Advances and Applications in Mathematical Sciences
Volume 21, Issue 3, January 2022, Pages 1601-1614

DIRICHLET-SCHWARZ MIXED BOUNDARY VALUE PROBLEM FOR POLYANALTIC FUNCTIONS ON UPPER HALF PLANE

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Abstract

Combination of n-Dirichlet and m-Schwarz is studied and an explicit representation of solution of inhomogeneous polyanalytic equation of order $(m+n)$ is given on the upper half plane $\mathbb{H}$.

1. Introduction

In this article, a combination of n-Dirichlet and m-Schwarz is studied and an explicit representation of solution of inhomogeneous polyanalytic equation of order $(m+n)$ is given on the upper half plane. Earlier 1-Dirichlet and n-Schwarz and reverse combinations were studied [2] but here a generalised result for every order is given. Dirichlet and Schwarz BVP's along with other similar BVP are studied independently on different domains like Upper Half Plane [1], Quarter Plane [5, 12, 13, 14, 15] and Unit Disc [5] etc. These type of boundary conditions are also studied on different- different domains and solved via different techniques [4, 5, 9, 10, 11]. The area integral written in Cauchy-Pompeiu formula is known as Pompeiu operator, was studied by Vekua see [7]. For a regular domain D, if $f \in L_p(D, \mathbb{C})$, $p > 1$, (where, $L_p(D, \mathbb{C})$ is the space of all equivalence classes of Lebesgue measurable

2020 Mathematics Subject Classification: 32A30, 30G20, 31A30.
Keywords: Schwarz, Dirichlet, Cauchy-Pompeiu, Gauss theorem, Boundary value problems.
Received November 1, 2021; Accepted December 20, 2021

functions f on D for which $|f|^p$ is integrable) then the Pompeiu operator Tf possesses weak derivatives and

$$\frac{\partial}{\partial \bar{z}}(Tf) = f, \; \frac{\partial}{\partial \bar{z}}(Tf) = \prod f$$

where $\prod f$ represents singular integral in the principal value sense. In case of upper half plane if $w : \mathbb{H} \to \mathbb{C}$ satisfies $w(x) \leq C|x|^{-\epsilon}$ for $|x| > K$, $\epsilon > 0$ and $w_{\bar{z}} \in L_1(\mathbb{H}, \mathbb{C})$, then the Cauchy-Pompeiu formula [3] is given by

$$w(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} w(t)\frac{dt}{t - z} - \frac{1}{\pi}\int_{0<\operatorname{Im}\zeta} w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

$$w(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} w(t)\frac{dt}{t - \bar{z}} - \frac{1}{\pi}\int_{0<\operatorname{Im}\zeta} w_{\zeta}(\zeta)\frac{\overline{d\xi d\eta}}{\zeta - z}$$

where $z \in \mathbb{H}$. In case of upper half plane $\mathbb{H}$, the Pompeiu operator T has the following form:

$$Tf(z) = -\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

and T satisfies the properties $\frac{\partial}{\partial \bar{z}}(Tf) = f, \; \frac{\partial}{\partial \bar{z}}(Tf) = \prod f$ where

$$\prod f(z) = -\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}$$

here derivatives are taken in distributional sense. We observe that for $z = x + iy \in \mathbb{H}$, $\gamma \in L^p(\mathbb{R}, \mathbb{C})$, $p \geq 1$

$$\lim_{z \to t_0} \frac{1}{\pi}\int_{-\infty}^{\infty} \gamma_0(t)\frac{y dt}{|t - z|^2} = \gamma_0(t_0).$$

For regular domains higher order of Pompeiu operators were studied in [6] and for upper half plane in [1].

2. Dirichlet-Schwarz Mixed Boundary Value Problem for Polyanaltic Functions

Theorem 1. *For* $m, n \geq 1$, *the mixed n-Dirichlet and m-Schwarz problem for the inhomogeneous polyanalytic equation in half plane*

$$\partial_{\bar{z}}^{m+n}(w) = f \text{ in } \mathbb{H} \tag{2.1}$$

$$\partial_{\bar{z}}^{i} w = \gamma_i \text{ on } \mathbb{R},\ 0 \leq i \leq n-1, \tag{2.2}$$

$$\operatorname{Re}(\partial_{\bar{z}}^{n+j} w) = \beta_j \text{ on } \mathbb{R} \tag{2.3}$$

$$\operatorname{Im}(\partial_{\bar{z}}^{n+j} w(i)) = C_j,\ 0 \leq j \leq m-1 \tag{2.4}$$

is uniquely solvable for $f \in L_{p,2}(\mathbb{H}, \mathbb{C})$, $p > 2$ *satisfying regularity conditions above and* $t^i \gamma_i(t) \in L^p(\mathbb{R}, \mathbb{C}) \cap C(\mathbb{R}, \mathbb{C})$, $0 \leq i \leq n-1$, $t^j \beta_j(t) \in L^p(\mathbb{R}, \mathbb{C}) \cap C(\mathbb{R}, \mathbb{C})$, $0 \leq j \leq m-1$ *and may be expressed as*

$$w(z) = i \sum_{\delta}^{n-1} \frac{c\delta}{\delta!} (z + \bar{z})^{\delta}$$

$$+ \sum_{\delta=0}^{n-1} \frac{1}{\pi i} \frac{(-1)^{\delta}}{\delta!} \int_{-\infty}^{\infty} \beta_{\delta}(s) \left(\frac{1}{(s-z)} + \frac{s}{(s^2+1)} \right) (2s - z - \bar{z})^{s} ds$$

$$+ \frac{(-1)^n}{(n-1)!} \sum_{\lambda=0}^{m-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{\lambda!} \int_{-\infty}^{+\infty} \left\{ \gamma_{\lambda}(t) \sum_{r=0}^{\lambda} \sum_{(a,b) \in T(n-1)} C_r^{\lambda} t^r N(a, b, n-1; z) (-1)^{n-1+\lambda-r} \right.$$

$$\left. \left(\frac{z^{a+b+\lambda-r+1} - z^a (\bar{z})^{\lambda+b-r+1}}{(\lambda+b-r+1)(t-z)} + \frac{1}{(t+i)(a+i)} \right) \right\} dt$$

$$+ \frac{(-1)^{n+m}}{(m-1)!\,(n-1)!} \frac{1}{\pi} \int_H f(\zeta) \sum_{r=0}^{m-1} \sum_{(a,b) \in T(n-1)} (-1)^{n+m-r-2} C_r^{m-1} (\bar{\zeta})^r \frac{N(a, b, n-1; z)}{m+b-r}$$

$$\left\{ (\zeta^{a+b+m-r} - \zeta^a (\bar{\zeta})^{m+b-r}) \left(\frac{\zeta}{\zeta^2+1} - \frac{1}{\zeta - z} \right) \left(\frac{z^{a+b+m-r} - z^a (\bar{z})^{m+b-r}}{\zeta - z} \right) \right\} \Bigg] d\xi d\eta$$

$$- \frac{(-1)^{n+m}}{(m-1)!\,(n-1)!} \frac{1}{\pi} \int_H \overline{f(\zeta)} \sum_{r=0}^{m-1} \sum_{(a,\, b) \in T(n-1)} (-1)^{n+m-r-2} (\zeta)^r \, \frac{N(a,\, b,\, n-1;\, z)}{m+a-r}$$

$$\left\{ \left(\zeta^{a+b+m-r} + \zeta^{a+m-r} (\overline{\zeta})^b \right) \left(\frac{1}{\overline{\zeta} - z} - \frac{\overline{\zeta}}{\overline{\zeta}^2 + 1} \right) + \frac{1}{\overline{\zeta} + i} \right\} \Bigg] d\xi d\eta \tag{2.5}$$

where

$$N(i,\, j,\, r,\, \overline{\zeta})\, \zeta^i \overline{\zeta}^j = \frac{r!\,(-1)^{i+j} (\widetilde{\zeta} + \overline{\overline{\zeta}})^{r-i-j}}{i!\, j!\, (r-i-j)!},$$

$$T(r) = \{(i,\, j) \in \mathbb{N}_0 \times \mathbb{N}_0 : i + j \leq r\}. \tag{2.6}$$

and

$$N(i,\, j,\, r,\, \overline{\zeta})\, \zeta^a \overline{\zeta}^b = \frac{\delta!\,(-1)^{a+b} (2s)^{r-a-b}}{a!\, b!\, (\delta - a - b)!},$$

$$T(\delta) = \{(a,\, b) \in \mathbb{N}_0 \times \mathbb{N}_0 : a + b \leq \delta\}. \tag{2.7}$$

if and only if

$$\sum_{\lambda=v}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda-v}}{(\lambda - v)!} \int_{-\infty}^{\infty} \gamma_\lambda(t) (t-z)^{\lambda-v} \frac{dt}{(t - \overline{z})}$$

$$+ \frac{(-1)^{n-v+m+1}}{(n-1-v)!\,(m-1)!} \frac{1}{\pi} \int_{\mathbb{H}} \sum_{(a,\, b) \in T(m-1)} N(a,\, b,\, m-1;\, \zeta)$$

$$\sum_{p=0}^{n-1-v} \frac{(-z)^{n-1-v-p}}{b+p+1} \left(\zeta^{a+b+p+1} - \zeta^a (\overline{\zeta})^{b+p+1} \right) f(\zeta) \frac{d\xi d\eta}{\zeta - \overline{z}} = 0 \tag{2.8}$$

Proof. We will break the equation into two equation first of order-n and second of order m and use substitution method to write the solution of given problem.

$$\partial_{\overline{z}}^{m+n} = \partial_{\overline{z}}^m (\partial_{\overline{z}}^n w) = f$$

Let

$$\partial_{\overline{z}}^n w = W \tag{2.9}$$

$$\partial_{\bar{z}}^{m} W = f \tag{2.10}$$

Solution of equation (2.9) is given by

$$w(z) = \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{\lambda!} \int_{-\infty}^{\infty} \gamma_{\lambda}(t)(t - \bar{z})^{\lambda} \frac{dt}{t - z}$$

$$+ \frac{(-1)^{n}}{(n-1)!} \frac{1}{\pi} \int_{\mathbb{H}} W(\zeta) \overline{(\zeta - z)}^{n-1} \frac{d\xi d\eta}{(\zeta - z)} \tag{2.11}$$

If and only if

$$\sum_{\lambda=v}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda - v}}{(\lambda - v)!} \int_{-\infty}^{\infty} \gamma_{\lambda}(t)(t - z)^{\lambda - v} \frac{dt}{t - z}$$

$$+ \frac{(-1)^{n-v}}{(n-1-v)!} \frac{1}{\pi} \int_{\mathbb{H}} W(\zeta)(\bar{\zeta} - z)^{n-1-v} \frac{d\xi d\eta}{(\zeta - \bar{z})} = 0.$$

Solution of equation (2.10) is given by

$$W(z) = i \sum_{\delta=0}^{m-1} \frac{c_{\delta}}{\delta!} (z + \bar{z})^{\delta}$$

$$+ \sum_{\delta=0}^{m-1} \frac{(-1)^{\delta}}{\pi i \delta!} \int_{-\infty}^{\infty} \beta_{\delta}(s) \left(\frac{1}{s - z} - \frac{s}{s^2 + 1} \right) (2s - z - \bar{z})^{\delta} ds$$

$$+ \frac{(-1)^{n}}{(m-1)!} \frac{1}{\pi} \int_{\mathbb{H}} \left(f(\zeta) \left(\frac{1}{\zeta - z} - \frac{\zeta}{\zeta^2 + 1} \right) - \overline{f(\zeta)} \left(\frac{1}{\bar{\zeta} - z} - \frac{\bar{\zeta}}{\overline{\zeta^2 + 1}} \right) \right)$$

$$(\zeta - z + (\overline{\zeta - z}))^{m-1} d\xi d\eta \tag{2.12}$$

Re-writing the above equation, we have

$$W(\zeta) = i \sum_{\delta=0}^{m-1} \frac{c_{\delta}}{\delta!} (\zeta + \bar{\zeta})^{\delta}$$

$$+ \sum_{\delta=0}^{m-1} \frac{(-1)^{\delta}}{\pi i \delta!} \int_{-\infty}^{\infty} \beta_{\delta}(s) \left(\frac{1}{s - \zeta} - \frac{s}{s^2 + 1} \right) (2s - \zeta - \bar{\zeta})^{\delta} ds$$

$$+\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\left(f(\tilde{\zeta})\left(\frac{1}{\tilde{\zeta}-\zeta}-\frac{\tilde{\zeta}}{\tilde{\zeta}^2+1}\right)-\overline{f(\tilde{\zeta})}\left(\frac{1}{\overline{\tilde{\zeta}}-z}-\frac{\overline{\tilde{\zeta}}}{\overline{\tilde{\zeta}}^2+1}\right)\right)$$

$$(\tilde{\zeta}-\zeta+\overline{(\tilde{\zeta}-\zeta))}^{m-1}d\tilde{\xi}d\tilde{\eta} \tag{2.13}$$

Now substituting the value from (2.13) in area integral of equation (2.11), we have

$$\frac{1}{\pi}\int_{\mathbb{H}}W(\zeta)(\overline{\zeta-z})^{n-1}\frac{d\xi d\eta}{(\zeta-z)}=\frac{1}{\pi}\int_{\mathbb{H}}\left[\left[i\sum_{\delta=0}^{m-1}\frac{c_\delta}{\delta!}(\zeta+\overline{\zeta})^{\delta}\right.\right.$$

$$\left.+\sum_{\delta=0}^{m-1}\frac{(-1)^{\delta}}{\pi i\delta!}\int_{-\infty}^{\infty}\beta_\delta(s)\left(\frac{1}{s-\zeta}-\frac{s}{s^2+1}\right)(2s-\zeta-\overline{\zeta})^{\delta}ds\right]$$

$$+\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{H}\left(f(\tilde{\zeta})\left(\frac{1}{\tilde{\zeta}-\zeta}-\frac{\tilde{\zeta}}{\tilde{\zeta}^2+1}\right)-\overline{f(\tilde{\zeta})}\left(\frac{1}{\overline{\tilde{\zeta}}-z}-\frac{\overline{\tilde{\zeta}}}{\overline{\tilde{\zeta}}^2+1}\right)\right)$$

$$\left.(\tilde{\zeta}-\zeta+\overline{(\tilde{\zeta}-\zeta))}^{m-1}d\tilde{\xi}d\tilde{\eta}\right]\frac{(\overline{\zeta-z})}{(\zeta-z)}d\xi d\eta \tag{2.11}$$

$$=i\sum_{\delta=0}^{m-1}\frac{c_\delta}{\delta!}\frac{1}{\pi}\int_{\mathbb{H}}(\zeta+\overline{\zeta})^{\delta}(\overline{\zeta}-\overline{z})^{n-1}\frac{d\xi d\eta}{(\xi-z)}$$

$$+\frac{1}{\pi}\left(\int_{\mathbb{H}}\left(\sum_{\delta=0}^{m-1}\frac{(-1)^{\delta}}{\pi i\delta!}\right)\int_{-\infty}^{\infty}\beta_\delta(s)\left(\frac{1}{s-\zeta}-\frac{s}{s^2-\zeta}\right)(2s-\zeta-\overline{\zeta})^{\delta}ds\right)\frac{(\overline{\zeta}-\overline{z})}{(\zeta-z)}d\xi d\eta$$

$$+\frac{1}{\pi}\int_{\mathbb{H}}\left(\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\left(\frac{f(\tilde{\zeta})}{\tilde{\zeta}-\zeta}-\frac{\tilde{\zeta}f(\tilde{\zeta})}{\tilde{\zeta}^2+1}\right)(\tilde{\zeta}-\zeta+\overline{\tilde{\zeta}-\zeta})^{m-1}d\tilde{\xi}d\tilde{\eta}\right)\frac{(\overline{\zeta-z})^{n-1}}{(\zeta-z)}d\xi d\eta$$

$$+\frac{1}{\pi}\int_{\mathbb{H}}\left(\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\left(\frac{f(\tilde{\zeta})}{\tilde{\zeta}-\zeta}-\frac{\overline{\tilde{\zeta}f(\tilde{\zeta})}}{\overline{\tilde{\zeta}}^2+1}\right)(\tilde{\zeta}-\zeta+\overline{\tilde{\zeta}-\zeta})^{m-1}d\tilde{\xi}d\tilde{\eta}\right)\frac{(\overline{\zeta}-\overline{z})^{n-1}}{(\zeta-z)}d\xi d\eta$$

$$=i\sum_{\delta=0}^{m-1}\sum_{r=0}^{\delta}\sum_{p=0}^{n-1}\frac{c_\delta}{\delta!}C_r^{\delta}C_p^{n-1}\overline{z}^{n-p-1}(-1)^{n-p-1}\frac{1}{\pi}\int_{\mathbb{H}}\zeta^{r}\overline{\zeta}^{-\delta-r+p}\frac{d\xi d\eta}{(\zeta-z)}$$

$$+\sum_{\delta=0}^{m-1}\frac{(-1)^{\delta}}{\pi i\delta!}\int_{-\infty}^{\infty}\beta_{\delta}(s)\left[\frac{1}{\pi}\int_{\mathbb{H}}\left(\frac{(2s-\zeta-\overline{\zeta})^{\delta}}{s-\zeta}-\frac{s}{s^2+1}(2s-\zeta-\overline{\zeta})^{\delta}\right)\frac{(\overline{\zeta}-\overline{z})^{n-1}}{(\zeta-z)}d\xi d\eta\right]ds$$

$$+\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}f(\widetilde{\zeta})\left(\frac{1}{\pi}\int_{\mathbb{H}}\frac{(\widetilde{\zeta}-\zeta+\overline{\widetilde{\zeta}-\zeta})^{m-1}(\overline{\zeta-z})^{n-1}}{(\widetilde{\zeta}-\zeta)(\zeta-z)}d\xi d\eta\right)d\widetilde{\xi}d\widetilde{\eta}$$

$$-\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{\widetilde{\zeta}f(\widetilde{\zeta})}{\widetilde{\zeta}^2+1}\left(\frac{1}{\pi}\int_{\mathbb{H}}\frac{(\widetilde{\zeta}-\zeta+\overline{\widetilde{\zeta}-\zeta})^{m-1}(\overline{\zeta-z})^{n-1}}{(\zeta-z)}d\xi d\eta\right)d\widetilde{\xi}d\widetilde{\eta}$$

$$-\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\overline{f(\widetilde{\zeta})}\left(\frac{1}{\pi}\int_{\mathbb{H}}\frac{(\widetilde{\zeta}-\zeta+\overline{\widetilde{\zeta}-\zeta})^{m-1}(\overline{\zeta-z})^{n-1}}{(\widetilde{\zeta}-\zeta)(\zeta-z)}d\xi d\eta\right)d\widetilde{\xi}d\widetilde{\eta}$$

$$+\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{\overline{\widetilde{\zeta}f(\widetilde{\zeta})}}{\overline{\widetilde{\zeta}}^2+1}\left(\frac{1}{\pi}\int_{\mathbb{H}}\frac{(\widetilde{\zeta}-\zeta+\overline{\widetilde{\zeta}-\zeta})^{m-1}(\overline{\zeta-z})^{n-1}}{(\zeta-z)}d\xi d\eta\right)d\widetilde{\xi}d\widetilde{\eta}$$

$$=i\sum_{\delta=0}^{m-1}\sum_{r=0}^{\delta}\sum_{p=0}^{n-1}\frac{c_{\delta}}{\delta!}C_r^{\delta}C_p^{n-1}\overline{z}^{n-p-1}(-1)^{n-p-1}(A)$$

$$+\sum_{\delta=0}^{m-1}\frac{(-1)^{\delta}}{\pi i\delta!}\int_{-\infty}^{\infty}\beta_{\delta}(s)(B)\,ds+\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}f(\widetilde{\zeta})(E)\,d\widetilde{\xi}d\widetilde{\eta}$$

$$-\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{\widetilde{\zeta}f(\widetilde{\zeta})}{\widetilde{\zeta}^2+1}(F)d\widetilde{\xi}d\widetilde{\eta}-\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\overline{f(\widetilde{\zeta})}(G)\,d\widetilde{\xi}d\widetilde{\eta}$$

$$+\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{\overline{\widetilde{\zeta}f(\widetilde{\zeta})}}{\overline{\widetilde{\zeta}}^2+1}(H)\,d\widetilde{\xi}d\widetilde{\eta} \tag{2.14}$$

Solving the above substitutions one by one we have the following expressions:

$$A=\frac{1}{\pi}\int_{\mathbb{H}}\zeta^r\overline{\zeta}^{\delta-r+p}=\frac{1}{\pi}\int_{\mathbb{H}}\frac{\partial}{\partial\overline{\zeta}}\left(\frac{\zeta^r\overline{\zeta}^{\delta-r+p+1}}{\delta-r+p+1}\right)\frac{d\xi d\eta}{\zeta-z}$$

$$B=\frac{1}{\pi}\int_{\mathbb{H}}\left(\frac{(2s-\zeta-\overline{\zeta})^{\delta}}{s-\zeta}\frac{(\overline{\zeta}-\overline{z})^{n-1}}{\zeta-z}\right)d\xi d\eta$$

$$- \frac{s}{s^2+1}\frac{1}{\pi}\int_{\mathbb{H}} \frac{(2s-\zeta-\overline{\zeta})^{\delta}(\overline{\zeta}-\overline{z})^{n-1}}{\zeta - z}\, d\xi d\eta$$

$$= \frac{1}{s-\zeta}\left[\frac{1}{\pi}\int_{\mathbb{H}} (2s-\zeta-\overline{\zeta})^{\delta}(\overline{\zeta}-\overline{z})^{n-1}\frac{d\xi d\eta}{\zeta - z}\right.$$

$$\left. - \frac{1}{\pi}\int_{H} (2s-\zeta-\overline{\zeta})^{\delta}(\overline{\zeta}-\overline{z})^{n-1}\frac{d\xi d\eta}{\zeta - z}\right]$$

$$- \frac{s}{s^2+1}\frac{1}{\pi}\int_{\mathbb{H}} (2s-\zeta-\overline{\zeta})^{\delta}(\overline{\zeta}-\overline{z})^{n-1}\frac{d\xi d\eta}{\zeta - z} \tag{2.15}$$

Using Cauchy Integral and Gauss theorem over $\mathbb{H}$, we may write the following values while evaluating integrals from domain towards its boundary.

$$(\tilde{\zeta} + \overline{\tilde{\zeta}} - \zeta - \overline{\zeta})^r = \sum_{(i,\, j)\in T(r)} N(i,\, j,\, r,\, \tilde{\zeta})\zeta^i\overline{\zeta}^j \tag{2.16}$$

where

$$N(i,\, j,\, r,\, \overline{\zeta})\zeta^i\overline{\zeta}^j = \frac{r!\,(-1)^{i+j}(\tilde{\zeta} + \overline{\tilde{\zeta}})^{r-i-j}}{i!\, j!\,(r-i-j)!},$$

$$T(r) = \{(i,\, j) \in \mathbb{N}_0 \times \mathbb{N}_0 : i + j \le r\}. \tag{2.17}$$

and

$$(2s - \zeta - \overline{\zeta})^{\delta} = \sum_{(a,\, b)\in T(\delta)} N(a,\, b,\, \delta,\, s)\zeta^a\overline{\zeta}^b \tag{2.18}$$

where

$$N(a,\, b,\, \delta,\, s)\zeta^a\overline{\zeta}^b = \frac{\delta!\,(-1)^{a+b}(2s)^{r-a-b}}{a!\, b!\,(\delta-a-b)!},$$

$$T(\delta) = \{(a,\, b) \in \mathbb{N}_0 \times \mathbb{N}_0 : a + b \le \delta\}. \tag{2.19}$$

Using Cauchy integral and Gauss theorem over $\mathbb{H}$, we may write:

$$(\overline{\zeta}-\overline{z})^{n-1}=\sum_{p=0}^{n-1}({}^{n-1}C_p)\overline{\zeta}^p(-\overline{\zeta})^{n-1-p} \tag{2.20}$$

Using above results we can write B as:

$$B=\sum_{(a,\,b)\in T(\delta)}N(a,\,b,\,\delta,\,s)\sum_{p=0}^{n-1}({}^{n-1}C_p)(-\overline{z})^{n-1-p}\,\frac{1}{s-z}$$

$$\left[\frac{1}{\pi}\int_{\mathbb{H}}\zeta^a\overline{\zeta}^{b+p}\,\frac{d\xi d\eta}{(\zeta-z)}-\frac{1}{\pi}\int_{\mathbb{H}}\zeta^a\overline{\zeta}^{b+p}\,\frac{d\xi d\eta}{(\zeta-s)}\right]$$

$$-\sum_{(a,\,b)\in T(\delta)}N(a,\,b,\,\delta,\,s)\sum_{p=0}^{n-1}({}^{n-1}C_p)(-\overline{z})^{n-1-p}\,\frac{s}{s^2+1}\left[\frac{1}{\pi}\int_{\mathbb{H}}\zeta^a\overline{\zeta}^{b+p}\,\frac{d\xi d\eta}{(\zeta-z)}\right]$$

$$=\sum_{(a,\,b)\in T(\delta)}N(a,b,\delta,s)\sum_{p=0}^{n-1}({}^{n-1}C_p)(-\overline{z})^{n-1-p}\,\frac{1}{s-z}\left[\frac{1}{\pi}\int_{\mathbb{H}}\frac{\partial}{\partial\overline{\zeta}}\left(\frac{\zeta^a\overline{\zeta}^{b+p+1}}{b+p+1}\right)\frac{d\xi d\eta}{(\zeta-z)}\right]$$

$$-\sum_{(a,\,b)\in T(\delta)}N(a,b,\delta,s)\sum_{p=0}^{n-1}({}^{n-1}C_p)(-\overline{z})^{n-1-p}\,\frac{s}{s^2+1}\left[\frac{1}{\pi}\int_{\mathbb{H}}\frac{\partial}{\partial\overline{\zeta}}\left(\frac{\zeta^a\overline{\zeta}^{b+p+1}}{b+p+1}\right)\frac{d\xi d\eta}{(\zeta-z)}\right]$$

$$=\sum_{(a,\,b)\in T(\delta)}N(a,\,b,\,\delta,\,s)\sum_{p=0}^{n-1}({}^{n-1}C_p)(-\overline{z})^{n-1-p}\left(\frac{1}{s-z}-\frac{s}{s^2+1}\right)$$

$$\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{s^{a+b+p+1}}{b+p+1}\,\frac{ds}{(s-z)}-\frac{z^a\overline{z}^{b+p+1}}{b+p+1}\right]$$

$$=\sum_{(a,\,b)\in T(\delta)}N(a,\,b,\,\delta,\,s)\sum_{p=0}^{n-1}({}^{n-1}C_p)(-\overline{z})^{n-1-p}\left(\frac{1}{s-z}-\frac{s}{s^2+1}\right)$$

$$\left(\frac{z^a\overline{z}^{b+p+1}}{b+p+1}-\frac{z^a\overline{z}^{b+p+1}}{b+p+1}\right) \tag{2.21}$$

$$E=\frac{1}{\pi}\int_{\mathbb{H}}\frac{(\widetilde{\zeta}-\zeta+\overline{\widetilde{\zeta}}-\overline{\zeta})^{m-1}(\overline{\zeta}-\overline{z})^{n-1}}{(\zeta-\widetilde{\zeta})(\zeta-z)}\,d\xi d\eta$$

$$= \frac{1}{(\tilde{\zeta} - z)}\left[\frac{1}{\pi}\int_{\mathbb{H}} \frac{(\tilde{\zeta} + \overline{\tilde{\zeta}} - \zeta - \bar{\zeta})^{m-1}(\bar{\zeta} - \bar{z})^{n-1}}{(\zeta - z)}\, d\xi d\eta\right.$$

$$\left.- \frac{1}{\pi}\int_{\mathbb{H}} \frac{(\tilde{\zeta} + \overline{\tilde{\zeta}} - \zeta - \bar{\zeta})^{m-1}(\bar{\zeta} - \bar{z})^{n-1}}{(\zeta - \tilde{\zeta})}\, d\xi d\eta\right]$$

$$\frac{1}{\tilde{\zeta} - z}\left[\sum_{(a,\, b)\in T(m-1)} N(a, b, m-1, \tilde{\zeta})\sum_{p=0}^{n-1}({}^{n-1}C_p)(-\bar{z})^{n+p-1}\left\{\frac{1}{\pi}\int_{\mathbb{H}} \frac{\partial}{\partial\bar{\zeta}}\left(\frac{\zeta^a\bar{\zeta}^{b+p+1}}{b+p+1}\right)\right.\right.$$

$$\left.\left.\frac{d\xi d\eta}{(\zeta - z)} - \frac{1}{\pi}\int_{\mathbb{H}} \zeta^a\bar{\zeta}^{b+p}\, \frac{d\xi d\eta}{(\zeta - \tilde{\zeta})}\right\}\right]$$

$$= \frac{1}{\tilde{\zeta} - z}\left[\sum_{(a,\, b)\in T(m-1)} N(a, b, m-1, \tilde{\zeta})\sum_{p=0}^{n-1}({}^{n-1}C_p)(-\bar{z})^{n+p-1}\left\{\frac{1}{\pi}\int_{\mathbb{H}} \frac{\partial}{\partial\bar{\zeta}}\left(\frac{\zeta^a\bar{\zeta}^{b+p+1}}{b+p+1}\right)\right.\right.$$

$$\left.\left.\frac{d\xi d\eta}{(\zeta - z)} - \frac{1}{\pi}\int_{\mathbb{H}} \frac{\partial}{\partial\bar{\zeta}}\frac{\zeta^a\bar{\zeta}^{b+p+1}}{b+p+1}\, \frac{d\xi d\eta}{(\zeta - \tilde{\zeta})}\right\}\right]$$

$$= \frac{1}{\tilde{\zeta} - z}\left[\sum_{(a,\, b)\in T(m-1)} N(a, b, m-1, \tilde{\zeta})\sum_{p=0}^{n-1}({}^{n-1}C_p)(-\bar{z})^{n+p-1}\left\{\frac{z^{a+b+p+1}}{b+p+1}\right.\right.$$

$$\left.\left.- \frac{z^a\bar{z}^{b+p+1}}{b+p+1} - \frac{\bar{\zeta}^{a+b+p+1}}{b+p+1} + \frac{\tilde{\zeta}^a\overline{\tilde{\zeta}}^{b+p+1}}{b+p+1}\right\}\right]$$

$$G = \frac{1}{\pi}\int_{\mathbb{H}} \frac{(\tilde{\zeta} - \zeta + \overline{\tilde{\zeta}} - \bar{\zeta})^{m-1}(\bar{\zeta} - \bar{z})^{n-1}}{(\zeta - \tilde{\zeta})(\zeta - z)}\, d\xi d\eta$$

$$= \frac{1}{(\overline{\tilde{\zeta}} - z)}\left(\frac{1}{\pi}\int_{\mathbb{H}} \frac{(\tilde{\zeta} - \zeta + \overline{\tilde{\zeta}} - \bar{\zeta})^{m-1}(\bar{\zeta} - \bar{z})^{n-1}}{(\zeta - z)}\, d\xi d\eta\right.$$

$$\left.- \frac{1}{\pi}\int_{\mathbb{H}} \frac{(\tilde{\zeta} - \zeta + \overline{\tilde{\zeta}} - \bar{\zeta})^{m-1}(\bar{\zeta} - \bar{z})^{n-1}}{(\zeta - \overline{\tilde{\zeta}})}\, d\xi d\eta\right)$$

$$= \frac{1}{(\overline{\tilde{\zeta}} - z)} \sum_{(a,\, b) \in T(m-1)} N(a,\, b,\, m-1,\, \tilde{\zeta}) \sum_{p=0}^{n-1} C_p (-\overline{\zeta})^{n+p-1}$$

$$\left\{ \frac{1}{\pi} \int_{\mathbb{H}} \zeta^a \overline{\zeta}^{b+p} \frac{d\xi d\eta}{(\zeta - z)} - \frac{1}{\pi} \int_{\mathbb{H}} \zeta^a (\overline{\zeta})^{b+p} \frac{d\xi d\eta}{(\zeta - z)} \right\}$$

$$= \frac{1}{(\overline{\tilde{\zeta}} - z)} \sum_{(a,\, b) \in T(m-1)} N(a,\, b,\, m-1,\, \tilde{\zeta}) \sum_{p=0}^{n-1} ({}^{n-1}C_p)(-\overline{z})^{n+p-1}$$

$$\left[\frac{z^{a+b+p+1}}{b+p+1} - \frac{z^a \overline{z}^{b+p+1}}{b+p+1} \right]$$

$$F = \frac{1}{\pi} \int_{\mathbb{H}} \frac{(\tilde{\zeta} - \zeta + \overline{\tilde{\zeta}} - \overline{\zeta})^{m-1} (\overline{\zeta} - \overline{z})^{n-1}}{(\zeta - z)} d\xi d\eta$$

$$= \sum_{(a,\, b) \in T(m-1)} N(a,\, b,\, m-1,\, \tilde{\zeta}) \sum_{p=0}^{n-1} C_p (-z)^{n+p-1} \frac{1}{\pi} \int_{\mathbb{H}} \zeta^a \overline{\zeta}^{b+p} \frac{d\xi d\eta}{(\zeta - z)}$$

$$= \sum_{(a,\, b) \in T(m-1)} N(a,\, b,\, m-1,\, \tilde{\zeta}) \sum_{p=0}^{n-1} C_p (-z)^{n+p-1} \left[\frac{z^{a+b+p+1}}{b+1} - \frac{z^{a+p} \overline{z}^{b+1}}{b+1} \right]$$

$$H - \frac{1}{\pi} \int_{\mathbb{H}} \frac{(\tilde{\zeta} - \zeta + \overline{\tilde{\zeta}} - \overline{\zeta})^{m-1} (\overline{\zeta} - \overline{z})^{n-1}}{(\zeta - z)} d\xi d\eta s$$

$$= \sum_{(a,\, b) \in T(m-1)} N(a,\, b,\, m-1,\, \tilde{\zeta}) \sum_{p=0}^{n-1} C_p (-\overline{z})^{n+p-1} \frac{1}{\pi} \int_{\mathbb{H}} \zeta^a \overline{\zeta}^{b+p} \frac{d\xi d\eta}{(\zeta - z)}$$

$$= \sum_{(a,\, b) \in T(m-1)} N(a,\, b,\, m-1,\, \tilde{\zeta}) \sum_{p=0}^{n-1} {}^{n}C_p (-\overline{z})^{n+p-1} \left[\frac{z^{a+b+p+1}}{b+p+1} - \frac{z^{a+p} \overline{z}^{b+1}}{b+p+1} \right]$$

Substituting the values of A, B, E, F and G in equation (2.14), we have

$$\frac{1}{\pi} \int_H W(\zeta) (\overline{\zeta - z})^{n-1} \frac{d\xi d\eta}{(\zeta - z)}$$

$$= i\sum_{\delta=0}^{m-1}\sum_{r=0}^{\delta}\sum_{p=0}^{n-1}({}^{\delta}C_r)({}^{n-1}C_p)\bar{z}^{n-p-1}(-1)^{n-p-1}\left[\frac{z^{\delta+p+1}}{\delta-r+p+1}-\frac{z^r\bar{z}^{\delta-r+p+1}}{\delta-r+p+1}\right]$$

$$+\sum_{\delta=0}^{m-1}\frac{(-1)^{\delta}}{\pi i\delta!}\int_{-\infty}^{\infty}\beta_{\delta}(s)\left[\sum_{(a,\, b)\in T(\delta)}N(a,\, b,\, \delta,\, s)\sum_{p=0}^{n-1}{}^{n-1}C_p(-\bar{z})^{n-1-p}\right.$$

$$\left.\left(\frac{1}{s-z}-\frac{s}{s^2+1}\right)\left(\frac{z^{a+b+p+1}}{b+p+1}-\frac{z^a\bar{z}^{b+p+1}}{b+p+1}\right)\right]ds$$

$$+\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{f(\tilde{\zeta})}{\tilde{\zeta}-z}\left[\sum_{(a,\, b)\in T(m-1)}N(a,\, b,\, m-1,\, \tilde{\zeta})\sum_{p=0}^{n-1}{}^{n}C_p(-\bar{z})^{n-p-1}\right.$$

$$\left.\left\{\frac{z^{a+b+p+1}}{b+p+1}-\frac{z^a\bar{z}^{b+p+1}}{b+p+1}-\frac{\tilde{\zeta}^{a+b+p+1}}{b+p+1}-\frac{\tilde{\zeta}^a\bar{\tilde{\zeta}}^{b+p+1}}{b+p+1}\right\}\right]d\tilde{\xi}d\tilde{\eta}$$

$$-\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{\tilde{\zeta}f(\tilde{\zeta})}{\tilde{\zeta}^2+1}\left[\sum_{(a,\, b)\in T(m-1)}N(a,\, b,\, m-1,\, \tilde{\zeta})\sum_{p=0}^{n-1}{}^{n}C_p(-\bar{z})^{n-p-1}\right.$$

$$\left.\left\{\frac{z^{a+b+p+1}}{b+1}-\frac{z^a\bar{z}^{b+p+1}}{b+1}\right\}\right]d\tilde{\xi}d\tilde{\eta}$$

$$+\frac{(-1)^m}{(m-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{\overline{\tilde{\zeta}f(\tilde{\zeta})}}{\overline{\tilde{\zeta}^2}+1}\left[\sum_{(a,\, b)\in T(m-1)}N(a,\, b,\, m-1,\, \tilde{\zeta})\sum_{p=0}^{n-1}{}^{n}C_p(-\bar{z})^{n-p-1}\right.$$

$$\left.\left\{\frac{z^{a+b+p+1}}{b+p+1}-\frac{z^a\bar{z}^{b+p+1}}{b+p+1}\right\}\right]d\tilde{\xi}d\tilde{\eta} \tag{2.22}$$

Similarly, we can evaluate the following integrals using Cauchy-Pompeiu and Gauss theorem over H.

$$\frac{1}{\pi}\int_{\mathbb{H}}\frac{(2s-\zeta-\bar{\zeta})^{\delta}}{s-\zeta}\frac{(\bar{\zeta}-z)^{n-1-v}}{(\zeta-\bar{z})}d\xi d\eta$$

$$-\frac{s}{s^2+1}\frac{1}{\pi}\int_{\mathbb{H}}\frac{(2s-\zeta-\bar{\zeta})^{\delta}(\bar{\zeta}-z)^{n-1-v}}{(\zeta-\bar{z})}d\xi d\eta$$

$$= \frac{s}{(s-\bar{z})}\left[\frac{1}{\pi}\int_{\mathbb{H}}(2s-\zeta-\bar{\zeta})^{\delta}(\bar{\zeta}-z)^{n-1-v}\frac{d\xi d\eta}{(\zeta-\bar{z})}\right.$$

$$\left.-\frac{1}{\pi}\int_{\mathbb{H}}(2s-\zeta-\bar{\zeta})^{\delta}(\bar{\zeta}-\bar{z})^{n-1-v}\frac{d\xi d\eta}{(\zeta-\bar{z})}\right]$$

$$-\frac{s}{s^2+1}\frac{1}{\pi}\int_{\mathbb{H}}(2s-\zeta-\bar{\zeta})^{\delta}(\bar{\zeta}-z)^{n-1-v}\frac{d\xi d\eta}{(\zeta-\bar{z})} = 0, \tag{2.23}$$

$$\frac{1}{\pi}\int_{\mathbb{H}}\frac{(\tilde{\zeta}-\zeta+\overline{\tilde{\zeta}-\zeta})^{m-1}(\bar{\zeta}-z)^{n-1-v}}{(\tilde{\zeta}-\zeta)(\zeta-\bar{z})}d\xi d\eta$$

$$= \frac{1}{\tilde{\zeta}-\bar{z}}\left[\frac{1}{\pi}\int_{\mathbb{H}}\frac{(\tilde{\zeta}-\overline{\tilde{\zeta}})^{m-1}(\bar{\zeta}-z)^{n-1-v}}{\zeta-\bar{z}}d\xi d\eta\right.$$

$$\left.-\frac{1}{\pi}\int_{\mathbb{H}}\frac{(\tilde{\zeta}-\overline{\tilde{\zeta}})^{m-1}(\bar{\zeta}-z)^{n-1-v}}{\zeta-\tilde{\zeta}}d\xi d\eta\right]$$

$$= \frac{-1}{\tilde{\zeta}-\bar{z}}\left[\sum_{(a,\,b)\in T(m-1)}N(a,\,b,\,m-1,\,\tilde{\zeta})\sum_{p=0}^{n-1-v}(-\bar{z})^{n-1-v-p}\left\{\frac{1}{\pi}\zeta^{a}\bar{\zeta}^{b+p}\frac{d\xi d\eta}{\zeta-\tilde{\zeta}}\right\}\right]$$

$$= \frac{-1}{\tilde{\zeta}-\bar{z}}\left[\sum_{(a,\,b)\in T(m-1)}N(a,b,m-1,\tilde{\zeta})\sum_{p=0}^{n-1-v}(-\bar{z})^{n-1-v-p}\left(\frac{\tilde{\zeta}^{a+b+p+1}}{b+p+1}-\frac{\tilde{\zeta}^{a}\overline{\tilde{\zeta}}^{b+p+1}}{b+p+1}\right)\right] \tag{2.25}$$

Equation (2.3) can be obtained from (2.7) using the similar technique used in equation (2.6). Verification of solution can be done using the similar techniques used in see [1], Cauchy-Pompeiu operators of higher order [5]. □

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