz \quad (3.3.1)$$

where $W_{\kappa,\tau}(z)$ represents Whittaker function.

Theorem 3.3.1. *Let* $x > 0, \alpha, \alpha', \beta, \beta', \eta, \rho, \delta, \nu \in C, \Re(\eta) > 0, \mu, \mu', \lambda,$ $m_i \in R^+$ *for* $i = 1, 2 \ldots, l$ *and* $|\arg z_j| < \frac{1}{2}\Omega_j \pi (j = 1, 2)$ *with* Ω_j *same as in the equations* (1.7) *and* (1.8). *Further more*

$$\mu \min_{1\le j\le m_2} \Re\left(-V_j \frac{f_j}{F_j}\right) + \mu' \min_{1\le j\le m_3} \Re\left(-Q_j \frac{h_j}{H_j}\right) < \Re(\rho) + \min\{0, \Re(\beta-\alpha),$$

$$\Re(\eta-\alpha-\alpha'-\beta')\}$$

Then

$$V\{D_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}(t^{\rho-1}S_N^M(\sigma(zt)^{\lambda})E_{\mu_i,c}^{\lambda_i}(\sigma_1(zt)^{m_1}, \ldots, \sigma_l(zt)^{m_l})I[\delta(zt)^{\mu}, \nu(zt)^{\mu'}])(x)\}$$

$$= \frac{x^{\rho+\alpha+\alpha'-\eta-1}}{s^q} \sum_{k=0}^{[N/M]} \sum_{k_1,\ldots,k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i,c}^{\lambda_i}\left(\frac{\sigma x^{\lambda}}{s^{\lambda}}\right)^k \prod_{i=1}^{l} \frac{1}{k_i!}\left(\frac{\sigma_i x^{m_i}}{s^{m_i}}\right)^{k_i}$$

$$\times I_{p+5,q+4:V}^{m,n+5:U}\left[\begin{matrix}\delta(s^{-1}x)^{\mu} \\ \nu(s^{-1}x)^{\mu'}\end{matrix}\left|\begin{matrix}J_1, J_2, J_3, J_9, J_{10}, A_1; A_2 : A_3; A_4 \\ B_1, K_1, K_2, K_3, K_8; B_2 : B_3; B_4\end{matrix}\right.\right] \tag{3.3.2}$$

where $J_1, J_2, J_3, K_1, K_2, K_3$ *are already given in the equations from* (2.2) *to* (2.7) *respectively and*

$$J_9 = \left(-\frac{1}{2} - q - \tau - \lambda k - \sum_{i=1}^{l} m_i k_i; \mu, \mu', 1\right), \tag{3.3.3}$$

$$J_{10} = \left(-\frac{1}{2} - q + \tau - \lambda k - \sum_{i=1}^{l} m_i k_i; \mu, \mu', 1\right) \tag{3.3.4}$$

$$K_8 = (\kappa - q - \lambda k - \sum_{i=1}^{l} m_i k_i; \mu, \mu', 1) \tag{3.3.5}$$

Proof. To prove this theorem, using the equation (3.3.1), we get

$$V\{D_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}(t^{\rho-1}S_N^M(\sigma(zt)^{\lambda})E_{\mu_i,c}^{\lambda_i}(\sigma_1(zt)^{m_1}, \ldots, \sigma_l(zt)^{m_l})I[\delta(zt)^{\mu}, \nu(zt)^{\mu'}])(x)\}$$

$$= \int_0^1 z^{q-1} e^{-\frac{1}{2}sz} W_{\kappa,\nu}(sz)\{D_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}(t^{\rho-1}S_N^M(\sigma(zt)^{\lambda})E_{\mu_i,c}^{\lambda_i}(\sigma_1(zt)^{m_1}, \ldots, \sigma_l(zt)^{m_l})$$

$$\times I[\delta(zt)^{\mu}, \nu(zt)^{\mu'}])(x)\}\, dz \tag{3.3.6}$$

Now using equations (1.1), (1.28) and (1.29) and interchanging the order of integration, which is permitted under the given absolute convergence conditions, we get

$$= \sum_{k=0}^{[N/M]} \sum_{k_1, \ldots, k_l = 0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \sigma^k \frac{\sigma_1^{k_1}}{k_1!} \cdots \frac{\sigma_l^{k_l}}{k_l!} \frac{1}{(2\pi i)^2}$$

$$\int_{L_1} \int_{L_2} \Phi(s_1, s_2) \theta_1(s_1) \theta_2(s_2) \delta^{s_1} \nu^{s_2}$$

$$\left[\int_0^1 z^{q + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2 - 1} e^{-\frac{1}{2} sz} W_{\kappa, \tau}(sz) dz \right]$$

$$(D_{0,x}^{\alpha, \alpha', \beta, \beta', \eta} t^{\rho + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2 - 1})(x) ds_1 ds_2 \tag{3.3.7}$$

now using the power function formula (1.23), we get

$$= x^{\rho + \alpha + \alpha' - \eta - 1} \sum_{k=0}^{[N/M]} \sum_{k_1, \ldots, k_l = 0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} (\sigma x^{\lambda})^k$$

$$\prod_{i=1}^{l} \frac{(\sigma_i x^{m_i})^{k_i}}{k_i!} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2)$$

$$\times \frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \beta + \mu s_1 + \mu' s_2)}$$

$$\frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \beta + \alpha + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \alpha' + \mu s_1 + \mu' s_2)}$$

$$\times \frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \alpha' + \beta' + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \beta' + \mu s_1 + \mu' s_2)}$$

$$\int_0^1 z^{q + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2 - 1} e^{-\frac{1}{2}} W_{\kappa, \tau}(sz) dz$$

$$\times \theta_1(s_1) \theta_2(s_2) (\delta x^{\mu})^{s_1} (\nu x^{\mu'})^{s_2} ds_1 ds_2 \tag{3.3.8}$$

Now computing the z-integral and putting in the above equation (3.3.8), we get

$$= \frac{x^{\rho+\alpha+\alpha'-\eta-1}}{s^q} \sum_{k=0}^{[N/M]} \sum_{k_1, \ldots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma x^\lambda}{s^\lambda} \right)^k$$

$$\prod_{i=1}^{l} \frac{1}{k_i!} \left(\frac{\sigma_i x^{m_i}}{s^{m_i}} \right)^{k_i} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2)$$

$$\times \frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \beta + \mu s_1 + \mu' s_2)}$$

$$\frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \beta + \alpha + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \alpha' + \mu s_1 + \mu' s_2)}$$

$$\times \frac{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \alpha' + \beta' + \mu s_1 + \mu' s_2)}{\Gamma(\rho + \lambda k + \sum_{i=1}^{l} m_i k_i - \eta + \alpha + \beta' + \mu s_1 + \mu' s_2)}$$

$$\frac{\Gamma\left(\frac{1}{2} + \tau + q + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2\right)}{\Gamma(1 - \kappa + q + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2)}$$

$$\times \Gamma\left(\frac{1}{2} - \tau + q + \lambda k + \sum_{i=1}^{l} m_i k_i + \mu s_1 + \mu' s_2\right)$$

$$\theta_1(s_1)\theta_2(s_2)(\delta(x/s)^{\mu})^{s_1}(\nu(x/s)^{\mu'})^{s_1} ds_1 ds_2 \tag{3.3.9}$$

Now explaining the above equation (3.3.9) with the help of equation (1.1) in terms of Mellin-Barnes contour integral, we get the required result (3.3.2).

Theorem 3.3.2. *Let* $x > 0$, $\alpha, \alpha', \beta, \beta', \eta, \rho, \delta, \nu \in C$, $\Re(\eta) > 0$, μ, μ', λ, $m_i \in R^+$ *for* $i = 1, 2 \ldots, l$ *and* $|\arg z_j| < \frac{1}{2}\Omega_j \pi (j = 1, 2)$ *with* Ω_j *same as in the equations* (1.7) *and* (1.8). *Furthermore*

$$1+\mu \min_{1\leq j\leq m_2} \Re\left(U_j \frac{1-e_j}{E_j}\right)+\mu' \min_{1\leq j\leq m_3} \Re\left(P_j \frac{1-g_j}{G_j}\right) > \Re(\rho)$$

$$-\min\{\Re(\eta-\alpha-\alpha'-m),\ \Re(-\alpha'-\beta+\eta),\ \Re(\beta')\}.$$

Then

$$V\{D_{x,\infty}^{\alpha,\alpha',\beta,\beta',\eta}(t^{\rho-1}S_N^M(\sigma(zt)^{\lambda})E_{\mu_i,c}^{\lambda_i}(\sigma_1(zt)^{m_1},\ \ldots,\ \sigma_l(zt)^{m_l})I_{p,q:V}^{m,n:U}[\delta(zt)^{-\mu},\ \nu(zt)^{-\mu'}])(x)\}$$

$$=\frac{x^{\rho+\alpha+\alpha'-\eta-1}}{q}\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_i,c}^{\lambda_i}\left(\frac{\sigma x^{\lambda}}{q^{\lambda}}\right)^{k}\prod_{i=1}^{l}\frac{1}{k_i!}\left(\frac{\sigma_i x^{m_i}}{q^{m_i}}\right)^{k_i}$$

$$\times I_{p+5,q+4:V}^{m,n+5:U}\left[\begin{matrix}\delta(sx)^{-\mu}\\ \nu(sx)^{-\mu'}\end{matrix}\middle|\begin{matrix}A_1,\ J_4,\ J_5,\ J_6,\ J_9,\ J_{10},\ A_2,\ A_3,\ A_4\\ B_1,\ K_4,\ K_5,\ K_6,\ K_8,\ B_2,\ B_3,\ B_4\end{matrix}\right] \quad (3.3.10)$$

where $J_4, J_5, J_6, J_9, J_{10}, K_4, K_5, K_6, K_8$ *are given above in the equations* (2.11), (2.12), (2.13), (3.3.3), (3.3.4), (2.14), (2.15), (2.16) *and* (3.3.5) *respectively*.

Conclusion

Our results are very general in nature. In this paper we have given two theorems for Merichev-Saigo-Maeda fractional differential operators involving general polynomial, multivariate Mittage-Leffler function and modified I-function of two variables and six theorems for beta, Laplace and Verma transforms of Merichev-Saigo-Maeda fractional differential operators involving a product of general polynomial, multivariable Mittage-Leffer function and modified I-function of two variables. On specifying parameters, all the results of this paper can also be obtained for modified H-function of two variables of Prasad and Prasad [8] and I-function of two variables of Shantha Kumari et al. [3].

References

[1] A. K. Rathie, A new generalization of generalized hypergeometric functions, Le Matematiche vol. LII, Fasc. II (1997), 297-310.

[2] B. L. J. Braaksma, Asymptotic expansions and analytic continuations for a class of Barnes-integrals, Compositio Math. 15 (1964), 239-341.

[3] K. Shantha Kumari, T. M. Vasudevan Nambisan and A. K. Rathie, A study of I-function of two variables, (2012). Le Matematiche Vol. LXIX(I) (2014), Fasc. 285-305. airXiv:1212.6717v1[math.CV],

[4] A. M. Mathai, R. K. Saxena and H. J. Haubold, The H-function: theory and applications, Springer, New York, (2010).

[5] O. I. Marichev, Voltera equation of Mellin convolution type with a Horn function in the kernel, Izv. Akad. Nauk. BSSR, Ser. Fiz-Mat. Nauk 1 (1974), 28-129.

[6] G. M. Mittage-Leffler, Sur la nouvelle function $E_\alpha(x)$, C. R. Acad. Sci. Paris 137 (1903), 554-558.

[7] T. R. Prabhakar, A singular integral equation with a generalized Mittage-Leffler function kernel, Yokohama Math. J. 19 (1971), 7-15.

[8] Y. N. Prasad and S. Prasad, J. of scientific research, Banaras Hindu University (1979), 67-76.

[9] M. Saigo and N. Maeda, More generalization of fractional calculus Transform Methods and special functions, Varna, Bulgaria (1996), 386-400.

[10] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives, Theory and Applications, Gordon and Breach, (1993).

[11] R. K. Saxena, S. K. Kalla and R. Saxena, Multivariable analogue of Mittage-Leffler function, Integral Transform and Special Functions 7(7) (2011), 533-548.

[12] I. L. Schiff, The Laplace Transform, Theory and Applications, Springer, New York, (1999).

[13] I. N. Sneddon, The use of integral transforms, Tata McGraw-Hill, New Delhi, (1979).

[14] H. M. Srivastava, A contour integral involving Fox's H-function, Indian J. Math. 14 (1972), 1-6.

[15] A. Wiman, Uber de fundamental satz in der theorie de funktionen $E_\alpha(x)$, Acta. Math. 29 (1905), 191-201.