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RADIUS OF CONVEXITY OF SECTIONS OF A SUBCLASS OF STARLIKE FUNCTIONS

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Abstract

In this article we show that every section of certain subclass $\mathcal{H}$ of the class $\mathcal{H}_n(\alpha, \beta)$ of starlike functions are convex in the disk $|z| < \frac{1}{4}$.

1. Introduction

Let $\mathcal{A}_n$ be the class of analytic functions f defined on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with Taylor's series expansion of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \ (n \in \mathbb{N} := \{1, 2, 3, \cdots\}). \tag{1}$$

Let $\mathcal{A}$ denote the subclass $\mathcal{A}_1$ of $\mathcal{A}_n$ consisting of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\mathcal{S}$ denote the subclass of it containing univalent functions [2]. In [3], Srivastava et al. introduced the subclass $\mathcal{H}_n(\alpha, \beta)$ of $\mathcal{A}_n$ consisting of functions in $\mathcal{A}_n$ satisfying

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$$\Re\left(\frac{\alpha z^2 f''(z) + zf'(z)}{f(z)}\right) > \alpha\beta\left(\beta + \frac{n}{2} - 1\right) + \beta - \frac{n\alpha}{2},$$

where $\alpha \geq 0$; $0 \leq \beta < 1$; $z \in \Delta$. Putting $n = 1, \alpha = 1, \beta = 0$ we obtain the subclass $\mathcal{H} = \mathcal{H}_1(1, 0)$ and $\mathcal{H}_n(\alpha, \beta)$ consisting of $f \in \mathcal{A}$ satisfying

$$\Re\left(\frac{z^2 f''(z) + zf'(z)}{f(z)}\right) > -\frac{1}{2}.$$

Functions in this subclass are known to be starlike [6]. The sections of $f \in \mathcal{A}$ are defined as

$$s_n(z) = z + \sum_{k=1}^{n} a_n z^n.$$

An interesting problem in univalent function theory is to estimate the radius of the largest disk inside the unit disk within which the sections of functions in a subclass satisfy certain geometric properties like univalence, starlikeness, convexity etc. Szego in [7] had proved that the partial sums of univalent functions $f \in \mathcal{S}$ are univalent in the disk $|z| < \frac{1}{4}$. Following this, several authors have obtained the radius of convexity of sections of functions in several subclasses [1, 4, 5]. In this paper, we prove that the sections of functions in the class $\mathcal{H}$ are convex in the disk $|z| < \frac{1}{4}$.

Lemma 1.1. *Let* $f = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{H}$. *Then*

(i) $|a_n| \leq \dfrac{\prod_{j=1}^{n-1} \frac{j^2 + 2}{j + 2}}{(n-1)!}$ *and equality holds for* $f(z) = z + \sum_{n=2}^{\infty} \dfrac{\prod_{j=1}^{n-1} \frac{j^2 + 2}{j + 2}}{(n-1)!} z^n$.

(ii) $\left|\frac{zf'(z)}{f(z)}\right| \leq \frac{\sum_{n=2}^{\infty}\frac{n}{(n-2)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}r^n}{r-\sum_{n=2}^{\infty}\frac{n}{(n-2)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}r^n}$ *for* $|z| = r < 1$. *The inequality is sharp.*

(iii) $1-\sum_{n=2}^{\infty}\frac{n}{(n-1)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}r^{n-1} \leq |f'(z)| \leq 1+\sum_{n=2}^{\infty}\frac{n}{(n-1)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}r^{n-1}$ *for* $|z| = r < 1$. *The inequality is sharp.*

(iv) *If* $f(z) = s_n(z) + \sigma_n(z)$, *with* $\sigma_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$, *then for* $|z| = r < 1$, *we have*

$$|\sigma_n(z)| \leq \sum_{k=n+1}^{\infty}\frac{k}{(k-1)!}\prod_{j=1}^{k-1}\frac{j^2+2}{j+2}r^{k-1}.$$

and

$$|z\sigma_n''(z)| \leq \sum_{k=n+1}^{\infty}\frac{k}{(k-2)!}\prod_{j=1}^{k-1}\frac{j^2+2}{j+2}r^{k-1}.$$

2 Main Theorem

Theorem 2.1. *Every section of* $f \in \mathcal{H}$ *is convex in the disk* $|z| < \frac{1}{4}$. *The radius* $\frac{1}{4}$ *cannot be replaced by a greater number.*

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be an arbitrary function in the class $\mathcal{H}$. Then $s_2(z) = z + a_2 z^2$ and

$$\Re\left(1 + \frac{zs''_2(z)}{s'_2(z)}\right) \geq 1 - \frac{2|a_2||z|}{1-2|a_2||z|}.$$

Since for $f \in \mathcal{H}$, $|a_2| \leq 1$ and hence

$$\Re\left(1+\frac{zs''_2(z)}{s'_2(z)}\right)\geq 1-\frac{2|z|}{1-2|z|}.$$

which is positive if $|z|<\frac{1}{4}$. Thus $s_2(z)$ is convex in the disk $|z|<\frac{1}{4}$. To prove that this is the best possible radius, consider the function

$$f_0(z)=z+\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}}{(n-1)!}z^n.$$

and the corresponding second partial sum $s_2(f_0)(z)=z+z^2$. At $z=-\frac{1}{4}$,

$$\Re\left(1+\frac{zs''_{2,f_0}(z)}{s'_{2,f_0}(z)}\right)=\Re\left(\frac{1+4z}{1+2z}\right)=0.$$

We now show that the third partial sum $s_3(z)=z+a_2z^2+a_3z^3$ of $f\in\mathcal{H}$ is convex in the disk $|z|<\frac{1}{4}$.

If $f\in\mathcal{H}$ then there is a positive function $p(z)=1+p_1z+p_2z^2+\cdots$ such that

$$\frac{1}{2}+\frac{zf'(z)+z^2f''(z)}{f(z)}=\frac{3}{2}p(z).$$

Substituting the Taylor's series for f and p and equating the coefficients gives

$$p_1=2a_2,\ p_2=\frac{14}{3}a_3-a_2^2.$$

Using the fact that $|p_1|\leq 2$ and $|p_2|\leq 2$ we obtain

$$a_2=\alpha,\ a_3=\frac{1}{7}(\beta+3\alpha^2)$$

where $|a|\leq 1$ and $|\beta|\leq 1$.

Now we show that

$$\Re\left(1 + \frac{zs''_3(z)}{s'_2(z)}\right) = \Re\left(\frac{1 + 4a_2z + 9a_3z^2}{1 + 2a_2z + 3a_3z^2}\right) > 0$$

in $|z| < \frac{1}{4}$.

Since $\Re\left(\frac{1 + 4a_2z + 9a_3z^2}{1 + 2a_2z + 3a_3z^2}\right)$ is harmonic in $|z| < \frac{1}{4}$, it is enough to check that

$$\Re\left(\frac{1 + 4a_2z + 9a_3z^2}{1 + 2a_2z + 3a_3z^2}\right) > 0 \tag{3}$$

for $|z| = \frac{1}{4}$.

Note that

$$\Re\left(\frac{1 + 4a_2z + 9a_3z^2}{1 + 2a_2z + 3a_3z^2}\right) = 2 - \Re\left(\frac{1 - 3a_3z^2}{1 + 2a_2z + 3a_3z^2}\right) \geq 2 - \left|\frac{1 - 3a_3z^2}{1 + 2a_2z + 3a_3z^2}\right|.$$

By considering a suitable rotation of f it is enough to verify (3) for $z = \frac{1}{4}$. In other words we must prove that

$$2 > \left|\frac{16 - 3a_3}{16 + 8a_2 + 3a_3}\right|.$$

Substituting the values of a_2 and a_3 from (2) it is enough to verify that

$$2\left|16 + 8\alpha + \frac{3}{7}\beta + \frac{9}{7}\alpha^2\right| > \left|16 - \frac{3}{7}\beta - \frac{9}{7}\alpha^2\right|.$$

for $|\alpha| = 1$ and $\beta = 1$. Using triangle inequality the above inequality follows if we prove

$$2\left|16 + 8\alpha + \frac{9}{7}\alpha^2\right| - \frac{9}{7} > \left|16 - \frac{9}{7}\alpha^2\right|.$$

Since $|\alpha| = 1$, the above inequality is equivalent to proving

$$2|112\overline{\alpha} + 56 + 9\alpha| - |112\overline{\alpha} - 9\alpha| > 9.$$

Putting $\Re(\alpha) = x$ it remains to show that

$$2\sqrt{4032x^2 + 13552x + 13745} - \sqrt{14641 - 4032x^2} > 9.$$

for $-1 \leq x \leq 1$.

Let $$G(x) = 2\sqrt{4032x^2 + 13552x + 13745} - \sqrt{14641 - 4032x^2},$$

$G_1(x) = 4032x^2 + 13552x + 13745$ and $G_2(x) = 14641 - 4032x^2$. Then

$$G(x) = \sqrt{G_1(x)} - \sqrt{G_2(x)}.$$

$G_1(x)$ is positive and increasing in $[-1, 1]$ with

$$G_1(x) \geq G_1(-1) = 4225.$$

Similarly $G_2(x)$ is also positive and attains its maximum at $x = 0$ in $[-1, 1]$. Hence $G_2(x) \leq G_2(0) = 14641$. Therefore for $x \in [-1, 1]$

$$G(x) = 2\sqrt{G_1(x)} - \sqrt{G_2(x)} > 2\sqrt{G_1(-1)} - \sqrt{G_2(0)} = 9.$$

Thus we have proved that $\Re\left(1 + \frac{zs''_3(z)}{s'_3(z)}\right) > 0$ in $|z| < \frac{1}{4}$.

We now prove that $\Re\left(1 + \frac{zs''_n(z)}{s'_n(z)}\right) > 0$ for $n \geq 4$ in $|z| < \frac{1}{4}$.

It is enough to show that $\Re\left(1 + \frac{zs''_n(z)}{s'_n(z)}\right) > 0$ hold for all $n \geq 4$ for $r = \frac{1}{4}$ thereby the conclusion follows by maximum principle.

Any $f(z) = z + \sum_{n=2}^{\infty} z^n$ can be written as

$$f(z) = S_n(z) + \sigma_n(z)$$

with $\sigma_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$. Then

$$\Re\left(1 + \frac{s''_n(z)}{s'_n(z)}\right) = \Re\left(1 + \frac{z[f''(z) - \sigma''_n(z)]}{[f'(z) - \sigma'_n(z)]}\right)$$

$$= \Re\left(1 + \frac{zf''(z)}{f'(z)} + \frac{\frac{zf''(z)}{f'(z)}\sigma_n'(z) - z\sigma''_n(z)}{f'(z) - \sigma_n'(z)}\right)$$

$$\geq 1 - \left|\frac{zf''(z)}{f'(z)}\right| - \frac{\left|\frac{zf''(z)}{f'(z)}\right| |\sigma_n'(z)| - |z\sigma''_n(z)|}{|f'(z)| - |\sigma_n'(z)|}$$

$$\geq 1 - \frac{\sum_{n=2}^{\infty} \frac{n}{(n-2)!} \prod_{j=1}^{n-1} \frac{j^2+2}{j+2} r^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{n}{(n-1)!} \prod_{j=1}^{n-1} \frac{j^2+2}{j+2} r^{n-1}}$$

$$\frac{\sum_{n=2}^{\infty} \frac{n}{(n-2)!} \prod_{j=1}^{n-1} \frac{j^2+2}{j+2} r^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{n}{(n-2)!} \prod_{j=1}^{n-1} \frac{j^2+2}{j+2} r^{n-1}}$$

$$- \frac{+\sum_{n=2}^{\infty} \frac{k}{(k-2)!} \prod_{j=1}^{k-1} \frac{j^2+2}{j+2} r^{k-1}}{1 - \sum_{k=2}^{n} \frac{k}{(k-2)!} \prod_{j=1}^{k-1} \frac{j^2+2}{j+2} r^{k-1}}$$

$$= \frac{1 - \sum_{n=2}^{\infty} \frac{n^2}{(n-2)!} \prod_{j=1}^{n-1} \frac{j^2+2}{j+2} r^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{n}{(n-2)!} \prod_{j=1}^{n-1} \frac{j^2+2}{j+2} r^{n-1}}$$

$$\frac{\sum_{n=2}^{\infty} \frac{n}{(n-2)!} \prod_{j=1}^{n-1} \frac{j^2+2}{j+2} r^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{n}{(n-2)!} \prod_{j=1}^{n-1} \frac{j^2+2}{j+2} r^{n-1}}$$

$$- \frac{+\sum_{n=2}^{\infty} \frac{k}{(k-2)!} \prod_{j=1}^{k-1} \frac{j^2+2}{j+2} r^{k-1}}{1 - \sum_{k=2}^{n} \frac{n}{(k-2)!} \prod_{j=1}^{k-1} \frac{j^2+2}{j+2} r^{k-1}}.$$

It is enough to show that

$$\frac{1-\sum_{n=2}^{\infty}\frac{n^2}{(n-2)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}r^{n-1}}{1-\sum_{n=2}^{\infty}\frac{n}{(n-2)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}r^{n-1}}$$

$$-\frac{\dfrac{\sum_{n=2}^{\infty}\frac{n}{(n-2)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}r^{n-1}}{1-\sum_{n=2}^{\infty}\frac{n}{(n-1)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}r^{n-1}}+\sum_{k=n+1}^{\infty}\frac{k}{(k-2)!}\prod_{j=1}^{k-1}\frac{j^2+2}{j+2}r^{k-1}}{1-\sum_{k=2}^{n}\frac{k}{(k-2)!}\prod_{j=1}^{k-1}\frac{j^2+2}{j+2}r^{k-1}}>0.$$

We prove that (4) holds for all $n \geq 4$ for $r = \frac{1}{4}$ which is same as proving

$$\frac{1-\sum_{n=2}^{\infty}\frac{n^2}{(n-2)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}\left(\frac{1}{4}\right)^{n-1}}{1-\sum_{n=2}^{\infty}\frac{n}{(n-2)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}\left(\frac{1}{4}\right)^{n-1}}$$

$$-\frac{\dfrac{\sum_{n=2}^{\infty}\frac{n}{(n-2)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}\left(\frac{1}{4}\right)^{n-1}}{1-\sum_{n=2}^{\infty}\frac{n}{(n-1)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}\left(\frac{1}{4}\right)^{n-1}}+\sum_{k=n+1}^{\infty}\frac{k}{(k-2)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}\left(\frac{1}{4}\right)^{n-1}}{1-\sum_{k=2}^{n}\frac{n}{(k-2)!}\prod_{j=1}^{k-1}\frac{j^2+2}{j+2}\left(\frac{1}{4}\right)^{k-1}}>0 \tag{4}$$

for $n \geq 4$. Let

$$G(n)=\frac{1-\sum_{n=2}^{\infty}\frac{n^2}{(n-2)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}\left(\frac{1}{4}\right)^{n-1}}{1-\sum_{n=2}^{\infty}\frac{n}{(n-1)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}\left(\frac{1}{4}\right)^{n-1}}$$

$$\frac{\sum_{n=2}^{\infty}\frac{n}{(n-2)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}\left(\frac{1}{4}\right)^{n-1}}{1-\sum_{n=2}^{\infty}\frac{n}{(n-1)!}\prod_{j=1}^{n-1}\frac{j^2+2}{j+2}\left(\frac{1}{4}\right)^{n-1}}+\sum_{k=n+1}^{\infty}\frac{k}{(k-2)!}$$

$$-\frac{\prod_{j=1}^{k-1}\frac{j^2+2}{j+2}\left(\frac{1}{4}\right)^{k-1}}{1-\sum_{k=2}^{n}\frac{n}{(k-2)!}\prod_{j=1}^{k-1}\frac{j^2+2}{j+2}\left(\frac{1}{4}\right)^{k-1}}.$$

Then we must prove that $G(n) > 0$ for $n \geq 4$. For $n = 4, 5, 6$ we have $G(4) \approx 0.23704$, $G(5) \approx 0.24598$, $G(6) \approx 0.24701$. Thus $G(n)$ is increasing and non-negative for all $n \geq 4$.

Hence for $f \in \mathcal{H}$, $s_n(z)$ is convex for all $n \geq 4$.

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