Advances and Applications in Mathematical Sciences
Volume 21, Issue 2, December 2021, Pages 985-992

A STUDY ON DEALS WITH GENERATING FUNCTION

P. SUGANYA, K. CHANDRAN, N. MALINI and P. MALLIGA

Assistant Professor
Department of Mathematics
Dhanalakshmi Srinivasan Arts
and Science College for Women (Autonomous)
Perambalur, Indian
E-mail: Pavalamp1994@gmail.com

Abstract

Generating function is just a different way of writing a sequence of number. Here we will be dealing mainly with sequence of number $\{a_n\}$. Which represent the number of object of size n for an enumeration problem the interest of the rotation is that certain natural operation on generating lead to powerful methods for dealing with recurrences on $\{a_n\}$.

Introduction

In dealing with generating function, we frequently want to use different transformation and manipulation that are illegal if the generating function are viewed as analytic function. Generating function is a method to solve the recurrence relations (1). In this chapter deals with solving recurrence and proving some of the combinatorial identities, finding asymptotic formulae for terms of sequence.

Definition. The sequence $a_0, a_1, a_2, \ldots, a_r,$ of real numbers. For some interval of real numbers containing zero whose value at t is given, the function $G(t)$ is defined by the series (2).

$$G(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_r t^r + \ldots$$

This function $G(t)$ is called the *Generating function* of the sequence a_r.

2020 Mathematics Subject Classification: 65Qxx.
Keywords: Recurrences relation, Generating function, Sequence.
Received November 2, 2021; Accepted November 15, 2021

Theorem. *If we know the generating function for the sequence* a_r, *then the generating function of sequences* a_{r+1}, a_{r+2} *can be find easily.*

Proof. Let us consider, that $G(t)$ be the generating function for the sequence, having the general term a_r, then we have

$$G(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_r t^r + \ldots$$

(or) $$\frac{G(t) - a_0}{t} = a_1 + a_2 t + a_3 t^2 + \ldots + a_r t^{r-1} + a_{r+1} t^r.$$

Thus $\frac{G(t) - a_0}{t}$ is the generating function for the sequence a_{r+1}.

In the same way,

$$\frac{G(t) - a_0 - a_1 t}{t} = a_2 + a_3 t + a_4 t^2 + \ldots$$ Is the generating function for a_{r+2}.

Example. Solve the recurrence relation $a_{r+2} - 3a_{r+1} + 2a_r = 0$ by the method of generating functions with the initial conditions $a_0 = 2$ and $a_1 = 3$.

Solution. Let us assume that

$$G(t) = \sum_{r=0}^{\infty} a_r t^r$$

Given $a_{r+2} - 3a_{r+1} + 2a_r = 0$. (1)

Multiply equation (1) by t^r and summing from $r = 0$ to ∞ we have

$$\sum_{r=0}^{\infty} a_{r+2} t^r - 3\sum_{r=0}^{\infty} a_{r+1} t^r + 2\sum_{r=0}^{\infty} a_r t^r = 0$$

$$(a_2 + a_3 t + a_4 t^2 + \ldots) - 3(a_0 + a_1 t + a_2 t^2 + \ldots) + 2(a_0 + a_1 t + a_2 t^2 \ldots) = 0$$

$$[\because G(t) = a_0 + a_1 t + a_2 t^2 \ldots]$$

$$\therefore \frac{G(t) - a_0 - a_1 t}{t^2} - 3\left(\frac{G(t) - a_0}{t}\right) + 2G(t) = 0. \quad (2)$$

Now put $a_0 = 2$ and $a_1 = 3$ in equation (2) and solving, we get

$$G(t) = \frac{2 - 3t}{1 - 3t + 2t^2} \text{ (or) } G(t) = \frac{2 - 3t}{(1 - t)(1 - 2t)}$$

Now,

Let $\frac{2 - 3t}{(1 - t)(1 - 2t)} = \frac{A}{1 - t} + \frac{B}{1 - 2t}$

$$2 - 3t = A(1 - 2t) + B(1 - t). \tag{3}$$

Put $t = 1$ on both sides of the equation (3) to find A

Hence -1 = -A

$\therefore A = 1$

Put $t = \frac{1}{2}$ on both sides of the equation (3) to find B.

Hence $\frac{1}{2} = \frac{1}{2}B \quad \therefore B = 1$

Thus $G(t) = \frac{1}{1 - t} + \frac{1}{1 - 2t}$

Hence, $a_r = 1 + 2^r$.

Example. Solve the recurrence relation

$$a_r - 7a_{r-1} + 10a_{r-2} = 0$$

By the method of generating functions with the initial conditions $a_0 = 3$ and $a_1 = 3$.

Solution. Given equation $a_r - 7a_{r-1} + 10a_{r-2} = 0$ (1)

Let us assume that

$$G(t) = \sum_{r=0}^{\infty} a_r t^r$$

Multiply equation (1) by t^r and summing from $r = 2$ to ∞, we have

$$\sum_{r=2}^{\infty} a_r t^r - 7\sum_{r=2}^{\infty} a_{r-1} t^r + 10\sum_{r=2}^{\infty} a_{r-2} t^r = 0$$

$$(a_2 t^2 + a_3 t^3 + \ldots) - 7(a_1 t^2 + a_2 t^3 + \ldots) + 10(a_0 t^2 + a_1 t^3 + \ldots) = 0$$

$$G(t) - a_0 - a_1 t - 7t[G(t) - a_0] + 10t^2 G(t) = 0. \tag{2}$$

Now, put $a_0 = 3$ and $a_1 = 3$ in equation (2) and solving, we get

$$G(t) = \frac{3 + 24t}{10t^2 - 7t + 1}$$

$$= \frac{3 + 24t}{(5t - 1)(2t - 1)}$$

$$G(t) = \frac{10}{(2t - 1)} - \frac{13}{(5t - 1)}$$

$$G(t) = \frac{13}{(1 - 5t)} - \frac{10}{(1 - 2t)}$$

Therefore, $a_r = 13(5)^r - 10(2)^r$.

Example. Solve the recurrence relation

$$a_{r+2} - 2a_{r+1} + a_r = 2^r$$

By the method of generating functions with the initial conditions $a_0 = 2$ and $a_1 = 1$.

Solution. Given equation $a_{r+2} - 2a_{r+1} + a_r = 2^r$ (1)

By taking the generating functions of equation (1) both the sides, we have

$$\frac{G(t) - a_0 - a_1 t}{t^2} - 2\left(\frac{G(t) - a_0}{t}\right) + G(t) = \frac{1}{1 - 2t} \tag{2}$$

Now, put $a_0 = 2$ and $a_1 = 2$ in equation (2) and solving

We get

$$G(t) - 2 - t - 2tG(t) - 2t \_ t^2 G(t) = \frac{t^2}{1-2t}$$

$$(t^2 - 2t + 1)\, G(t) = 2 + 3t + \frac{t^2}{1-2t}$$

$$(1 - t^2)\, G(t) = 2 + 3t + \frac{t^2}{1-2t}$$

$$G(t) = \frac{2}{(1-t)^2} + \frac{3t}{(1-t)^2} + \frac{t^2}{(1-2t)(1-t)^2}$$

By partial fractions

$$\frac{t^2}{(1-2t)(1-t)^2} = \frac{1}{(1-2t)} - \frac{1}{(1-t)^2}$$

Hence, $G(t) = \frac{1}{(1-t)^2} + \frac{3t}{(1-t)^2} + \frac{1}{(1-2t)}$

Therefore, $a_r = (r+1) + 3r + 2^r$

$$a_r = 1 + 4r + 2^r.$$

Example. Solve the recurrence relation $a_{r+2} - 5a_{r+1} + 6a_r = 2$ by the method of generating functions satisfying the initial conditions $a_0 = 1$ and $a_1 = 2$.

Solution. Given equation $a_{r+2} - 5a_{r+1} + 6a_r = 2$ (1)

Let us assume that

$$G(t) = \sum_{r=0}^{\infty} a_r t^r$$

Now, by taking generating functions of equation (1), we have

$$\frac{G(t) - a_0 - a_1 t}{t^2} - 5\left(\frac{G(t) - a_0}{t}\right) + 6G(t) = \frac{2}{1-t}$$

Put $a_0 = 1$ and $a_1 = 3$ in the above equation and solving, we get

$$G(t) = \frac{5t^2 - 4t + 1}{(1-t)(1-2t)(1-3t)}$$

By partial fractions

$$G(t) = \frac{1}{(1-t)} + \frac{1}{(1-2t)} + \frac{1}{(1-3t)}$$

Therefore, the solution after applying inverse transformations.

$$a_r = 1 - 2^r + 3^r.$$

Example. Find the generating function of the sequence $\{0^2, 1^2, 2^2, 3^2, \ldots\}$.

Solution. We have

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots$$

Differentiating both sides

$$\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots$$

$$\frac{1+x}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots$$

$$\frac{x(1+x)}{(1-x)^3} = 0x^0 + 1^2x + 2^2x^2 + 3^2x^3 + 4^2x^3 +$$

$\{0^2, 1^2, 2^2, 3^2, \ldots\}$ has the generating function $\frac{x(1+x)}{(1-x)^3}$.

Example. Evaluate the sum: $1^2 + 2^2 + 3^2 + \ldots + r^2$ using generating function.

Solution. We know that $\frac{1}{1-z} = 1 + z + z^2 + \ldots + z^r + \ldots$

Differentiating both sides with respect to z, we get

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \ldots + rz^{r-1} + \ldots$$

Multiplying both sides by z, we get

$$\frac{z}{(1-z)^2} = z + 2z^2 + 3z^2 + \ldots + rz^r + \ldots$$

Differentiating both sides with respect to z, we get

$$\frac{1+z}{(1-z)^2} = 1^2 + 2^2 z + 3^2 z^2 + \ldots + r^2 z^r + \ldots$$

Multiplying both sides by z, we get

$$\frac{z(1+z)}{(1-z)^2} = 0^2 + 1^2 z + 2^2 z^2 + 3^2 z^3 \ldots + r^2 z^r + \ldots$$

This shows that $z(1+z)(1-z)^3$ is the generating function of the numeric function $0^2, 1^2, 2^2, \ldots, r^2$.

Now

$$\frac{1}{1-z} \times \frac{z(1+z)}{(1-z)^3} = (1 + z + z^2 + \ldots + 2^r + \ldots) \times (0^2 + 1^2 z + \ldots)$$

$$= 0^2 + (0^2 + 1^2)z + (0^2 + 1^2 + 2^2)z^2 + \ldots + (0^2 + 1^2 + \ldots + r^2)z^r + \ldots$$

Thus $0^2 + 1^2 + 2^2 + \ldots + r^2 =$ coefficient of z^r in the expansion of $z(1+z)/(1-z)^4$.

But from binomial theorem we know that the coefficient of z^r in the expansion of $1/(1-z)^4$ is

$$\frac{(-4)(-4-1)\ldots(-4-r+1)}{r!}(-1)^r = \frac{4 \cdot 5 \cdot 6 \cdot \ldots (r+3)}{r!}$$

$$= \frac{(r+1)(r+2)(r+3)}{1 \cdot 2 \cdot 3}$$

So, the coefficient of z^r in the expansion $z(1+z)/(1-z)^4$ is

$$\frac{r(r+1)(r+2)}{1 \cdot 2 \cdot 3} + \frac{r(r-1)(r+1)}{1 \cdot 2 \cdot 3} = \frac{r(r+1)(2r+1)}{6}$$

Hence $1^2 + 2^2 + 3^2 + \ldots + r^2 = \dfrac{r(r+1)(2r+1)}{6}$.

Conclusion

We have discussed some basic application of generating functions (3), as a method to solve a linear recurrence or combinatorial problems. However there are certainly many more aspects that are not discussed here. Readers interested to learn more are invited to read for a very extensive treatment of generating functions (4), (5).

References

[1] J. K. Sharma, Discrete mathematics second edition faculty of Management studies University of Delhi.

[2] K. Chandrasekhara Rao, Discrete mathematics, Narosa Publishing House, New Delhi, Chennai, Mumbai, Kolkata.

[3] Satinder Bal Gupta, Discrete mathematics and structures, University Science Press (an imprint of Laxmi publications pvt.ltd.) India, USA.

[4] K. D. Joshi, Foundations of discrete mathematics, Department of mathematics, Indian Institute of Technology, Powai, Bombay, India.

[5] Richard Johnsonbaugh, Discrete mathematics, sixth edition, Depaul University, Chicago.