Advances and Applications in Mathematical Sciences
Volume 19, Issue 12, October 2020, Pages 1267-1277

ON THE NUMERICAL SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND

AMMAR BENDJABRI and MOSTEFA NADIR

Department of Mathematics
University of Laghoat, Algeria
E-mail: bedjabriammar@yahoo.fr

Department of Mathematics
University of Msila, Algeria
E-mail: mostefa.nadir@univ-msila.dz
mostefanadir@yahoo.fr

Abstract

In this work we concern with the approximate solution of the linear equation $Af = f$ where A is injective and compact operator, this equation admits a unique solution in direct sense or in the least square sense provided the right-hand side f is in $R(A)$ or in $R(A) + R(A)^{\perp}$, respectively. Due to the nonclosed range $R(A)$ the solution is not stable. Besides, if A is positive de.nite we can replace the original equation by the auxiliary one $\alpha\varphi_\alpha + A\varphi_\alpha = f$ where its solution φ_α exist, stable and converges to the exact solution φ of the original equation as α tends to zero.

1. Introduction

The inverse problem takes a considerable part in the domain of differential and partial differential equations. As prominent example of the ill posed problem we find integral equation of the first kind where the most problems of engineering and mathematical physics can be modelled in this equation, many methods study its approximation solution and stability. In [1] the authors solved the integral equation of the first kind by Chebyshev wavelet constructed in the bounded interval and used the Galerkin technical in order to reduce the integral equation into an algebraic linear system. On

2010 Mathematics Subject Classification: 65N20, 65M25, 65D30.
Keywords: first-kind Fredholm integral equations, Lavrentiev method, Tikhonov regularization, numerical quadrature, discrete approximation.
Received June 15, 2020; Accepted July 5, 2020

the other hand, in [3, 4] the collocation method with Legendre wavelets are used directly for this equation and convert it to an algebraic linear system. The use of the since basis functions method for solving the first kind integral equation was find in [5]. The method proposed by authors in [10] is to use the Hermite polynomial with least square method in order to solve integral equation of the first kind with degenerate kernel supported by Galerkin and collocation methods.

Let A be a linear compact operator de.ned from Hilbert space H to itself over the field $\mathbb{R}$. We explicit the linear inverse problem of a first kind by

$$A\varphi = f, \tag{1}$$

where f is the data function and φ the unknown potential one, suppose that A is injective, then the equation (1) admits a unique solution in direct sense or in the last square sense provided the right-hand side f is in $R(A)$ or in $R(A) + R(A)^{\perp}$, respectively. Due to the nonclosed range $R(A)$ the solution is not stable. Besides, if A is positive definite we can replace the original equation by the auxiliary one

$$\alpha\varphi_{\alpha} + A\varphi_{\alpha} = f, \tag{2}$$

where we add the term $\alpha\varphi$ to the operator $A\varphi$ for α positive and small, the equation (2) admits a stable solution φ_{α}. Noting that the function φ_{α} converges to the exact solution φ of equation (1) as α tends to zero [8].

Lavrentiev method

The Lavrentiev method for the equation (1) is to replace the equation by the following one $A_{\varphi} = f_{\delta}$, with $\| f - f_{\delta} \| \leq \delta$, if the right-hand side f_{δ} is not in the range $R(A)$, Lavrentiev changes the equation $A\varphi = f_{\delta}$ by its auxiliary equation

$$\alpha\varphi_{\alpha\delta} + A\varphi_{\alpha\delta} = f_{\delta},\ \alpha > 0. \tag{3}$$

It is clear that, if the operator A is positive definite, the problem of the second kind (3) is well-posed. Lavrentiev proves that the solution $\varphi_{\alpha\delta}$ of the equation (3) tends to the exact solution φ of the equation (1) with conditions α and δ tend to zero.

Tikhonov Regularization Method

The Tikhonov regularization of the equation (1) corresponds to the regularization operators

$$R_\alpha = (\alpha I + A^* A)^{-1} A^*, \text{ for } \alpha > 0, \tag{4}$$

which approximates the unbounded operator A^{-1} on $R(A)$. Noting that, the solution $\varphi_\alpha = R_\alpha f$ represents the unique solution of the equation

$$\alpha\varphi_\alpha + A^* A\varphi_\alpha = A^* f, \tag{5}$$

and depends continuously on f, for all $f \in H$ and $\alpha > 0$. Also this solution is the unique minimum of the Tikhonov functional

$$J_\alpha(X) = \| A\varphi - f \|^2 + \alpha\| \varphi \|^2, \text{ for } \varphi \in H \text{ and } \alpha > 0.$$

2. Main Results

In this work we focus our study to the Fredholm integral equations of the first kind

$$A\varphi(x) = \int_a^b k(x, t)\varphi(t)dt = f(x), \; a \leq x \leq b$$

where $k(x, t)$ and f are given continuous functions and $\varphi(x) \in H([a, b])$ is the unknown potential function to be determined.

Lemma 1 [9]. *The problem* (2) *is well posed with the norm* $\| (\alpha I + A)^{-1} \| = O\left(\frac{1}{\sqrt{\alpha}}\right)$ *provided* A *is injective and positive definite operator*.

Proposition. *The injectivity and the positivity of the compact operator* A *lead to the existence and uniqueness of the solution of the auxiliary problem* (2)

$$\alpha\varphi_\alpha + A\varphi_\alpha = f, \; \alpha > 0.$$

Besides, the solution φ_α *converges to the exact solution* φ *of the initial problem* (1), *as* α *goes to zero, say*

$$\lim_{\alpha \to 0} \| \varphi - \varphi_\alpha \| = 0.$$

Proof.

Indeed,

$$\begin{aligned} \varphi - \varphi_\alpha &= \varphi - (\alpha I + A)^{-1} f \\ &= \varphi - (\alpha I + A)^{-1} A\varphi \\ &= \alpha(\alpha I + A)^{-1}\varphi. \end{aligned}$$

Therefore

$$\| \varphi - \varphi_\alpha \| \leq \alpha \| (\alpha I + A)^{-1} \| \| \varphi \|$$

$$\| \varphi - \varphi_\alpha \| = O(\sqrt{\alpha}).$$

Nyström method

Using the quadrature rule to approximate $\int_a^b k(x, t)\varphi(t)dt$ say

$$A\varphi(x) = \int_a^b k(x, t)\varphi(t)dt \simeq \sum_{j=1}^{n} w_j k(x, t_j)\varphi(t_j).$$

So, the equation (1) can be replaced by

$$A_n\varphi(x) = \sum_{j=1}^{n} w_j k(x, t_j)\varphi(t_j) = f(x), \; 0 \leq x \leq 1. \tag{6}$$

In the collocation method the values of $\varphi(t_j)$, $j = 1, 2, \ldots, n$ are found so that the equation (6) is verified for all points $x_1, x_2, \ldots, x_m$, in [0, 1]. It is not necessary to take $m = n$, but often m and n are chosen to be equal, and x_i is chosen as $x_i = t_i$, $i = 1, 2, \ldots, n$.

$$\sum_{j=1}^{n} w_j k(x, t_j)\varphi(t_j) = f(x_i),\ i = 1, 2, \ldots, n. \tag{7}$$

Taking $\mathcal{A} = (a_{ij})$ the $n \times n$ matrix such that $a_{ij} = w_j k(x_i, t_j)$ for $1 \leq i, j \leq n$, the unknown vector $\vec{\Phi} = (\varphi(t_1), \varphi(t_2), \ldots, \varphi(t_n))^T = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T$ and the right-hand side vector $\vec{F} = (f(x_1), f(t_2), \ldots, f(x_n))^T = (f_1, f_2, \ldots, f_n)^T$, Then the auxiliary equation (2) can be approximated by the matrix equation

$$(\alpha I + \mathcal{A})\overrightarrow{\Phi_\alpha} = \vec{F}. \tag{8}$$

The algebraic system (8) admits a unique solution $\overrightarrow{\Phi_\alpha}$ converges to the solution $\vec{\Phi}$ of the system $\mathcal{A}\overrightarrow{\Phi_\alpha} = \vec{F}$ as $\alpha \to 0$.

Lemma 2. *The norm* $\| (\alpha I + A)^{-1} \| \leq \frac{1}{\alpha}$ *provided A is positive definite operator. Indeed, A is positive and injective, it follows*

$$\langle \varphi, \varphi \rangle = \langle (\alpha I + A)(\alpha I + A)^{-1}\varphi, (\alpha I + A)(\alpha I + A)^{-1}\varphi \rangle$$

$$= \langle \alpha(\alpha I + A)^{-1}\varphi + A(\alpha I + A)^{-1}\varphi, \alpha(\alpha I + A)^{-1}\varphi + A(\alpha I + A)^{-1}\varphi \rangle$$

$$= \| \varphi \|^2 = \alpha^2 \| (\alpha I + A)^{-1}\varphi \|^2 + 2\alpha \langle A(\alpha I + A)^{-1}\varphi, (\alpha I + A)^{-1}\varphi \rangle$$

$$+ \| A(\alpha I + A)^{-1}\varphi \|^2, \| \varphi \|^2 \geq \alpha^2 \| (\alpha I + A)^{-1}\varphi \|^2.$$

Therefore, we obtain

$$\| (\alpha I + A)^{-1} \| \leq \frac{1}{\alpha}.$$

Lemma 3. *The operator $(\alpha I + A_n)$ is invertible on the Hilbert space H to itself if $\| (A - A_n)A_n \| \leq \alpha^2$. Besides, if $(\alpha I + A_n)\varphi_n = f$ and $(\alpha I + A)\varphi = f$, then*

$$\| \varphi - \varphi_n \| = O(\alpha).$$

Indeed,

$$\frac{1}{\alpha}[I-(\alpha I+A)^{-1}A_n](\alpha I+A_n)=I-(\alpha I+A)^{-1}A_n+\frac{1}{\alpha}[I-(\alpha I+A)^{-1}A_n]A_n$$

$$=I-(\alpha I+A)^{-1}A_n+\frac{1}{\alpha}[(\alpha I+A)^{-1}((\alpha I+A)-A_n)]A_n$$

$$=I-(\alpha I+A)^{-1}A_n+\frac{1}{\alpha}(\alpha I+A)^{-1}(\alpha I+A-A_n)A_n$$

$$=\left(I+\frac{1}{\alpha}(\alpha I+A)^{-1}(A-A_n)A_n\right).$$

Noting that, the right side of the last expression is invertible, for

$$\frac{1}{\alpha}\|(\alpha I+A)^{-1}(A-A_n)A_n\|\leq\frac{1}{\alpha}\|(\alpha I+A)^{-1}\|\|(A-A_n)A_n\|<1.$$

The injectivity of the right side involves the one of $(\alpha I+A_n)$ and so its bijectivity. For the error, we get

$$\|\varphi_n-\varphi\|=\|((\alpha I+A_n)^{-1}-(\alpha I+A)^{-1}f\|$$

$$=\|((\alpha I+A)^{-1}(A_n-A)(\alpha I+A_n)^{-1})f\|$$

$$=\|((\alpha I+A)^{-1}(A-A_n)\varphi_n\|$$

$$\leq\|(\alpha I+A)^{-1}\|\|(A-A_n)\|\|\varphi_n\|$$

$$=O(\alpha).$$

3. Explanatory Examples

Example 1. Consider the first-kind integral equations of Fredholm

$$\int_0^1\exp(xt)\varphi(t)dt=\frac{\exp(x+1)-1}{x+1},$$

where $0\leq t,\ x\leq 1$, and the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x)=\exp(x).$$

Table 1. We present the exact solution φ and its approximate one φ_α as well as the absolute error $|\varphi - \varphi_\alpha|$ of the example 1 in some arbitrary points for $N = 10$ and $\alpha = 10^{-6}$, the error is compared with the Chebyshev Wavelet Method [1] and the Haar wavelets method [6].

Val of x	Exact sol φ	App sol φ_α	$\|\varphi - \varphi_\alpha\|$	Error [1]	Error [6]
0.000	1.00e+00	9.99e-01	1.21e-06	1.46e - 05	7.85e-03
0.200	1.22e+00	1.22e+00	8.22e-07	1.73e - 05	5.69e-03
0.400	1.49e+00	1.49e+00	3.09e-07	1.57e - 05	2.31e-03
0.600	1.82e+00	1.82e+00	1.05e-06	1.30e - 06	2.86e-03
0.800	2.22e+00	2.22e+00	1.80e-06	1.52e - 05	1.04e-02
1.000	2.71e+00	2.71e+00	7.95e-06	1.04e - 05	4.98e-03

Example 2. Consider the first-kind integral equations of Fredholm

$$\int_0^1 \exp(t \sin x)\varphi(t)dt = \frac{1}{1 + \sin^2 x}(\exp(\sin x)(\cos 1 \sin x + \sin 1) - \sin x),$$

where $0 \le t, x \le 1$, and the function $f(x)$ is chosen so that the exact solution is given by $\varphi(x) = \cos x$.

Table 2. We present the exact solution φ and its approximate one φ_α as well as the absolute error $|\varphi - \varphi_\alpha|$ of the example 2 in some arbitrary points for $N = 10$ and $\alpha = 10^{-6}$, the error is compared with Legendre wavelets collocation method [3].

Val of x	Exact sol φ	App sol φ_α	$\|\varphi - \varphi_\alpha\|$	Error [3]

0.000	1.000e+00	9.991e-01	8.828e-04	1.488e-03
0.200	9.800e-01	9.803e-01	2.667e-04	2.531e-03
0.400	9.210e-01	9.207e-01	3.320e-04	4.740e-03
0.600	8.253e-01	8.255e-01	2.007e-04	2.920e-03
0.800	6.967e-01	6.966e-01	4.073e-05	4.951e-03
0.900	5.816e-01	5.815e-01	9.163e-05	1.239e-03

Example 3. Consider the first-kind integral equations of Fredholm

$$\int_0^1 (x^2 + 2tx + t^2)\varphi(t)dt = \frac{x^2}{2} + \frac{2x}{3} + \frac{1}{4},$$

where $0 \leq t, x \leq 1$, and the function $f(x)$ is chosen so that the exact solution is given by $\varphi(x) = x$.

Table 3. We present the exact solution φ and its approximate one φ_α as well as the absolute error $|\varphi - \varphi_\alpha|$ of the example 3 in some arbitrary points for $N = 10$ and $\alpha = 10^{-6}$, the error is compared with Hermite polynomial method [9].

Val of x	Exact sol φ	App sol φ_α	$\vert \varphi - \varphi_\alpha \vert$	Error [10]
0.000	0.000e+00	1.781e-07	1.781e-07	1.6e-03
0.200	2.000e-01	2.000e-01	2.158e-08	6.0e-04
0.400	4.000e-01	4.000e-01	4.687e-08	5.0e-04
0.600	6.000e-01	5.999e-01	8.356e-08	5.0e-04
0.800	8.000e-01	8.000e-01	7.016e-09	4.0e-04
1.000	1.000e+00	1.000e+00	1.140e-07	1.7e-03

Example 4. Consider the first-kind integral equations of Fredholm

$$\int_0^1 \cos(x-t)\varphi(t)dt = \frac{1}{4}\cos x + \frac{1}{2}\sin x - \frac{1}{4}\cos(x-2),$$

where $0 \le t,\ x \le 1$, and the function $f(x)$ is chosen so that the exact solution is given by $\varphi(x) = \sin x$.

Table 4. We present the exact solution φ and its approximate one φ_α as well as the absolute error $|\varphi - \varphi_\alpha|$ of the example 4 in some arbitrary points, the error is calculated for $N = 10$ and $\alpha = 10^{-6}$,

Val of x	Exact sol φ	App sol φ_α	$\lvert \varphi - \varphi_\alpha \rvert$
0.000	0.000e+00	1.384e-08	1.384e-08
0.200	1.986e-01	1.986e-01	8.067e-09
0.400	3.894e-01	3.894e-01	5.170e-09
0.600	5.646e-01	5.646e-01	1.680e-08
0.800	7.173e-01	7.173e-01	1.941e-08
1.000	8.414e-01	8.414e-01	3.261e-08

Example 5. Consider the first-kind integral equations of Fredholm

$$\int_0^1 \sinh(x-t)\varphi(t)dt = -\frac{1}{8}\exp(x)(\exp(-2) - 3 + \exp(-2x) + \exp(2)\exp(-2x)),$$

where $0 \le t,\ x \le 1$, and the function $f(x)$ is chosen so that the exact solution is given by $\varphi(x) \cosh x$.

Table 5. We present the exact solution φ and its approximate one φ_α as well as the absolute error $|\varphi - \varphi_\alpha|$ of the example 5 in some arbitrary points, the error is calculated for $N = 10$ and $\alpha = 10^{-6}$,

Val of x	Exact sol φ	App sol φ_α	$\lvert \varphi - \varphi_\alpha \rvert$
0.000	1.000e+00	9.999e-01	5.193e-08
0.200	1.020e+00	1.020e+00	3.413e-08

0.400	1.081e+00	1.081e+00	2.062e-08
0.600	1.185e+00	1.185e+00	1.546e-09
0.800	1.337e+00	1.337e+00	2.289e-08
1.000	1.543e+00	1.543e+00	2.962e-08

4. Conclusion

This numerical technique for solving Fredholm integral equations of first kind, concentrated on the few modification of Lavrentiev classical method supported by the modified Simpson approximation [7], the approximate solution φ_α is measurably close to the exact solution φ of the given equation on the whole interval [0, 1]. This method is tested by solving some examples for which the exact solution is known and proves its efficiency compared with other methods.

References

[1] H. Adibi and P. Assari, Chebyshev wavelet method for numerical solution of Fredholm integral equations of the first kind, Mathematical Problems in Engineering, Volume 2010, Article ID 138408, 17 pages

[2] M. Bahmanpour and M. A. F. Araghi, A method for solving Fredholm integral equation of the first kind based on Chebyshev wavelets, Analysis in Theory and Applications 29(3) (2013), 197-207.

[3] J. Kumar, P. Manchanda and Pooja, Numerical solution of Fredholm integral equations of the first kind using Legendre wavelets collocation method, International Journal of Pure and Applied Mathematics 117(1) (2017), 33-43.

[4] K. Maleknejad and R. Dehbozorgi, Legendre wavelets direct method for the numerical solution of Fredholm integral equation of the first kind, Proceedings of the World Congress on Engineering 2016, Vol. 1, London, U.K.

[5] Y. Maleknejad, H. S. Rostami, and Kalalagh, Numerical solution for first kind Fredholm integral equations by using sinc collocation method, International Journal of Applied Physics and Mathematics 6(3) (2016), 120-128.

[6] K. Maleknejad, R. Mollapourasl and K. Nouri, Convergence of numerical solution of the Fredholm integral equation of the first kind with degenerate kernel, Applied Mathematics and Computation 181(2) (2006), 1000-1007.

[7] M. Nadir and A. Rahmoune, Modified method for solving linear Volterra integral equations of the second kind using Simpsons rule, International Journal Mathematical Manuscripts 1(2) (2007), 133-140.

[8] M. Nadir and N. Djaidja, Approximation method for Volterra integral equation of the first kind, International Journal of Mathematics and Computation 29(4) (2018), 67-72.

[9] M. Nadir and N. Djaidja, Comparison between Taylor and perturbed method for Volterra integral equation of the first kind, Italian Journal of Pure and Applied Mathematics, in press.

[10] N. A. Sulaiman and T. I. Hassan, Numerical solutions of Fredholm integral equation of the first kind with degenerate kernel by using Hermite polynomial, J. Edu. Sci. 18(4) (2006), 63-73.