Advances and Applications in Mathematical Sciences
Volume 21, Issue 2, December 2021, Pages 1021-1027

TAYLOR SERIES

A. SARANYA, C. MAHENDRAN, P. PRIYA and R. RAMYA

Department of Mathematics
Dhanalakshmi Srinivasan College of Arts
and Science for Women (Autonomous)
Perambalur, India
E-mail: kasaranya92@gmail.com,
priyaperiyasamy13@gmail.com
ramya25071987@gmail.com

Abstract

Taylor series method and computer closed from solution of system of lane Emden equation subject to given initial condition. In this work this method is further explored and extended to a class of non-linear ODE, PDEs, system of nonlinear ODEs, PDEs subject to certain initial condition and boundary condition. In some case we could find exact solution and if that is not possible then we compute approximate solution we have compared these solutions with other existing techniques and showed that method is simple and superior to other existing iterative techniques. We have also provided Mathematics codes which user may find useful and can compute solutions as a per this need.

Introduction

In the Mathematics the Taylor series of a function is infinite sum of terms that are expressed in terms of function's derivates at single point. For most common function and sum of its Taylor series are equal near this point. Taylor's series are named after brook Taylor who introduced them in 1715.

As the degree of the Taylor polynomial rises, it approaches the correct function. This image shows sin x and its Taylor approximations by polynomials of degree 1, 3, 5, 7, 9, 11 and 13 at $x = 0$.

If zero is the point where the derivates are considered, a Taylor series is

2020 Mathematics Subject Classification: 30Kxx.
Keywords: Taylor series, Maclaurin's series.
Received November 2, 2021; Accepted November 15, 2021

also called Maclaurin series, after Collin Maclaurin's who made extensive use of this special case of Taylor series in 18th century.

The partial sum formed by the first $n+1$ terms of Taylor series is a polynomial of the degree n that is called the nth Taylor polynomial of the function. A function may differ from the sum of its Taylor series, even if its Taylor series is convergent.

Definition. The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series.

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a).$$

Where $n!$ denotes the factorial of n, in the more compact sigma notation, this can be written as

$$\sum\_(n=0)^\infty(f^(n))(a)/n!\,[(x-a)]^n$$

Where $f^((n))(a)$ denote the nth derivates of f evaluated at the point a. The derivate of order zero of f is defined to be itself and $[(x-a)]^0$ and $0!$ Are both defined to be1).

When $a=0$, the series is also called a Maclurin series.

Theorem. Taylor's theorem. *Let* $f(z)$ *be analytic in region* D *containing* z_0. *Then* $f(z)$ *can be represented as a power series in* $z-z_0$ *given by*

$$f(z)=f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z)}{2!}(z-z_0^2)+\ldots+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+\ldots$$

The expansion is valid in the largest open disc with centre z_0 *contained in* *D*.

Proof. Let $r>0$ be such that disc $|z-z_o|<r$ is contained in D.

let $0<r_1$ let C_1 be the circle $|z-z_o|<r_1$.

By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi} \oint \frac{f(\tau)}{\tau - z} dz \ldots (1)$$

By the theorem on higher derivates we have,

$$\frac{1}{\tau - z} = \frac{1}{(\tau - z_0) - (\tau - z_0)}$$

$$= \frac{1}{\tau - z}\left[1 + \left(\frac{z - z_0}{\tau - z_0}\right) + \left(\frac{z - z_0}{\tau - z_0}\right)^2 + \left(\frac{z - z_0}{\tau - z_0}\right)^3 + \ldots + \left(\frac{z - z_0}{\tau - z_0}\right)^{n-1} + \frac{\left(\frac{z - z_0}{\tau - z_0}\right)^n}{1 - \left(\frac{z - z_0}{\tau - z_0}\right)}\right]$$

$$= \frac{1}{\tau - z_0} + \frac{z - z_0}{(\tau - z_0)^2} + \frac{(z - z_0)^2}{(\tau - z_0)^3} + \ldots + \frac{(z - z_0)^{n-1}}{(\tau - z_0)^n} + \frac{(z - z_0)^n}{(\tau - z_0)^n (\tau - z_0)^2}$$

Now, multiplying throughout by $\frac{f(\tau)dr}{2\pi i}$, integrating over C_1 and using 1 and 2 we get

$$f(z) = f(z_0) + f'(z_0)(z_0 - z_0)$$

$$+ \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{f^{(n-1)}}{(n-1)!}(z - z_0)^{n-1} + R_n \ldots (1)$$

Where $R_n = \frac{(z - z_0)^n}{2\pi i} \int \frac{f(\tau)d\tau}{(\tau - z)(\tau - z_0)^n}$

Here τ lies C_1 and z lies in the interior of C_1 so that $|\tau - z_0| = r_n$ and $|z - z_0| = r_1$ and $|z - z_0| < r_1$

$$\frac{1}{|\tau - z|} \leq \frac{1}{r1 - |z - z_0|}$$

Let M denote the maximum value of $|f(z)|$ on C_1

Then $| R_n | \leq \frac{| z - z_0 |^n}{2\pi} \frac{M(2\pi r)}{(r1 - | z - z_0 |) r^n}$

Also $\frac{| z - z_0 |}{r1} < 1$ hence $\lim_{n \to \infty} R_n = 0$

Taking limit $n \to \infty$ in (3) we get

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \ldots$$

The above series is called Taylor's series of $f(z)$ about the z_0. These if $f(z)$ is analytic at point z_0 then $f(z)$ can be represented as Taylors series about z_0 which is series is nonnegative powers of $z - z_0$. The expansion is valid in some neighbourhood of z_0.

The Taylor's series expansion of $f(z)$ about point zero is called Maclaurin's series.

Example. Let $f(z) = e^z$

Then $f^{(n)}(z) = e^z$ for all n and hence $f^{(n)}(0) = 1$

Hence the Maclaurin's series for e^z is given by

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots$$

2. The Taylor's series for $f(z) = \frac{1}{z}$ about $z = 1$ is given by

$$\frac{1}{z} = f(1) + \frac{f'(1)}{1!}(z - 1) + \frac{f''(1)}{2!}(z - 1)^2 + \frac{f'''(1)}{3!}(z - 1)^3 +$$

Now,

$$f(z) = \frac{1}{z} f(1) = 1$$

$$f'(z) = -\frac{1}{z^2} f'(1) = -1$$

$$f''(z) = -\frac{2}{z^3}\, f''(1) = 2$$

$f'''(z) = -\frac{6}{z^4}\, f'''(1) = -6.$ by

$$\ldots$$

Hence the Taylor's series expansion for $\frac{1}{z}$ about 1 is

$$\frac{1}{z} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \ldots$$

This expansion is valid in the disc $|z-1| < 1$,

Similarly the Taylor's series for $f(z) = 1/z$ about $z = 1$ is given by

$$\frac{1}{z} = \frac{1}{i} - \frac{z-i}{i^2} + \frac{(z-i)^2}{i^3} - \frac{(z-i)^3}{i^4} + \ldots$$

And the expansion is valid in the disc $|z-i| < 1$ (verify)

Problem: Expand $f(z) = \frac{z-1}{z+1}$ as taylor's series

1. about the point $z = 0$

2. about the point $z = 1$. Determine the region of conver gence in each case.

Solution

(i) $f(z) = \frac{z-1}{z+1}$

$$= (z-1)(1+z)^{-1}$$

$$= (z-1)(1 - z + z^2 - z^3 + \ldots) \text{ if } |z| < 1.$$

$$= (z - z^2 + z^3 - \ldots) - (1 - z + z^2 - z^2 - z^3 - \ldots)$$

$$= -1 + 2z - 2z^2 + 2z^3 + \ldots$$

(ii) $f(z) = \frac{z-1}{z+1}$

$$= \frac{z-1}{(2+z-1)}$$

$$= \frac{z-1}{2\left(1+\frac{z-1}{2}\right)}$$

$$= \frac{z-1}{2}\left(1+\frac{z-1}{2}\right)^{-1}$$

$$= \frac{z-1}{2}\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+\ldots\right] \text{if } \left|\ \frac{z-1}{2}\ \right| < 1,$$

$$= \frac{z-1}{2} - \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} - \ldots$$

The region of convergence is given by $\left|\ \frac{z-1}{2}\ \right| < 1$ which is same as the circular disc $|\ z-1\ | < 2$

Expand $\cos z$ into a Taylor's series about the point $z = \frac{\pi}{2}$ and determined the region of convergence.

Solution

$$f(z)\cos z$$

The Taylors series for $f(z)$ about $z = \frac{\pi}{2}$ is

$$f(z) = f\left(\frac{\pi}{2}\right) + \frac{\left(z-\frac{\pi}{2}\right)}{1!}F'\left(\frac{\pi}{2}\right) + \frac{\left(z-\frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right) + \frac{\left(z-\frac{\pi}{2}\right)}{3!}f''' + \left(\frac{\pi}{2}\right) + \ldots$$

Now

$$f(z) = \cos z.\ \textit{hence}\ f\left(\frac{\pi}{2}\right) = 0$$

$$f'(z) = -\sin z. \text{ hence } f'\left(\frac{\pi}{2}\right) = -1$$

$$f''(z) = -\cos z. \text{ hence } f''\left(\frac{\pi}{2}\right) = 0$$

$$f'''(z) = \sin z. \text{ hence } f'''(z) = 1.$$

The Taylor's series for $\cos z$ about $z = \frac{\pi}{2}$ is

$$\cos z = \frac{\left(z - \frac{\pi}{2}\right)}{1!} + \frac{\left(z - \frac{\pi}{2}\right)^3}{3!} - \frac{\left(z - \frac{\pi}{2}\right)^5}{5!} + \dots$$

The expansion is valid throughout the complex plane.

Conclusion

It is often possible to represent a function by way of an infinite series.

Near a specified point we can tell the value of a function just by knowing the value of function at the specified point and all its derivative's values at the point. Some approximating polynomials are only convergent in certain intervals. We can use the series to approximate the value of difficult in integral to a desired accuracy. We can evaluate limits by substituting approximate series representations.

References

[1] S. Arumugam, A. Thangapandi Isaac and A. Somasundaram, Complex analysis, New Scitech publication (India) Pvt. Ltd November (2003).

[2] James Ward Brown and Ruel V. Churchill, Complex variables and applications, McGraw-Hill Companies, Inc. 2009.

[3] Robert E. Greene and Steven G. Krantz, Function theory of one complex variable, (third edition), Graduate Studies in Mathematics 40, 2006.

[4] S. Punnuswamy, Foundations complex analysis, Narosa, New Delhi, (1995)

[5] Thomas, George B, Jr. and Finney, Ross l. Calculus and Analytic Geometry (9th ed.), Addison Wesley, 1996, ISBN 0-201-53174-7.