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CONNECTEDNESS OF INDEPENDENCE FRACTALS

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Abstract

Fractal Geometry was introduced by Benoit B Mandelbrot in the 20th century. All objects that we see in nature are not regular geometric shapes of the standard geometry. Fractal geometry is a mathematical tool for dealing with this irregularity. Most of the fractals have highly irregular structures while it is also a union of many smaller copies of itself. This paper covers the connectedness of graphs and fractals using the independence polynomial of a graph. We also explain the concepts of the independence fractals of a graph.

1. Introduction

Fractal geometry is a study of fractal objects or simply fractals. Fractal is a union of smaller copies of itself. There are so many ways to construct a fractal object. Many fractal objects may be constructed with geometric constructions. The most important fractals constructed using this are the Mandelbrot set and Julia set. To construct the Mandelbrot set, we have to use the recursive formula $z_n = z_{n-1}^2 + C$, where C is a complex number and set z_0 as zero. The Julia sets are strictly connected with the Mandelbrot set. The recursive formula that is used to construct the Julia set is the same as that of type Mandelbrot set. In Julia set, C must be constant during the whole

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generation process, while the value of z_0 varies. The value of C determines the shape of the Julia set; in other words, each point of the complex plane is associated with a particular Julia set ([2] and [3]).

An independent set of a graph G is a set of vertices in G, no two of which are adjacent. The size or order of an independent set is the number of vertices it contains. A maximal independent set is an independent set that is not a proper subset of any other independent set. The size of a maximal independent set is the independence number of G and is denoted by $\alpha(G)$ or α. The independence polynomial of G is given by $I_G(x) = \sum_{k=0}^{\alpha} i_k x^k$, where i_k be the number of independent sets of order k. The reduced polynomial of graph G is $f_G(x) = I_G(x) - 1$.

The set of roots of f_{G^k}, where G^k is the k-times lexicographic product of graph G with itself, is finite and hence a compact subset of the complex plane for each k. The sequence of roots of f_{G^k} is known to converge as k tends to ∞, with respect to the Hausdorff metric defined on the set of all compact subsets of C. The limiting set is usually a fractal and is known as the independence fractal of G. We will discuss the independence fractal of some families of graphs.

2. Preliminaries

Definition 2.1 [4]**.** Let G be a graph. The independence polynomial of G, denoted by $i_G(x)$, is given by $i_G(x) = \sum_{k=0}^{\alpha} i_k x^k$, where i_k be the number of independent sets of order k and α be the independence number of G.

For any graph G $i_0 = 1$ and $i_1 = n(G)$, where $n(G)$ is the number of vertices of G.

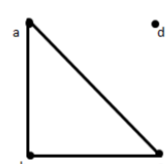

Figure 1.

Example 2.1. In Figure 1 independent set of order 2 are $\{a, d\}, \{b, d\}, \{c, d\}$ hence $i_2 = 3$. There is no independent set of order greater than 2 and hence $i_k = 0 \; \forall k > 2$. Therefore the independence polynomial of G is given by $i_G(x) = \sum_{k=1}^{\alpha} i_k x^k = 1 + 4x^1 + 3x^2 = 1 + 4x + 3x^2$.

Definition 2.2 [4]**.** The lexicographic product or composition $G[H]$ of graphs G and H is a graph such that

1. The vertex set of $G[H]$ is the cartesian product $V(G) \times V(H)$ and

2. Any two vertices (u, v) and (x, y) are adjacent in $G[H]$ if and only if either u is adjacent with x in G or $u = x$ and v is adjacent with y in H.

Theorem 2.1 [5]**.** *The independence polynomial of* $G[H]$ *is given by* $i_{G[H]}(x) = i_G(I_H(x) - 1)$.

Example 2.2. To find $i_{G[H]}(x)$ in Figure 2 where $i_G(x) = 1 + 4x + 3x^2$ $i_H(x) = 1 + 4x + 2x^2$

$$
\begin{aligned}
i_{G[H]}(x) &= i_G(i_H(x) - 1) \\
&= i_G(4x + 2x^2) \\
&= 1 + 4(4x + 2x^2) + 3(4x + 2x^2)^2 \\
&= 1 + 16x + 8x^2 + 3(16x^2 + 16x^3 + 4x^4) \\
&= 1 + 16x + 56x^2 + 48x^3 + 12x^4
\end{aligned}
$$

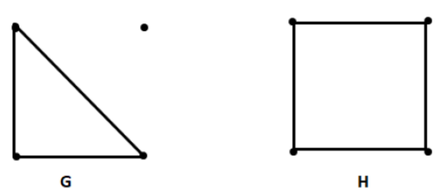

Figure 2.

Definition 2.3 [5]**.** The reduced independence polynomial of G as the function $f_G(x) = i_G(x) - 1$.

Corollary 2.1.1 [5]. $f_{G[H]} = f_G(f_H(x))$.

Reduced independence polynomials are closed under lexicographic powers: $f_{G^k}(x) = f_G(f_G(f_G(\ldots)))_k$, $k = 2, 3, \ldots$.

Definition 2.4 [1]. The forward orbit of a point z_0 with respect to f is the set $\mathcal{O}^+(z_0) = \{f^k(z_0)\}_{k=0}^{\infty}$. For a polynomial f, its filled Julia set $K(f)$ is the set of all points z whose forward orbit $\mathcal{O}^+(z)$ is bounded in $(\mathbb{C}, |\cdot|)$. The Julia set of $J(f)$ is the boundary $\partial K(f)$. The Fatou set $F(f)$ is the complement of $J(f)$ in $\mathbb{C}$.

Example 2.3. For the polynomial $x^3 + 2x^2 + x$, let $z_0 = 1$, then by definition $\mathcal{O}^+(1) = \{f^0(1), f^1(1), f^2(1), \ldots\} = \{1, 4, 100, \ldots\}$ which is clearly unbounded, and therefore 1 is not in the filled Julia set of f. Let $z_0 = -1$ then by definition $\mathcal{O}^+(-1) = \{f^0(-1), f^1(-1), f^2(-1), \ldots\} = \{-1, -0, 0, 0, \ldots\}$ which is bounded so -1 is in $K(f)$.

Definition 2.5 [1]. If g is a map of a set X into itself, a subset A of X is

1. forward invariant if $g(E) = E$

2. backward invariant if $g^{-1}(E) = E$

3. completely invariant if $g(A) = A = g^{-1}(A)$.

If A is completely invariant then the complement of A, the interior of A, the closure of A and the boundary of A are all completely invariant as well.

Theorem 2.2 [1]. *The sets $K(f)$, $F(f)$ and $J(f)$ are all completely invariant.*

Definition 2.6 [1]. A family F of maps from a metric space (X, d) to a metric space (X, d_1) is equicontinuous at the point x_0 in X if, for every positive ϵ, there is some positive δ such that for every x in X and for all f in $\mathcal{F}$; $d(x_0, x) < \delta$ implies $d_1(f(x_0), f(x)) < \epsilon$.

Definition 2.7 [1]**.** For a non-constant polynomial f, $F(f)$ is the maximal open subset of $(\mathbb{C}, |\cdot|)$ on which $\{f^k\}_{k=0}^{\infty}$ is equicontinuous. $J(f)$ is the complement of $F(f)$ in $(\mathbb{C}, |\cdot|)$.

By this, definition $F(f)$ is an open subset of $(\mathbb{C}, |\cdot|)$ which makes $J(f)$ a closed subset of $(\mathbb{C}, |\cdot|)$.

It can be shown that $J(f)$ is a perfect set. So $J(f)$ is closed, bounded, and uncountable.

Definition 2.8 [1]**.** A point ζ is a fixed point of f if $f(\zeta) = \zeta$.

Note. If we assume that for some choice of z_0 the sequence $\{f^k(z_0)\}_{k=0}^{\infty}$ converges to a point w, then we have $w = \lim_{n\to\infty} f^n(z_0) = f(\lim_{n\to\infty} f^{n-1}(z_0))$ $= f(w)$. So w is a fixed point of f and all forward orbit of points in $\mathbb{C}$, if they converge, converge to a fixed point of f.

We can characterize the fixed points of a polynomial f by the behavior of the derivative of the function at the point ζ. For all polynomials the derivative $f'(\zeta)$ is defined for any fixed point ζ, so we define ζ to be

1. a super attracting fixed point if $|f'(\zeta)| = 0$

2. an attracting fixed point if $|f'(\zeta)| < 1$

3. a repelling fixed point if $|f'(\zeta)| < 1$

4. an indifferent fixed point if $|f'(\zeta)| = 1$, which includes the following cases

(a) a rationally indifferent fixed point if $|f'(\zeta)| = 1$ and $|f'(\zeta)|$ is a root of unity

(b) an irrationally indifferent fixed point if $|f'(\zeta)| = 1$ and $|f'(\zeta)|$ is not a root of unity.

Definition 2.9 [1]**.** A critical point z of f is a point that has no local inverse, that is f fails to be injective in any neighborhood of z.

The distinction between super-attracting fixed points and attracting fixed points is that if ζ is super-attracting it is also a critical point of *f*, while attracting fixed points are not critical points.

Theorem 2.3 (Attracting Fixed Point Theorem) [6]**.** *Suppose ζ is an attracting (or super-attracting) fixed point for f. Then there is an interval I that contains ζ in its interior and in which the following condition is satisfied*:

$$\text{if } x \in I \text{ then } f^n(x) \in I \ \forall n \text{ and } f^n(x) \to \zeta \text{ as } n \to \infty.$$

Theorem 2.4 (Repelling Fixed Point Theorem). [6] *Suppose ζ is a repelling fixed point for f. Then there is an interval I that contains ζ in its interior and in which the following condition is satisfied*:

$$\text{if } x \in I \text{ and } x \neq \zeta, \text{ then there is an integer } n > 0 \text{ such that } f^n(x) \in I.$$

Definition 2.10 [1]**.** A point z_0 is a periodic point of *f* if for some positive integer k, $f^k(z_0) = z_0$. The smallest such k is the period of z_0. If $k = 1$ then z_0 is a fixed point of *f*.

Definition 2.11 [1]**.** The cycle is said to be

1. super-attracting cycle if $|\lambda| = 0$

2. an attracting cycle if $0 < |\lambda| < 1$

3. a repelling cycle if $|\lambda| > 1$

4. a rationally indifferent cycle if $|\lambda| = 1$ and λ is a root of unity and

5. an irrationally indifferent cycle if $|\lambda| = 1$ but λ is not a root of unity.

Another characterization of the Julia set can be found by using the backward orbit of a point.

Definition 2.12 [1]**.** For a point, $z_0 \in \mathbb{C}$ its backward orbit with respect to *f* is the set $\mathcal{O}^-(z_0) = \bigcup_{k=0}^{\infty} f^{-k}(z_0)$.

A polynomial *f* has at most one exceptional point whose backward orbit is finite, and an exceptional point for *f*, if it exists, lies in $F(f)$.

Theorem 2.5 [1]. *For a polynomial f (of degree > 1), a nonempty open set W which meets* $J(f)$, *and for all sufficiently large integers* n, $f^n(W) \supset J(f)$.

Theorem 2.6 [1]. *For a polynomial f of degree at least* 2

1. *If z is not exceptional, then* $J(f)$ *is contained in the closure of* $\mathcal{O}^-(z)$.

2. *if* $z \in J(f)$ *then* $J(f)$ *is the closure of* $\mathcal{O}^-(z)$.

Definition 2.13 [1]. A Mobius map is a rational map of the form $\phi(z) = \dfrac{az + b}{cz + d}$, $ad - bc \neq 0$, for a, b, c and d fixed complex numbers. The condition $ad - bc \neq 0$ ensures that ϕ is one to one and therefore invertible.

Definition 2.14 [1]. Two polynomials f and g are conjugate if there exist a Mobius map ϕ such that $g = \phi \circ f \circ \phi^{-1}$.

For two conjugate polynomial f and g we have $g^k = (\phi \circ f \circ \phi^{-1})^k = \phi \circ f^k \circ \phi^{-1}$.

Theorem 2.7 [1]. *If* $g = \phi \circ f \circ \phi^{-1}$ *for some Mobius map* ϕ, *then* $F(g) = \phi(F(f))$ *and* $J(g) = \phi(J(f))$. *The sets* $J(g)$ *and* $J(f)$ *are then said to be analytically conjugate, as are* $F(g)$ *and* $F(f)$.

Theorem 2.8 [6]. *Let f be a polynomial and* z_0 *a point which does not lie in any attracting cycle or Siegel disk of f, then* $\lim_{k \to \infty} f^{-k}(z_0) = J(f)$, *where the limit is taken with respect to the Hausdorff metric on the compact subspace of* $(\mathbb{C}, |\cdot|)$.

Attracting cycles are also contained in $F(f)$, *so we have for any* $z_0 \in J(f) \lim_{k \to \infty} f^{-k}(z_0) = J(f)$.

3. Independence Fractals

Definition 3.1 [1]. *The independence fractal of a graph G is defined as the set* $\mathcal{F}(G) = \lim_{k \to \infty}$ *Roots* (f_{G^k}).

The following theorem shows that the independence fractals exist for all graph G.

Theorem 3.1 [1]. *The independence fractal* $\mathcal{F}(G)$ *of a graph* $G \neq K_1$ *is precisely the Julia set* $J(f_G)$ *of its reduced independence polynomial* $f_G(x)$. *Equivalently* $\mathcal{F}(G)$ *is the closure of the union of the reduced independence roots of powers of* G^k, $k = 1, 2, 3, \ldots$.

Theorem 3.2 [1]. *Let f be a polynomial of degree at least two.*

1. *Its Julia set* $J(f)$ *is connected if and only if the forward orbit of each of its critical points is bounded in* $(\mathbb{C}, |\cdot|)$.

2. *Its Julia set* $J(f)$ *is totally disconnected if (but not only if) the forward orbit of each of its critical points is unbounded in* $(\mathbb{C}, |\cdot|)$.

Example 3.1. For the polynomial $f(x) = 4x^2 + 4x$ has one critical point at $\frac{-1}{2}$. $\mathcal{O}^+(-0.5) = \{f^k(-0.5)\}_{k=0}^{\infty} = \{-0.5, -1, 0, 0, \ldots\}$ which is bounded. Therefore the independence fractal of *f* is connected.

Example 3.2. The polynomial $f(x) = 4x + 3x^2$ has one critical point at $x = \frac{-2}{3}$ and $\mathcal{O}^+\left(\frac{-2}{3}\right) = \left\{\frac{-2}{3}, \frac{-4}{3}, \frac{32}{3} \ldots\right\}$ which is unbounded. So the independence fractal of *f* is completely disconnected.

Theorem 3.3 [1]. *Every graph G with an independence number at least* 2 *is an induced subgraph of a graph H with the same independence number, whose independence fractal is disconnected.*

Now we are going to discuss these results on the group or family of graphs. There arise a question such as for which graph G is $F(G)$ *connected?*

3.1 The Family K_n and K_n^c

In this section, we consider the family of complete graphs K_n with n vertices. The independence polynomial of K_n is $I_{K_n}(x) = nx + 1$ and hence the reduced polynomial becomes $f_{K_n} = nx$. Also the composition

$f_{K_n}^k(x) = n^k x$. Therefore the independence fractal of K_n is $\mathcal{F}(K_n) = \{0\}$ and which is totally disconnected.

Next, consider the family of null graph K_n^c with n vertices. The independence polynomial is given by $I_{K_n^c}(x) = (x+1)^n$ so that the reduced polynomial will be $f_{K_n^c}(x) = (x+1)^n - 1$. Clearly the only critical point of $f_{K_n^c}$ is -1. $f_{K_n^c}(-1) = (1+-1)^n - 1 = -1$, $f_{K_n^c}^2(-1) = -1 \ldots \{f_{K_n^c}^2(-1)\} = \{-1, -1, -1, -1, \ldots\}$. That is the forward orbit of -1 is bounded and hence the independence fractal $\mathcal{F}(K_n^c)$ of K_n^c is connected.

3.2 The Family $K_{n,n}$

Consider $K_{n,n}$, the bipartite graph of $2n$ vertices. The independence polynomial of $K_{n,n}$ is given by $I_{K_{n,n}}(x) = 2(1+x)^n - 1$. Therefore the reduced polynomial of $K_{n,n}$ is $f_{K_{n,n}}(x) = 2(1+x)^n - 2$. We examine the connectedness of the independence fractal of $K_{n,n}$, for that first find the critical point of $f'K_{n,n}$.

$f'K_{n,n} = 2n(1+x)^{n-1}$, whose only critical point is -1.

$\mathcal{F}(K_{n,n})$ will be either connected or totally disconnected, depending on whether the forward orbit of $z = -1$ is bounded or unbounded, respectively in $(\mathbb{C}, |\cdot|)$.

Case 1. n is even. $f_{K_{n,n}}(-1) = 2(1+-1)^n - 2 = -2$, $f_{K_{n,n}}(-2) = 2(1-2)^n - 2 = 2(-1)^n - 2 = 2-2 = 0$, since n is even. And $f_{K_{n,n}}(0) = 2(1-0)^n - 2 = 2-2 = 0$. That is $\{f_{K_{n,n}}(-1)\}_{k=0}^{\infty} = \{-1, -2, 0, 0, 0, \ldots\}$, this shows that the forward orbit of -1 is bounded in $(\mathbb{C}, |\cdot|)$, so that the independence fractal $\mathcal{F}(K_{n,n})$, n is even, is connected.

Case 2. n is odd. $f_{K_{n,n}}(-1) = -2$, $f_{K_{n,n}}(-2) = 2(1 + -2)^n - 2 = 2(1)^n - 2 = -2 - 2 = -4$. Clearly $f_{K_{n,n}}(-1) \leqslant -2 f_{K_{n,n}}(-2) < -3$ if $z < -2$ then $f_{K_{n,n}}(-z) = 2(1 + -z)^n - 2 = 2(z+1)^n - 2 < z - 1$, since $z + 1 < -1$. Hence the forward orbit of -1 is unbounded in $(\mathbb{C}, |\cdot|)$ and therefore the independence fractal $\mathcal{F}(K_{n,n})$, n is odd, is totally disconnected.

This serves as the proof of the following theorem.

Theorem 3.4. *The independence fractal of* $K_{n,n}$ *is connected if n is even and totally disconnected if n is odd.*

We can find a bound for the independence fractal of $K_{n,n}$ *from the theorem stated below.*

Theorem 3.5. $\mathcal{F}(K_{n,n})$ *lies in the disk* $|z + 1| \leq 1$.

Proof. Let F_k be the zero of $f^k_{K_{n,n}}$. To prove this theorem we can use induction on k. Let F_1 be the zero of $f_{K_{n,n}}$. That is $f_{K_{n,n}}(F_1) = 0$. This gives $2(1 + F_1)^n - 2 = 0 \Rightarrow (1 + F_1)^n = 1 \Rightarrow |1 + F_1| = 1$. It follows that F_1 lies in the disk $|z + 1| \leq 1$. Now assume that the result is true for k-1. That is $|F_{k-1} + 1| \leq 1$ where F_{k-1} is the zero of $f^{k-1}_{K_{n,n}}$. Let F_k be a zero of $f^k_{K_{n,n}}$. Therefore $f^k(F_k) = 0 \Rightarrow f^{k-1}(f(F_k)) = 0$. It follows that $f(F_k)$ is a zero of $f^{k-1}_{K_{n,n}}$. Then by the induction hypothesis we have $f(F_k)$ lies in the disk $|z + 1| \leq 1$.

$$|f(F_k) + 1| \leq 1$$

$$|2(1 + F_k)^n - 2 + 1| \leq 1$$

$$|2(1 + F_k)^n - 1| \leq 1$$

$$-1 \leq |2(1 + F_k)^n - 1| \leq 1$$

$$0 \leq |\, 2(1+F_k)^n \,| \leq 2$$

$$0 \leq |\, 2(1+F_k)^n \,| \leq 1$$

$$|\, 1+F_k \,| \leq 1$$

Hence F_k lies in the disk $|z+1| \leq 1$. Then by the principle of mathematical induction every zeros of $f^k_{K_{n,\,n}}$ lies in the unit disk.

As we did earlier, we have to find out the connectedness of the independence fractal of paths, this will give in the next section.

3.3 The family of paths P_n

For the path P_n, we have to use the recursive formula to find the independence polynomial of P_n which is given by

$$I_{P_n}(x) = I_{P_{n-1}}(x) = xI_{P_{n-2}}(x).$$

Independence polynomial of P_3 is $I_{P_3}(x) = 1 + 4x + 3x^2$ and the reduced polynomial $f(x) = 4x + 3x^2$, so f has the critical point at $x = -\frac{4}{6}$ and $\mathcal{O}^{+}\left(-\frac{2}{3}\right) = \left\{f^k\left(-\frac{2}{3}\right)\right\}_{k=0}^{\infty} = \left\{-\frac{2}{3}, -\frac{4}{3}, 0, 0, 0, 0, \ldots\right\}$ is bounded. Hence the independence fractal of P_3 is connected. The independence fractals of path P_n, $4 \leq n \leq 25$, are not connected (see Appendix 1). We conclude this section with a theorem which states the connectedness of independence fractal of paths.

Theorem 3.6. *The independence fractal of a path* P_n, $3 \leq n \leq 25$, *is connected only if* $n = 3$.

4. Appendix 1

Independence polynomial of P_4 is $I_{P_4}(x) = 1 + 5x + 6x^2 + x^3$ and the reduced polynomial $f(x) = 5x + 6x^2 + x^3$, f has the critical points at $x = -0.4725$ and $x = -3.5275$ and $\mathcal{O}^{+}(0.4725) = \{-0.4725, -1.12845,$

$0.561179, 4.87215, 282.442, \ldots\}$ and $\mathcal{O}^{+}(-3.52275) = \{-3.5275, -13.1285, 3362.58, \ldots\}$. That is the forward orbit of each critical points are unbounded. Hence the independence fractal of P_4 is totally disconnected.

Independence polynomial of P_5 is $I_{P_5}(x) = 1 + 6x + 10x^2 + 4x^3$ and the reduced polynomial $f(x) = 6x + 10x^2 + 4x^3$, f has the critical points at $x = -0.3924$ and $x = -1.2743$ and $\mathcal{O}^{+}(-0.3924) = \{-0.3924, -1.05631, 0.1055640, 0.749527, 11.79940, 8034.180, \ldots\}$ and $\mathcal{O}^{+}(-1.2743) = \{-1.2743, -0.31556, 3.0149, 218.603, \ldots\}$. That is the forward orbit of each critical points are unbounded. Hence the independence fractal of P_5 is totally disconnected.

Independence polynomial of P_6 is $I_{P_6}(x) = 1 + 7x + 15x^2 + 10x^3 + x^4$ and the reduced polynomial $f(x) = 7x + 15x^2 + 10x^3 + x^4$, f has the critical point at $x = -0.78481$, $x = -0.3503$ and $x = -6.3648$ and $\mathcal{O}^{+}(-0.3503)$ $\{-0.3503, -1.02624, -1.08502, -1.323750, -3.107119, 83.7067 \ldots\}$, $\mathcal{O}^{+}(-6.3648) = \{-6.3648, -374.202, 190857, \ldots\}$ and $\mathcal{O}^{+}(-0.7848) = \{-0.7848, -0.709258, -0.733947, -0.720894, -0.727261, \ldots\}$. That is the forward orbit of the critical point $x = -0.3503, -6.3648$ are unbounded and $x = 0.7848$ is bounded. Hence the independence fractal of P_6 is neither connected nor totally disconnected.

Independence polynomial of P_7 is $I_{P_7}(x) = 1 + 8x + 21x^2 + 20x^3 + 5x^4$ and the reduced polynomial $f(x) = 8x + 21x^2 + 20x^3 + 5x^4$, f has the critical point at $-0.3251, -0.5902$ and -2.0847 and $\mathcal{O}^{+}(-0.5902) = \{-0.5902, -0.911612, -1.53968, -7.43522, -8161.47, \ldots\}$, $\mathcal{O}^{+}(-2.0847) = \{-2.0847, -12.1757, 76802.3, \ldots\}$ and $\mathcal{O}^{+}(-0.3251) = \{-0.3251, -1.0126, -2.0773, -12.1743, -76763.5, \ldots\}$. That is the forward orbit of the critical point are unbounded. Hence the independence fractal of P_7 is neither connected nor totally disconnected.

Independence polynomial of P_8 is $I_{P_8}(x) = 1 + 9x + 28x^2 + 35x^3 + 15x^4 + x^5$ and the reduced polynomial $f(x) = 9x + 28x^2 + 35x^3 + 15x^4 + x^5$, f has the critical point at $-0.3086, -0.49003, -10.01256$ and -1.18881 and $\mathcal{O}^+(-0.3086) = \{-0.3086, 1.00623, -2.01857, 23.5754, \ldots\}$, $\mathcal{O}^+(-1.18881) = \{-1.1888, -2.34605, 64.389, \ldots\}$ and $\mathcal{O}^+(-10.01256) = \{-10.01256, -17710.3, \ldots\}$. That is the forward orbit of the critical point are unbounded. Hence the independence fractal of P_8 is neither connected nor totally disconnected.

Independence polynomial of P_9 is $I_{P_9}(x) = 1 + 10x + 36x^2 + 56x^3 + 35x^4 + 6x^5$ and the reduced polynomial $f(x) = 10x + 36x^2 + 56x^3 + 35x^4 + 6x^5$, f has the critical point at $-0.29713, -0.43049, -0.84122$ and -3.09782 and $\mathcal{O}^+(0.29713) = \{-0.29713, 1.0031, -0.987426, 1.0475, -0.7667910, 1.239350, 1.32930, 342, \ldots\}$, $\mathcal{O}^+(-0.43049) = \{-0.43049, -0.9876, -1.0468, -0.770776$ $-1.24257, 1.38422, 390.334 \ldots\}$, $\mathcal{O}^+(-0.84122) = \{-0.84122, -1.27361, 1.95238, 1252.25, \ldots\}$ and $\mathcal{O}^+(-3.09782) = \{-3.09782, -161.24, \ldots\}$. That is the forward orbit of the critical point are unbounded. Hence the independence fractal of P_9 is neither connected nor totally disconnected.

Independence polynomial of P_{10} is $I_{P_{10}}(x) = 1 + 11x + 45x^2 + 84x^3 + 70x^4 + 21x^5 + x^6$ and the reduced polynomial $f(x) = 11x + 45x^2 + 84x^3 + 70x^4 + 21x^5 + x^6$, the critical point of f are $-0.28881, -0.39169, -0.66526, 1.68347$ and -14.47077 and $\mathcal{O}^+(-0.28881) = \{-0.28881, -1.00156, -0.0125238, 0.144987, 2.82912, 11096.6, \ldots\}$, $\mathcal{O}^+(-39169) = \{-0.39169, -0.994835, -0.0408407, -0.379719, -0.995045, -0.0391988, -0.366938, 0.995683, -0.034201, -32684, -0.999263, 0.00588623, -0.06206, -0.535608, -1.03034, 0.25944, 7.69174, , \ldots\}$, $\mathcal{O}^+(-0.66526) = \{-0.66526, -1.07271, -0.678558, -72.3876, \ldots\}$, $\mathcal{O}^+(-1.68347) = \{-1.68347, 9.29233, \ldots\}$ and $\mathcal{O}^+(-14.47077) = \{-14.47077, -1318850, \ldots\}$. That is the forward orbit of the critical point are unbounded. Hence the independence fractal of P_{10} is neither connected nor totally disconnected.

Independence polynomial of P_{11} is $I_{P_{11}}(x) = 1 + 12x + 55x^2 + 120x^3 + 126x^4 + 56x^5 + 7x^6$ and the reduced polynomial $f(x) = 12x + 55x^2 + 120x^3 + 126x^4 + 56x^5 + 7x^6$, the critical point of f are $-0.28256, -0.36475, -0.56182, -1.14392$ and -4.31363 and $\mathcal{O}^+(-0.28256) = \{-0.28256, -1.00079, -0.00315749, 0.038442, 0.549679, 57.6505, \ldots\}$, $\mathcal{O}^+(-0.36475) = \{-0.36475, -0.997754, -0.00900377, -0.103673, -0.772752, -0.81431, -0.705064, -0.93901, -0.251338, -0.998522, -0.0059206, 0.69442, -0.603656, -1.01743, 0.683093, 1.11743, 557.16, \ldots\}$, $\mathcal{O}^+(-0.56182) = \{-0.56182, -1.02267, 0.0881918, 1.57632, 2055.95, \ldots\}$, $\mathcal{O}^+(-1.14392) = \{-1.14392, 0.366182, 20.1168, \ldots\}$ and $\mathcal{O}^+(-4.31363) = \{-4.31363, -3574.52, \ldots\}$. That is the forward orbit of the critical point are unbounded. Hence the independence fractal of P_{11} is totally disconnected.

Independence polynomial of P_{12} is $I_{P_{12}}(x) = 1 + 13x + 66x^2 + 165x^3 + 210x^4 + 126x^5 + 28x^6 + x^7$ and the reduced polynomial $f(x) = 13x + 66x^2 + 165x^3 + 210x^4 + 126x^5 + 28x^6 + x^7$, the critical point of f are $-0.2773, -0.34515, -0.49491$ $-0.87422, -2.26848$ and -19.73952 and $\mathcal{O}^+(-0.27773) = \{-0.27773, -1.0004, -1.002, 1.01012, -1.05373, -1.36485, -10.1675, 8087747 \ldots\}$, $\mathcal{O}^+(-0.34515) = \{-0.34515, -0.998993372, -0.994997198, -0.97572934, -0.895473146, \ldots\}$, $\mathcal{O}^+(-0.49491) = \{-0.49491, -1.007851773, -1.041137551, -1.260731378, -5.53649638, 162779.782, 782, \ldots\}$, $\mathcal{O}^+(-0.87422) = \{-0.87422, -0.7356601, -0.857051, 0.73571, 0.853339, \ldots\}$, $\mathcal{O}^+(-2.26848) = \{-2.26848, -117.5268005, \ldots\}$ and $\mathcal{O}^+(-19.73952) = \{-19.73952, -141701425.3, \ldots\}$. That is the forward orbit of the critical point are unbounded. Hence the independence fractal of P_{12} is totally disconnected.

Independence polynomial of P_{13} is $I_{P_{13}}(x) = 1 + 14x + 78x^2 + 220x^3 + 330x^4 + 252x^5 + 84x^6 + 8x^7$ and the reduced polynomial $f(x) = 14x + 78x^2 + 220x^3 + 330x^4 + 252x^5 + 84x^6 + 8x^7$, the critical point of f are -0.27392, -0.33037, -0.44866, -0.71719, -1.49777 and -5.73209 and $\mathcal{O}^+(-0.27392) = \{-0.27392, -1, -2, 92, \ldots\}$, $\mathcal{O}^+(-0.33037) = \{-0.33037, -1, -2, 92, \ldots\}$, $\mathcal{O}^+(-0.44866) = \{-0.44866, -1, -2, 92, \ldots\}$, $\mathcal{O}^+(-0.71719) = \{-0.71719, -1, -2, 92, \ldots\}$, $\mathcal{O}^+(-1.49777) = \{-1.49777, -10.8757, -3.9 \times 10^7, \ldots\}$ and $\mathcal{O}^+(-5.73209) = \{-5.73209, 110873, \ldots\}$. That is the forward orbit of the critical point are unbounded. Hence the independence fractal of P_{13} is totally disconnected.

Independence polynomial of P_{14} is $I_{P_{14}}(x) = 1 + 15x + 91x^2 + 286x^3 + 495x^4 + 462x^5 + 210x^6 + 36x^7 + x^8$ and the reduced polynomial $f(x) = 15x + 91x^2 + 286x^3 + 495x^4 + 462x^5 + 210x^6 + 36x^7 + x^8$, the critical point of f are -0.27242, -0.31228, -0.4382, -0.57138, -1.20938, -2.75264 and -26.44372 and $\mathcal{O}^+(-0.27242) = \{-0.27242, -1, -2, 270, \ldots\}$, $\mathcal{O}^+(-0.31228) = \{-0.31228, -1, -2, 270, \ldots\}$, $\mathcal{O}^+(-0.43820) = \{-0.43820, -1, -2, 270, \ldots\}$, $\mathcal{O}^+(-0.57138) = \{-0.57138, -1, -2, 270, \ldots\}$, $\mathcal{O}^+(-1.20938) = \{-1.20938, -1.86871, 158.2364, \ldots\}$ and $\mathcal{O}^+(-26.44372) = \{-26.44372, -2 \times 10^{10}, \ldots\}$. That is the forward orbit of the critical point are unbounded. Hence the independence fractal of P_{14} is totally disconnected.

Independence polynomial of P_{15} is $I_{P_{15}}(x) = 1 + 16x + 105x^2 + 364x^3 + 715x^4 + 792x^5 + 462x^6 + 120x^7 + 9x^8$ and the reduced polynomial $f(x) = 16x + 105x^2 + 364x^3 + 715x^4 + 792x^5 + 462x^6 + 120x^7 + 9x^8$, the critical point of f are -0.26836, -0.30983, -0.38985, -0.54702, -0.54702, -1.90254 and -7.35321 and $\mathcal{O}^+(-0.26836) = \{-0.26836, -1, -1, \ldots\}$, $\mathcal{O}^+(-0.30983) = \{-0.30983, -1, -1, \ldots\}$, $\mathcal{O}^+(-0.38985) = \{-0.38985, -1, -1, \ldots\}$, $\mathcal{O}^+(-0.54702) =$

$\{-0.54702, -1, -1, \ldots\}$, $\mathcal{O}^+(-1.90254) = \{-1.90254, 96.485, \ldots\}$ and $\mathcal{O}^+(-7.35321) = \{-7.35321, -4603551, \ldots\}$. That is the forward orbit of the critical point are unbounded. Hence the independence fractal of P_{15} is totally disconnected.

Independence polynomial of P_{16} is $I_{P_{16}}(x) = 1 + 17x + 120x^2 + 455x^3 + 1001x^4 + 1287x^5 + 924x^6 + 330x^7 + 45x^8 + x^9$ and the reduced polynomial $f(x) = 17x + 120x^2 + 455x^3 + 1001x^4 + 1287x^5 + 924x^6 + 330x^7 + 45x^8 + x^9$, the critical point of f are -0.26629, -0.30248, -0.37027, -0.49692, -0.75522, -1.3911, 3.70905 and -32.70868 and $\mathcal{O}^+(-0.26629) = \{-0.26629, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.30248) = \{-0.30248, -1, 0, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.37027) = \{-0.37027, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.49692) = \{-0.49692, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.75522) = \{-75522, -1.0593, 0.8776, 2251, \ldots\}$, $\mathcal{O}^+(-1.3911) = \{-1.3911, 8.63532, \ldots\}$, $\mathcal{O}^+(-3.70905) = \{-3.70905, 37696.2, \ldots\}$ and $\mathcal{O}^+(-32.70868) = \{-32.70868, 3.97 \times 10^{12}, \ldots\}$. That is the forward orbit of the critical point are unbounded. Hence the independence fractal of P_{16} is totally disconnected.

Independence polynomial of P_{17} is $I_{P_{17}}(x) = 1 + 18x + 136x^2 + 560x^3 + 1365x^4 + 2002x^5 + 1716x^6 + 792x^7 + 165x^8 + 10x^9$ and the reduced polynomial $f(x) = 18x + 136x^2 + 560x^3 + 1365x^4 + 2002x^5 + 1716x^6 + 792x^7 + 165x^8 + 10x^9$, the critical point of f are -0.26455, -0.29645, -0.35474, -0.4593, -0.65901, -1.09752, -2.35813 and -9.17697 and $\mathcal{O}^+(-0.26455) = \{-0.26455, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.29645) = \{-0.29645, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.35474) = \{-0.35474, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.45930) = \{-0.45930, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.65901) = \{-0.65901, -1.01565, 0.0919501, 3.35213, \ldots\}$, $\mathcal{O}^+(-1.09752) = \{-1.09752, 0.383285, 110.912, \ldots\}$, $\mathcal{O}^+(-2.35813) = \{-2.35813, -1255.92, \ldots\}$ and $\mathcal{O}^+(-9.17697) = \{-9.17697, 2.46333 \times 10^8, \ldots\}$. That is the forward orbit of the critical point are unbounded. Hence the independence fractal of P_{17} is totally disconnected.

Independence polynomial of P_{18} is $I_{P_{18}}(x) = 1 + 19x + 153x^2 + 680x^3 + 1820x^4 + 3003x^5 + 3003x^6 + 1716x^7 + 495x^8 + 55x^9 + x^{10}$ and the reduced polynomial $f(x) = 19x + 153x^2 + 680x^3 + 1820x^4 + 3003x^5 + 3003x^6 + 1716x^7 + 495x^8 + 55x^9 + x^{10}$, the critical point of f are -0.26308, -0.29143, -0.34216, -0.4302, -0.58976, -0.91116, -1.69858, -4.56452 and -40.4091 and $\mathcal{O}^+(-0.26308) = \{-0.26308, -1, -1, \ldots\}$, $\mathcal{O}^+(-0.29143) = \{-0.29143, -1, -1, \ldots\}$, $\mathcal{O}^+(-0.34216) = \{-0.34216, -1, -1, \ldots\}$, $\mathcal{O}^+(-0.4302) = \{-0.4302, -1, -1 \ldots\}$, $\mathcal{O}^+(-0.58976) = \{-0.58976, -1, -1 \ldots\}$, $\mathcal{O}^+(-0.91116) = \{-0.91116, 0.745462, -0.9402230, -0.7653940, \ldots\}$, $\mathcal{O}^+(-1.69858) = \{-1.69858, -88.18820, \ldots\}$, $\mathcal{O}^+(-4.56452) = \{-4.56452, 992233, \ldots\}$, and $\mathcal{O}^+(-40.4091) = \{-40.4091, -9.61736 \times 10^{14}, \ldots\}$. That is the forward orbit of some of the critical point are unbounded. Hence the independence fractal of P_{18} is not connected.

Independence polynomial of P_{19} is $I_{P_{19}}(x) = 1 + 20x + 171x^2 + 816x^3 + 2380x^4 + 436x^5 + 5005x^6 + 3432x^7 + 1287x^8 + 220x^9 + 11x^{10}$ and the reduced polynomial $f(x) = 20x + 171x^2 + 816x^3 + 2380x^4 + 436x^5 + 5005x^6 + 3432x^7 + 1287x^8 + 220x^9 + 11x^{10}$, the critical point of f are -0.26182, -0.2872, -0.33182, -0.40714, -0.53793, -0.78426, -1.32197, -2.86448 and -11.20339 and $\mathcal{O}^+(-0.26182) = \{-0.26182, -1, -2, 1540, \ldots\}$, $\mathcal{O}^+(-0.2872) = \{-0.2872, -1, -2, 1540, \ldots\}$, $\mathcal{O}^+(-0.33182) = \{-0.33182, -1, -2, 1540, \ldots\}$, $\mathcal{O}^+(-0.40714) = \{-0.40714, -1, -2, 1540, \ldots\}$, $\mathcal{O}^+(-0.53793) = \{-0.53793, -1, -2, 1540, \ldots\}$, $\mathcal{O}^+(-0.78426) = \{-0.78426, -0.94417, -1.39879, -7.62536, \ldots\}$, $\mathcal{O}^+(-1.32197) = \{-1.32197, -10.4796, \ldots\}$, $\mathcal{O}^+(-2.86448) = \{-2.86448, -20388.6, \ldots\}$ and $\mathcal{O}^+(-11.20339) = \{-11.20339, -1.65027 \times 10^{10}, \ldots\}$. That is the forward orbit of some of the critical point are unbounded. Hence the independence fractal of P_{19} is totally disconnected.

Independence polynomial of P_{20} is $I_{P_{20}}(x) = 1 + 21x + 190x^2 + 969x^3 + 3060x^4 + 6188x^5 + 8008x^6 + 6435x^7 + 3003x^8 + 715x^9 + 66x^{10} + x^{11}$ and the reduced polynomial $f(x) = 21x + 190x^2 + 969x^3 + 3060x^4 + 6188x^5 + 8008x^6 + 6435x^7 + 3003x^8 + 715x^9 + 66x^{10} + x^{11}$, the critical point of f are -0.26074, -0.2836, -0.3232, -0.38849, -0.49794, -0.69325, -1.8398, -2.03863, -5.51009 and -48.92009 and $\mathcal{O}^+(-0.26074) = \{-0.26074, -1, -2, 966, \ldots\}$, $\mathcal{O}^+(-0.2836) = \{-0.2836, -1, -2, 966, \ldots\}$, $\mathcal{O}^+(-0.3232) = \{-0.3232, -1, -2, 966, \ldots\}$, $\mathcal{O}^+(-0.38849) = \{-0.38849, -1, -2, 966, \ldots\}$, $\mathcal{O}^+(-0.49794) = \{-0.49794, -1, -2, 966, \ldots\}$, $\mathcal{O}^+(-0.69325) = \{-0.96325, -0.986054, -1.9015, 792.009, \ldots\}$, $\mathcal{O}^+(-1.08398) = \{-1.08398, -2.38929, -4200.68, \ldots\}$, $\mathcal{O}^+(-2.03863) = \{-2.03863, -945.881, \ldots\}$, $\mathcal{O}^+(-5.51009) = \{-5.51009, -3.2397 \times 10^7, \ldots\}$, and $\mathcal{O}^+(-48.92009) = \{-48.92009, 2.87741 \times 10^{17}, \ldots\}$. That is the forward orbit of some of the critical point are unbounded. Hence the independence fractal of P_{20} is totally disconnected.

Independence polynomial of P_{21} is $I_{P_{21}}(x) = 1 + 22x + 210x^2 + 1140x^3 + 3876x^4 + 8568x^5 + 12376x^6 + 11440x^7 + 6435x^8 + 2002x^9 + 286x^{10} + 12x^{11}$ and the reduced polynomial $f(x) = 22x + 210x^2 + 1140x^3 + 3876x^4 + 8568x^5 + 12376x^6 + 11440x^7 + 6435x^8 + 2002x^9 + 286x^{10} + 12x^{11}$, the critical point of f are -0.2598, -0.28051, -0.31592, -0.37315, -0.46629, -0.62534, -0.92254, -1.56912, -3.42155 and -13.43244 and $\mathcal{O}^+(-0.25980) = \{-0.25980, -1, -1, \ldots\}$, $\mathcal{O}^+(-0.28051) = \{-0.28051, -1, -1, \ldots\}$, $\mathcal{O}^+(-0.31592) = \{-0.31592, -1, -1, \ldots\}$, $\mathcal{O}^+(-0.37315) = \{-0.37315, -1, -1, \ldots\}$, $\mathcal{O}^+(-0.46629) = \{-0.466269, -1, -1, \ldots\}$, $\mathcal{O}^+(-0.62534) = \{-0.62534, -1, -1, \ldots\}$, $\mathcal{O}^+(-0.92254) = \{-0.92254, -1.25163, 13.6553, \ldots\}$, $\mathcal{O}^+(-1.56912) = \{-1.56912, -79.7738, \ldots\}$, $\mathcal{O}^+(-3.42155) = \{-3.42155, -407709, \ldots\}$, and $\mathcal{O}^+(-13.43244) = \{-13.43244, 1.35279 \times 10^{12}, \ldots\}$. That is the forward orbit of some of the critical point are unbounded. Hence the independence fractal of P_{21} is not connected.

Independence polynomial of P_{22} is $I_{P_{22}}(x) = 1 + 23x + 231x^2 + 1330x^3 + 4845x^4 + 11628x^5 + 18564x^6 + 19448x^7 + 12870x^8 + 5005x^9 + 1001x^{10} + 78x^{11} + x^{12}$ and the reduced polynomial $f(x) = 23x + 231x^2 + 1330x^3 + 4845x^4 + 11628x^5 + 18564x^6 + 19448x^7 + 12870x^8 + 5005x^9 + 1001x^{10} + 78x^{11} + x^{12}$, the critical point of f are $-0.25898, -0.27784, -0.30972, -0.36036, -0.44074, -0.57307, -0.80715, -1.27356, -2.41119, -6.54575$ and -58.24165 and $\mathcal{O}^+(-0.25898) = \{-0.25898, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.27784) = \{-0.27784, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.30972) = \{-0.30972, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.36036) = \{-0.360636, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.44074) = \{-0.44074, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.57307) = \{-0.57307, -0.998851, -0.182733, -0.998022, -0.0313204, -0.999992, -0.000095997, 0.0022058, -0.0496236, 0.708885, -1.01966, 0.348045, 270.743 \ldots\}$, $\mathcal{O}^+(-0.80715) = \{-0.80715, -1.05333, 1.11042, 150113, \ldots\}$, $\mathcal{O}^+ 27356 = \{-1.27356, -8.37082, \ldots\}$, $\mathcal{O}^+(-2.41119) = \{-2.41119, -13681.3,$ $\mathcal{O}^+(-6.54575) = \{-6.54575, -1.28431 \times 10^9 \ldots\}$, and $\mathcal{O}^+(-58.24165) = \{-58.24165, 1.04188 \times 10^{20}, \ldots\}$. That is the forward orbit of some of the critical point are unbounded. Hence the independence fractal of P_{22} is not connected.

Independence polynomial of P_{23} is $I_{P_{23}}(x) = 1 + 24x + 253x^2 + 1540x^3 + 5985x^4 + 15504x^5 + 27132x^6 + 31824x^7 + 24310x^8 + 11440x^9 + 3003x^{10} + 364x^{11} + 13x^{12}$ and the reduced polynomial $f(x) = 24x + 253x^2 + 1540x^3 + 5985x^4 + 15504x^5 + 27132x^6 + 31824x^7 + 24310x^8 + 11440x^9 + 3003x^{10} + 364x^{11} + 13x^{12}$, the critical point of f are $-0.25826, -0.27551, -0.30438, -0.34956, -0.41975, -0.53179, -0.72129, -1.07374, -1.83891, -4.02933$, and -15.86414 and $\mathcal{O}^+(-0.25826) = \{-0.25826, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.27551) = \{-0.27551, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.30438) = \{-0.30438, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.34956) = \{-0.34956, -1, 0, 0, \ldots\}$, $\mathcal{O}^+(-0.41975) = \{-0.41975, -1.00002, 0.00015999, -00384641, 0.0961459, 6.6778, \ldots\}$, $\mathcal{O}^+(-0.53179) = \{-0.53179, -0.999635,$

-0.00292105, -0.0679844, -0.838372, -0.916267, -0.611496, 1.00202, -0.016125, 0.459663, 1253.59, $\ldots\}$, $\mathcal{O}^{+}(-0.72129) = \{-0.72129, -1.01275, 0.100087, 7.26755, \ldots\}$, $\mathcal{O}^{+}(-1.07374) = \{-1.07374, -0.393927, 599.267, \ldots\}$, $\mathcal{O}^{+}(-1.83891) = \{-1.83891, -837.361, \ldots\}$, $\mathcal{O}^{+}(-4.02933) = \{-4.02933, 9.83295 \times 10^{6}, \ldots\}$, and $\mathcal{O}^{+}(-15.86414) = \{-15.86414, -1.33182 \times 10^{14}, \ldots\}$. That is the forward orbit of some of the critical point are unbounded. Hence the independence fractal of P_{23} is not connected.

Independence polynomial of P_{24} is $I_{P_{24}}(x) = 1 + 25x + 276x^2 + 1771x^3 + 7315x^4 + 20349x^5 + 38760x^6 + 50388x^7 + 43758x^8 + 24310x^9 + 8008x^{10} + 1365x^{11} + 91x^{12} + x^{13}$ and the reduced polynomial $f(x) = 25x + 276x^2 + 1771x^3 + 7315x^4 + 20349x^5 + 38760x^6 + 50388x^7 + 43758x^8 + 24310x^9 + 8008x^{10} + 1365x^{11} + 91x^{12} + x^{13}$, the critical point of f are -0.25763, -0.27346, -0.29975, -0.34035, -0.40226, -0.49852, -0.65536, -0.93133, -1.47983, -2.81625, -7.67148 and -15.86414 and $\mathcal{O}^{+}(-0.25763) = \{-0.25763, -1, -1, \ldots\}$, $\mathcal{O}^{+}(-0.27346) = \{-0.27346, -1, -1, \ldots\}$, $\mathcal{O}^{+}(-0.29975) = \{-0.29975, -1, -1, \ldots\}$, $\mathcal{O}^{+}(-0.34035) = \{-0.34035, -1, -1, \ldots\}$, $\mathcal{O}^{+}(-0.40226) = \{-0.40226, -1, -1, \ldots\}$, $\mathcal{O}^{+}(-0.49852) = \{-0.49852, -0.999878, -0, -0.998904, -0.990265, -0.9221165, \ldots\}$, $\mathcal{O}^{+}(-0.65536) = \{-0.65536, -1.010335, 1.03138, -1.40537, -64.3914, \ldots\}$, $\mathcal{O}^{+}(-0.93133) = \{-0.93133, -0.750578, -0.981796, -0.869028, -0.825799, \ldots\}$, $\mathcal{O}^{+}(-1.47983) = \{-1.47983, -77.4068, -1.27518 \times 10^{23}, \ldots\}$, $\mathcal{O}^{+}(-2.81625) = \{-2.81625, 227198, \ldots\}$, $\mathcal{O}^{+}(-7.67148) = \{-7.67148, -6.07485 \times 10^{23}, \ldots\}$ and $\mathcal{O}^{+}(-68.37378) = \{-68.37378, -4.49029 \times 10^{22}, \ldots\}$. That is the forward orbit of some of the critical point are unbounded. Hence the independence fractal of P_{24} is not connected.

Independence polynomial of P_{25} is $I_{P_{25}}(x) = 1 + 26x + 300x^2 + 2024x^3 + 8855x^4 + 26334x^5 + 54264x^6 + 77520x^7 + 75582x^8 + 48620x^9 + 19448x^{10} + 4368x^{11} + 1455x^{12} + 14x^{13}$ and the reduced polynomial $f(x) = 26x + 300x^2$

$+ 2024x^3 + 8855x^4 + 26334x^5 + 54264x^6 + 77520x^7 + 75582x^8 + 48620x^9 + 19448x^{10} + 4368x^{11} + 1455x^{12} + 14x^{13}$, the critical point of f are -0.25706, -0.27166, -0.2957, -0.38749, -0.47123, -0.60341, -0.82562, -1.23778, -2.1313, -4.6878 and -18.49849 and $\mathcal{O}^+(-0.25706) = \{-0.25706, -1, -2, 7916, \ldots\}$, $\mathcal{O}^+(-0.27166) = \{-0.27166, -1, -2, 7916, \ldots\}$, $\mathcal{O}^+(-0.2957) = \{-0.2957, -1, -2, 7916, \ldots\}$, $\mathcal{O}^+(-0.38749) = \{-0.38749, -1, -2, 7916, \ldots\}$, $\mathcal{O}^+(-0.47123) = \{-0.47123, -0.999957, -1.99923, 7894.46, \ldots\}$, $\mathcal{O}^+(-0.60341) = \{-0.60341, -1.00095, -2.0172, -8388.94, \ldots\}$, $\mathcal{O}^+(-0.82562) = \{-0.82562, -0.948557, -1.32847, -0.873677, -0.873677, 0.981707, -1.7055, -1489.2, \ldots\}$, $\mathcal{O}^+(-1.23778) = \{-1.23778, -10.2907, \ldots\}$, $\mathcal{O}^+(-2.1313) = \{-2.1313, 10331.2, \ldots\}$, $\mathcal{O}^+(-4.6878) = \{-4.6878, -2813510000, \ldots\}$, $\mathcal{O}^+(-7.67148) = \{-7.67148, -6.07485 \times 10^{23}, \ldots\}$ and $\mathcal{O}^+(-18.49849) = \{-18.49849, 1.55055 \times 10^{16}, \ldots\}$. That is the forward orbit of some of the critical point are unbounded. Hence the independence fractal of P_{25} is totally disconnected.

Conclusion

In this paper, we studied the independence fractals of some graphs and have presented some results on connectedness of Independence fractals for the family of graphs K_n, K_n^c, $K_{n,\,n}$ and P_n.

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