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TRUNCATIONS AND TOTALLY REGULAR PROPERTIES OF LEXICOGRAPHIC MIN-PRODUCT OF TWO FUZZY GRAPHS

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Abstract

In this paper, truncations of lexicographic min-product of fuzzy graphs and totally regular properties of lexicographic min- product of two given fuzzy graphs are discussed.

1. Introduction

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975. The properties of fuzzy graphs have been studied by Azriel Rosenfeld [9]. Later on, Bhattacharya [7] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by J. N. Mordeson and C. S. Peng [3]. The conjunction of two fuzzy graphs was defined by A. Nagoor Gani and K. Radha [4]. Properties of truncations on fuzzy graphs were introduced by A. Nagoorgani and K. Radha [5]. The concept of Double vertex fuzzy graph and complete Double vertex fuzzy graph were studied by K. Radha and S. Arumugam [10]. In this paper we discussed about some properties of truncations on double vertex fuzzy graphs and complete double vertex fuzzy graphs.

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First let us recall some preliminary definitions that can be found in [1-10].

A fuzzy graph G is a pair of functions (σ, μ) where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ. The underlying crisp graph of $G : (\sigma, \mu)$ is denoted by $G^* : (V, E)$ where $E \subseteq V \times V$. A fuzzy graph G is an effective fuzzy graph if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $uv \in E$ and G is a complete fuzzy graph if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$. Therefore G is a complete fuzzy graph if and only if G is an effective fuzzy graph and G^* is complete.

The degree of a vertex u of a fuzzy graph $G : (\sigma, \mu)$ with underlying crisp graph $G^* : (V, E)$ is defined as $d_G(u) = \sum \mu(uv)$ where the summation runs over all $uv \in E$.

The lower and upper truncations of σ at a level t, $0 < t \leq 1$, are the fuzzy subsets $\sigma_{(t)}$ and $\sigma^{(t)}$ defined respectively by

$$\sigma_{(t)}(u) = \begin{cases} \sigma(u), \text{ if}, \ \sigma(u) > t \\ 0, \text{ if } \ \sigma(u) \leq t \end{cases}$$

$$\sigma^{(t)}(u) = \begin{cases} t, \text{ if}, \ \sigma(u) > t \\ \sigma(u), \text{ if } \ \sigma(u) \leq t. \end{cases}$$

Let $G_1 : (\sigma_1, \mu_1)$ and $G_1 : (\sigma_2, \mu_2)$ denote two fuzzy graphs. The lexicographic min-product $G_1[G_2]_{\min} = G : (\sigma, \mu)$ with underlying crisp graph $G^* : (V, E)$ where $V = V_1 \times V_2$. $E = \{(u_1, v_1)(u_2, v_2)/u_1 u_2 \in E_1$ or $u_1 = u_2$ and $v_1 v_2 \in E_2)$ is given by $\sigma(u_1, v_1) = \sigma_1(u_1) \vee \sigma_2(v_1)$ for all $(u_1, v_1) \in V_1 \times V_2$ and

$$\mu((u_1, v_1)(u_2, v_2)) = \begin{cases} \mu(u_1, u_2), \text{ if } \ u_1u_2 \in E_1 \\ \sigma(u_1) \wedge \mu_2(v_1v_2), \text{ if } u_1 = u_2, \ v_1v_2 \in E_2. \end{cases}$$

2. Truncations of the Lexicographic Min-Product

Theorem 2.1. *The lexicographic min-product of the lower truncation of* *two fuzzy graphs* G_1 *and* G_2 *is the fuzzy subgraph of the lower truncation of lexicographic min-product of* G_1 *and* G_2 *that is* $G_{1(t)}[G_{2(t)}]_{\min}$ *is a fuzzy sub graph of* $(G_{1(t)}[G_2]_{\min})_{(t)}$.

Proof. First we prove that $\sigma_{1(t)}[\sigma_{2(t)}]_{\min} \leq (\sigma_1[\sigma_2]_{\min})_{(t)}$. Let $(u, v) \in V_1 \times V_2$ be any vertex.

By the definition of lexicographic min product $\sigma(u, v) = \sigma_1(u) \vee \sigma_2(v)$. Without loss of generality assume that $\sigma_1(u) \leq \sigma_2(v)$. Let $0 < t \leq 1$ be arbitrary.

There are three possibilities to consider:

$t \leq \sigma_1(u) \leq \sigma_2(v)$, $\sigma_1(u) \leq t \leq \sigma_2(v)$ and $\sigma_1(u) \leq \sigma_2(v) \leq t$

Case 1. $t \leq \sigma_1(u) \leq \sigma_2(v)$.

Then $\sigma_{1(t)}(u) = \sigma_1(u)$, $\sigma_{2(t)}(v) = \sigma_2(v)$.

Therefore

$$
\begin{aligned}
(\sigma_{1(t)}[\sigma_{2(t)}]_{\min}(u, v) &= \sigma_{1(t)}(u) \vee \sigma_{2(t)}(v) \\
&= \sigma_1(u) \vee \sigma_2(v) \\
&= \sigma_2(v).
\end{aligned}
$$

Since $(\sigma_{1(t)}[\sigma_{2(t)}]_{\min}(u, v) = \sigma_2(v) \geq t)$, $(\sigma_1[\sigma_2]_{\min})_{(t)}(u, v) = \sigma_2(v))$.

Therefore $(\sigma_{1(t)}[\sigma_{2(t)}]_{\min}(u, v)) = (\sigma_1[\sigma_2]_{\min})_{(t)}(u, v)$.

Case 2. $\sigma_1(u) \leq t \leq \sigma_2(v)$.

Then $\sigma_{1(t)}(u) = 0$ and $\sigma_{2(t)}(v) = \sigma_2(v)$.

Since $\sigma_{1(t)}(u) = 0$, $u \in V_{(t)}$ and hence $(u, v) \notin V_{1(t)} \times V_{2(t)}$.

Therefore $(\sigma_{1(t)}[\sigma_{2(t)}]_{mn})(u, v) = 0$

Since

$$(\sigma_1[\sigma_2]_{\min})(u, v) = \sigma_1(u) \vee \sigma_2(v)$$

$$= \sigma_2(v) \geq t$$

$$(\sigma_1[\sigma_2]_{\min})_{(t)}(u, v) = \sigma_2(v).$$

Therefore, in this case $(\sigma_{1(t)}[\sigma_{2(t)}]_{\min})(u, v) < (\sigma_1[\sigma_2]_{\min})_{(t)}(u, v)$.

Case 3. $\sigma_1(u) \leq \sigma_2(v) \leq t$.

Here $\sigma_{1(t)}(u) = 0$ and $\sigma_{2(t)}(v) = 0$.

Therefore $u \notin V_{1(t)}$ and $v \notin V_{2(t)}$ and hence $(u, v) \notin V_{1(t)} \times V_{2(t)}$.

Therefore $(\sigma_{1(t)}[\sigma_{2(t)}]_{\min})(u, v) = 0$.

Also since $\sigma_1(u) \vee \sigma_2(v) = \sigma_2(v) \leq t$, $(\sigma_1[\sigma_2]_{\min})_{(t)}(u, v) = 0$.

Therefore $(\sigma_{1(t)}[\sigma_{2(t)}]_{\min})(u, v) = 0\,(\sigma_1[\sigma_2]_{\min})_{(t)}(u, v)$.

Hence in all the three cases, $(\sigma_{1(t)}[\sigma_{2(t)}]_{\min})(u, v) \leq (\sigma_1[\sigma_2]_{\min})_{(t)}(u, v)$.

The proof is similar if $\sigma_2(v) \leq \sigma_1(u)$.

Next we prove that $(\mu_1[\mu_2]_{\min})_{(t)}(\mu_{1(t)}[\mu_{2(t)}]_{\min})_{\min}$.

Consider the edge $e = (u, v)(u, w)$ of $G_1[G_2]$.

$$(\mu_1[\mu_2]_{\min})(e) = \sigma_1(u) \wedge \mu_2(vw).$$

Assume that $\sigma_1(u) \leq \mu_2(vw)$.

There are three possibilities to consider:

(i) $t \leq \sigma_1(u) \leq \mu_2(v, w)$;

(ii) $\sigma_1(u) \leq t \leq \mu_2(vw)$;

(iii) $\sigma_1(u) \leq \mu_2(vw) \leq t$

Case (i). $t \leq \sigma_1(u) \leq \mu_2(vw)$.

Then $\sigma_{1(t)}(u) = \sigma_1(u)$ and $\mu_{2(t)}(vw) = \mu_2(vw)$.

Therefore

$$(\mu_{1(t)}[\mu_{2(t)}]_{\min})(e) = \sigma_{1(t)}(u) \wedge \mu_{2(t)}(vw)$$

$$= \sigma_1(u).$$

Since

$$(\mu_1[\mu_2]_{\min})(e) = \sigma_1(u) \wedge \mu_2(vw)$$

$$= \sigma_1(u) \geq t$$

$$(\mu_1[\mu_2]_{\min})_{(t)}(e) = \sigma_1(u).$$

Therefore $(\mu_{1(t)}[\mu_{2(t)}])_{\min}(e) = (\mu_1[\mu_2]_{\min})_{(t)}(e)$.

Case (ii). $\sigma_1(u) \leq t \leq \mu_2(v, w)$.

Proceeding as in the above case (i),

$$(\mu_{1(t)}[\mu_{2(t)}]_{\min})(e) = 0 < \mu_2(v, w) = (\mu_1[\mu_2]_{\min})_{(t)}(e).$$

Case (iii). $\sigma_1(u) \leq \mu_2(v, w) \leq t$ proceeding as in the above case 3,

$(\mu_{1(t)}[\mu_{2(t)}])_{\min}(e) = 0 = (\mu_{1(t)}[\mu_2]_{\min})_{(t)}(e)$.

Hence $(\mu_1[\mu_2]_{\min})_{(t)}(e) \leq (\mu_{1(t)}[\mu_{2(t)}]_{\min})(e)$.

The proof is similar if $\sigma_1(u) \geq \mu_2(vw)$.

Now consider the edge of the form $e = (u, x)(v, y)$ where $uv \in E_1$.

Then $(\mu_1[\mu_2]_{\min})_{(t)}(e) = \mu_1(uv) = (\mu_{1(t)}[\mu_{2(t)}]_{\min})(e)$ if $\mu_1(uv) < t$ and $(\mu_1[\mu_2]_{\min})_{(t)}(e) = 0 = (\mu_{1(t)}[\mu_{2(t)}]_{\min})(e)$ if $\mu_1(uv) < t$. Therefore in all the cases, $(\mu_1[\mu_2]_{\min})_{(t)}(e) \leq (\mu_{1(t)}[\mu_{2(t)}]_{\min})(e)$. Hence $(G_1\, [G_2]_{\min})_{(t)}$ is a fuzzy subgraph of $(G_1\, [G_2]_{\min})_{(t)}$ where $0 < t \leq 1$.

Theorem 2.2. *The lexicographic min-product of the upper truncation of two fuzzy graphs G_1 and G_2 is the fuzzy subgraph of the upper truncation of the lexicographic min-product of G_1 and G_2. That is, $G_1^{(t)}[G_2^{(t)}]_{\min}$ is the* *same as* $(G_1[G_2]_{\min})^{(t)}$. *In other words,* $G_1^{(t)}[G_2^{(t)}]_{\min}$ *and* $(G_1[G_2]_{\min})^{(t)}$ *are isomorphic*.

Proof. First, we prove that $\sigma_1^{(t)}[\sigma_2^{(t)}] = (\sigma_1[\sigma_1])^{(t)}$. Let $(u, v) \in V_1 \times V_2$ be any vertex. By the definition of lexicographic min-product $\sigma(u, v) = \sigma_1(u) \vee \sigma_2(v)$. Without loss of generality assume that $\sigma_1(u) \leq \sigma_2(v)$. Let $0 < t \leq 1$ be arbitrary. There are three possibilities to consider $t \leq \sigma_1(u) \leq \sigma_2(v)$; $\sigma_1(u) \leq t \leq \sigma_2(v)$; $\sigma_2(v)$; $\sigma_1(u) \leq \sigma_2(v) \leq t$.

Case 1. $t \leq \sigma_1(u) \leq \sigma_2(v)$.

Then $\sigma_1^{(t)}(u) = t$, $\sigma_2^{(t)}(v) = t$. Therefore $(\sigma_1^{(t)}[\sigma_2^{(t)}]_{\min})(u, v) = \sigma_1^{(t)}(u) \vee \sigma_2^{(t)}(v) = t \vee t = t$.

Since $(\sigma_1[\sigma_2]_{\min})(u, v) = \sigma_2(v) \geq t$

$$(\sigma_1[\sigma_2]_{\min})^{(t)}(u, v) = t.$$

Therefore $(\sigma_1^{(t)}[\sigma_2^{(t)}])_{\min}(u, v) = (\sigma_1[\sigma_2]_{\min})^{(t)}(u, v)$.

Case 2. $\sigma_1(u) \leq t \leq \sigma_2(v)$. Then $\sigma_1^{(t)}(u) = \sigma_1(u)$ and $\sigma_2^{(t)}(v) = t$. Therefore

$$\begin{aligned}(\sigma_1^{(t)}[\sigma_2^{(t)}]_{\min})(u, v) &= \sigma_1^{(t)}(u) \vee \sigma_2^{(t)}(v)\\ &= \sigma_1(u) \vee t\\ &= t.\end{aligned}$$

Since $(\sigma_1[\sigma_2]_{\min})(u, v) = \sigma_1(u) \vee \sigma_2(v) = \sigma_2(v) \geq t$.

$$(\sigma_1[\sigma_2]_{\min})^{(t)}(u, v) = t.$$

Therefore in this case $(\sigma_1^{(t)}[\sigma_2^{(t)}])_{\min} = (\sigma_1[\sigma_2]_{\min})^{(t)}$.

Case 3. $\sigma_1(u) \leq \sigma_2(v) \leq t$. Here $\sigma_1^{(t)}(u) = \sigma_1(u)$ and $\sigma_2^{(t)}(v) = \sigma_2(v)$.

Therefor $(\sigma_1^{(t)}[\sigma_2^{(t)}]_{\min})(u, v) = \sigma_1^{(t)}(u) \vee \sigma_2^{(t)}(v) = \sigma_1(u) \vee \sigma_2(v) = \sigma_2(v) \leq t$.

Hence $\sigma_1^{(t)}[\sigma_2^{(t)}]_{\min}(u, v) = \sigma_2(v)$. Since

$$(\sigma_1[\sigma_2]_{\min})^{(t)}(u, v) = \sigma_1(u) \vee \sigma_2(v) = \sigma_2(v) \leq t,$$

$$(\sigma_1[\sigma_2]_{\min})^{(t)}(u, v) = \sigma_2(v).$$

Hence in this case also $\sigma_1^{(t)}[\sigma_2^{(t)}]_{\min} = (\sigma_1[\sigma_2]_{\min})^{(t)}$. This proof is similar if $\sigma_2(v) \leq \sigma_2(u)$. Hence $\sigma_1^{(t)}[\sigma_2^{(t)}]_{\min}(u, v) = (\sigma_1[\sigma_2]_{\min})^{(t)}(u, v)$ for every vertex (u, v).

Next we prove that

$$(\mu_1[\mu_2]_{\min})^{(t)} \leq \mu_1^{(t)}[\mu_2^{(t)}]_{\min}.$$

Consider the edge $e = (u, v)(u, w)$ of $G_1[G_2]_{\min}$.

$$(\mu_1[\mu_2])_{\min}(e) = \sigma_1(u) \wedge \mu_2(vw).$$

Assume that $\sigma_1(u) \leq \mu_2(v, w)$.

There are three possibilities to consider:

(i) $t \leq \sigma_1(u) \leq \mu_2(v, w)$; (ii) $\sigma_1(u) \leq t \leq \mu_2(vw)$; (iii) $\sigma_1(u) \leq \mu_2(vw) \leq t$.

Case (i). $t \leq \sigma_1(u) \leq \mu_2(vw)$.

Then

$\sigma_1^{(t)}(u) = t$ and $\mu_2^{(t)}(vw) = t$.

Therefore $(\mu_1^{(t)}[\mu_2^{(t)}]_{\min})(e) = \sigma_1^{(t)}(u) \wedge \mu_1^{(t)}(vw) = t$.

Since $(\mu_1[\mu_2]_{\min})(e) = \sigma_1(u) \wedge \mu_2(vw) = \mu_2(vw) \geq t$,

$$(\mu_1[\mu_2]_{\min})^{(t)}(e) = t.$$

Therefore $(\mu_1^{(t)}[\mu_2^{(t)}]_{\min})(e) = (\mu_1[\mu_2]_{\min})^{(t)}(e)$.

Case. (ii) $\sigma_1(u) \leq t \leq \mu_2(vw)$.

Then $\sigma_1^{(t)}(u) = \sigma_1(u)$ and $\sigma_2^{(t)}(vw) = t$.

Therefore $(\mu_1^{(t)}[\mu_2^{(t)}]_{\min})(e)\,\sigma_1^{(t)}(u) \vee \mu_2^{(t)}(vw) = \sigma_1(u) \vee t = t.$

Since $(\mu_1[\mu_2]_{\min})(e) = \sigma_1(u) \vee \mu_2(vw) = \mu_2(vw) \geq t$, $(\mu_1[\mu_2]_{\min})^{(t)}(e) = t$.

Therefore $(\mu_1^{(t)}[\mu_2^{(t)}]_{\min})(e) = (\mu_1[\mu_2]_{\min})^{(t)}(e)$.

Case (iii) $\sigma_1(u) \leq \mu_2(v, w) \leq t$ proceeding as above $(\mu_1^{(t)}[\mu_2^{(t)}])(e) = \mu_2(vw) = (\mu_1[\mu_2])^{(t)}(e)$. Now consider the edge of the form $e = (u, x)(v, y)$ where $uv \in E_1$. Then $(\mu_1[\mu_2]_{\min})(e) = \mu_1(vu)$. Therefore $(\mu_1^{(t)}[\mu_2^{(t)}])(e) = \mu_2(uv) = (\mu_1[\mu_2])^{(t)}(e)$, if $\mu_1(uv) \geq t$ and $(\mu_1^{(t)}[\mu_2^{(t)}]_{\min})(e) = \mu_1(uv) = (\mu_1[\mu_2])^{(t)}(e)$, if $\mu_1(uv) \geq t$. Hence $(\mu_1[\mu_2]_{\min})^{(t)} = (\mu_1^{(t)}[\mu_2^{(t)}])_{\min}$ in all the cases.

The proof is similar if $\sigma_1(u) \geq \mu_2(v, w)$. Hence $G_1^{(t)}[G_2^{(t)}]_{\min}$ is the same as the fuzzy graph $(G_1[G_2]_{\min})^{(t)}$ where $0 < t \leq 1$.

3. Total Degree of a Vertex in Lexicographic Min-Product

The total degree of any vertex in the lexicographic min-product $G_1[G_2]_{\min}$ of the fuzzy graph $G_1 : (\sigma_1, \mu_1)$ with $G_2 : (\sigma_2, \mu_2)$ is given by,

$$d_{G_1[G_2]_{\min}}(u_1, v_j) = \sum_{u_i u_k \in E_1,\, v_l \in V_2} \mu_1(u_i u_k) + \sum_{u_i = u_k,\, v_j v_l \in E_2} \sigma_1(u_i) \wedge \mu_2(v_j v_l) + \sigma_1(u_i) \vee \sigma_2(v_j).$$

Theorem 3.1. *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are two fuzzy graphs such that $\sigma_1 \geq \mu_2$, then the total degree of a vertex in the lexicographic min-product $G_1[G_2]_{\min}$ of the fuzzy graph $G_1 : (\sigma_1, \mu_1)$ with $G_2 : (\sigma_2, \mu_2)$ is given by,*

$$td_{G_1[G_2]_{\min}}(u_i, v_j) = |\, V_1 \,| d_{G_1}(u_i) + d_{G_2}(v_j) + \sigma_1(u_i) \vee \sigma_2(v_j).$$

Proof. Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are two fuzzy graphs such that $\sigma_1 \geq \mu_2$. This implies that $\sigma_1 \wedge \mu_2 = \mu_2$. Then the total degree of any vertex $(u_i, v_j) \in V_1 \times V_2$ is given by,

$$td_{G_1[G_2]_{\min}}(u_i, v_j) = \sum_{u_i u_k \in E_1, v_l \in V_2} \mu_1(u_i, u_k) + \sum_{u_i = u_k, v_j v_l \in E_2} \sigma(u_i) \wedge \mu_2(v_j v_l)$$

$$+ \sigma_1(u_i) \vee \sigma_2(v_j)$$

$$= |V_2| \sum_{u_i u_k \in E_1} \mu_1(u_i, u_k) + \sum_{u_i = u_k, v_j v_l \in E_2} \mu_2(v_j v_l) + \sigma_1(u_i) \vee \sigma_2(v_j)$$

$$= |V_2| d_{G_1}(u_i) + d_{G_2}(v_j) + \sigma_1(u_i) \vee \sigma_2(v_j).$$

Corollary 3.2. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are two fuzzy graphs such that* $\sigma_1 \geq \mu_2$, σ_1, σ_2 *and* μ_2 *are constant functions of value* c_1, c_2 *and* c *respectively, then the total degree of a vertex in the lexicographic min-product* $G_1[G_2]_{\min}$ *of the two fuzzy graphs* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *is given by,*

$$td_{G_1[G_2]_{\min}}(u_i, v_j) = |V_2| d_{G_1}(u_i) + d_{G_2^*}(v_j)c + c_1 \vee c_2.$$

Proof. When μ_2 is a constant function of value 'c', then

$$\sum_{u_i = u_k,\, v_j v_l \in E_2} \mu_2(v_j v_l) = \sum_{u_i = u_k,\, v_j v_l \in E_2} c = d_{G_2^*}(v_j)c.$$

and $\sigma_1(u_i) \vee \sigma_2(v_j) = c_1 \vee c_2$.

Hence the result follows from Theorem 3.1.

Corollary 3.3. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are two fuzzy graphs such that* $\sigma_1 \geq \mu_2 \sigma_1$, σ_2 *and* μ_2 *are constant functions of value* c_1, c_2 *and* c *respectively, then the total degree of a vertex in the lexicographic min-product* $G_1[G_2]_{\min}$ *of the two fuzzy graphs* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *is given by,*

$$d_{G_1[G_2]_{\min}}(u_i, v_j) = |V_2| d_{G_1}(u_i) + d_{G_2^*}(v_j)c - |V_2| c_1 + c_1 \vee c_2.$$

Proof. Since $|V_2| d_{G_1}(u_i) = |V_2| (d_{G_1}(u_i) - \sigma_1(u_i))$, the result follows from Corollary 3.2.

Theorem 3.4. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are two fuzzy graphs such that* $\sigma_1 \geq \mu_2$, *then the total degree of a vertex in the lexicographic min-product* $G_1[G_2]_{\min}$ *of the fuzzy graphs* $G_1 : (\sigma_1, \mu_1)$ *with* $G_2 : (\sigma_2, \mu_2)$ *is given by*,

$$td_{G_1[G_2]_{\min}}(u_i, v_j) = |V_2| td_{G_1}(u_i) + td_{G_2}(v_j) - |V_2| \sigma_1(u_i) - \sigma_2(v_j) + \sigma_1(u_i) \vee \sigma_2(v_j).$$

Proof. Since $|V_2| d_{G_1}(u_i) = |V_2| (td_{G_1}(u_i) - \sigma_1(u_i))$ and $d_{G_2}(v_j) = td_{G_2}(v_j) - \sigma_2(v_j)$, the result follows from Theorem 3.1.

Corollary 3.5. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are two fuzzy graphs such that* $\sigma_1 \geq \mu_2 \sigma_1$, σ_2 *and* μ_2 *are constant functions of value* c_1, c_2 *and* c *respectively, then the total degree of a vertex in the lexicographic min-product* $G_1[G_2]_{\min}$ *of the two fuzzy graphs* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *is given by*,

$$td_{G_1[G_2]_{\min}}(u_i, v_j) = |V_2| d_{G_1}(u_i) + d_{G_2}(v_j) - |V_2| c_1 - c_2 + c_1 \vee c_2.$$

Proof. Since $|V_2| d_{G_1}(u_i) = |V_2| (td_{G_1}(u_i) - \sigma_1(u_i))$, the result follows from corollary 3.4.

Theorem 3.6. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are two fuzzy graphs such that* $\sigma_1 \geq \mu_2$ *then the total degree of a vertex in the lexicographic min-product* $G_1[G_2]_{\min}$ *of the fuzzy graph* $G_1 : (\sigma_1, \mu_1)$ *with* $G_2 : (\sigma_2, \mu_2)$ *is given by*,

$$d_{G_1[G_2]_{\min}}(u_i, v_j) = |V_2| d_{G_1}(u_i) + d_{G_2^*}(v_j)\sigma_1(\mu_i) + \sigma_1(\mu_i) \vee \sigma_2(v_j).$$

Proof. $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are two fuzzy graphs such that $\sigma_1 \geq \mu_2$. This implies that $\sigma_1 \wedge \mu_2 = \sigma_1$. Then, the total degree of any vertex $(u_i, v_j) \in V_1 \times V_2$ is given by,

$$td_{G_1[G_2]_{\min}}(u_i, v_j) = |V_2| d_{G_1}(u_i) + \sum_{u_i = u_k, v_j v_l \in E_2} \sigma_1(u_i) \wedge \mu_2(v_j v_l) + \sigma_1(u_i) \vee \sigma_2(v_j)$$

$$= |V_2| d_{G_1}(u_i) + \sum_{u_i = u_k,\ v_j v_l \in E_2} \sigma_1(\mu_i) + \sigma_1(\mu_i) + \sigma_1(u_i) \vee \sigma_2(v_j)$$

$$= |V_2| d_{G_1}(u_i) + d_{G_2^*}(v_j)\sigma_1(\mu_i) + \sigma_1(u_i) \vee \sigma_2(v_j).$$

Corollary 3.7. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are two fuzzy graphs such that* $\sigma_1 \geq \mu_2$ *and* σ_1 *and* σ_2 *are constant functions of values* c_1 *and* c_2 *respectively, then the total degree of a vertex in the lexicographic min-product* $G_1[G_2]_{\min}$ *of the fuzzy graph* $G_1 : (\sigma_1, \mu_1)$ *with* $G_2 : (\sigma_2, \mu_2)$ *is given by,*

$$td_{G_1[G_2]_{\min}}(u_i, v_j) = |V_2| d_{G_1}(u_i) + d_{G_2^*}(v_j)c_1 + c_1 \vee c_2.$$

Theorem 3.8. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are two fuzzy graphs such that* $\sigma_1 \geq \mu_2$ *then the total degree of a vertex in the lexicographic min-product* $G_1[G_2]_{\min}$ *of the fuzzy graph* $G_1 : (\sigma_1, \mu_1)$ *with* $G_2 : (\sigma_2, \mu_2)$ *is given by,*

$$d_{G_1[G_2]_{\min}}(u_i, v_j) = |V_2| td_{G_1}(u_i) + d_{G_2^*}(v_j)\sigma_1(\mu_i) - |V_2| \sigma_1(u_i) + \sigma_1(u_i) \vee \sigma_2(v_j).$$

Proof. Since $|V_2| d_{G_1}(u_i) = |V_2| (td_{G_1}(u_i) - \sigma_1(u_i))$, the result follows from Theorem 3.8.

Corollary 3.9. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are two fuzzy graphs such that* $\sigma_1 \geq \mu_2$ *and* σ_1 *and* σ_2 *are constant functions of values* c_1 *and* c_2 *respectively, then the total degree of a vertex in the lexicographic min-product* $G_1[G_2]_{\min}$ *of the fuzzy graph* $G_1 : (\sigma_1, \mu_1)$ *with* $G_2 : (\sigma_2, \mu_2)$ *is given by,*

$$td_{G_1[G_2]_{\min}}(u_i, v_j) = |V_2| td_{G_1}(u_i) + d_{G_2^*}(v_j)c_1 - |V_2| c_1 + c_1 \vee c_2.$$

4. Totally Regular Properties of Lexicographic Min-product

Theorem 4.1. *Let* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are two fuzzy graphs such that* $\sigma_1 \geq \mu_2, \sigma_1, \sigma_2$ *and* μ_2 *are constant functions of value* c_1, c_2 *and* c *respectively. Then the lexicographic min-product* $G_1[G_2]_{\min}$ *is totally regular if and only if* G_1 *is a regular fuzzy graph and* G_2^* *is a regular graph.*

Proof. Assume that $G_1[G_2]_{\min}$ is a k-regular fuzzy graph.

First, we prove that G_1 is a regular fuzzy graph.

Let u and w be any two vertices of G_1.

Fix a vertex $v \in V_2$. Since $G_1[G_2]_{\min}$ is a k-regular fuzzy graph, using corollary 3.2,

$$d_{G_1[G_2]_{\min}}(u, v) = d_{G_1[G_2]_{\min}}(w, v)$$

implies $|V_2|d_{G_1}(u) + d_{G_2^*}(v)c + c_1 \vee c_2 = |V_2|d_{G_1}(w) + d_{G_2^*}(v)c + c_1 \vee c_2$ which gives $|V_2|d_{G_1}(u) = |V_2|d_{G_1}(w)$ and hence $d_{G_1}(u) = d_{G_1}(w)$.

This is true for every pair of vertices of G_1. Hence G_1 is a regular fuzzy graph. Let u and w be any two vertices of G_2.

Fix a vertex $v \in V_1$ $td_{G_1[G_2]_{\min}}(v, u) = td_{G_1[G_2]_{\min}}(v, w)$ implies $|V_2|d_{G_1}(v) + d_{G_2^*}(u)c + c_1 \vee c_2 = |V_2|d_{G_1}(v) + d_{G_2^*}(w)c + c_1 \vee c_2$ which gives $d_{G_2^*}(u)c = d_{G_2^*}(w)c$ and hence $d_{G_2^*}(u) = d_{G_2^*}(w)$. This is true for every pair of vertices of G_2. Hence G_2^* is a regular graph.

Conversely assume that G_1 is a k-regular fuzzy graph and G_2^* is a m-regular graph. Then for any vertex (u, v) of $G_1[G_2]_{\min}$,

$$\begin{aligned} td_{G_1[G_2]_{\min}}(u, v) &= |V_2|d_{G_1}(u) + d_{G_2^*}(u)c + c_1 \vee c_2 \\ &= |V_2|k + mc + c_1 \vee c_2. \end{aligned}$$

Hence $G_1[G_2]_{\min}$ is a $(|V_2|k + mc + c_1 \vee c_2)$- regular fuzzy graph.

Theorem 4.2. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are two fuzzy graphs such that $\sigma_1 \geq \mu_2, \sigma_1, \sigma_2$ and μ_2 are constant functions of value c_1, c_2 and c respectively. Then the lexicographic min-product $G_1[G_2]_{\min}$ is totally regular if and only if G_1 is a totally regular fuzzy graph and G_2^* is a regular graph.*

Proof. The proof follows by proceeding as in Theorem 4.1 by using Corollary 3.3.

Theorem 4.3. *Let* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are two fuzzy graphs such that* $\sigma_1 \geq \mu_2, \sigma_1, \sigma_2$ *and* μ_2 *are constant functions of value* c_1, c_2 *and c respectively. Then the lexicographic min-product* $G_1[G_2]_{\min}$ *is totally regular if and only if* G_1 *and* G_2 *are totally regular fuzzy graphs.*

Proof. The proof follows by proceeding as in Theorem 4.1 by using Corollary 3.5.

Theorem 4.4. *Let* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are two fuzzy graphs such that* $\sigma_1 \geq \mu_2$ *and* σ_1 *and* σ_2 *are constant functions of values* c_1 *and* c_2 *respectively. Then the lexicographic min-product* $G_1[G_2]_{\min}$ *is totally regular if and only if* G_1 *is a regular fuzzy graph and* G_2^* *is a regular graph.*

Proof. The proof follows by proceeding as in Theorem 4.1 by using Corollary 3.7.

Theorem 4.5. *Let* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are two fuzzy graphs such that* $\sigma_1 \geq \mu_2$ *and* σ_1 *and* σ_2 *are constant functions of values* c_1 *and* c_2 *respectively. Then the lexicographic min-product* $G_1[G_2]_{\min}$ *is totally regular if and only if* G_1 *is a totally regular fuzzy graph and* G_2^* *is a regular graph.*

Proof. The proof follows by proceeding as in Theorem 4.1 by using Corollary 3.9.

Conclusion

In this paper we have discussed the truncation properties of lexicographic min-product of two fuzzy graphs and the totally regular property of lexicographic min-product of two fuzzy graphs. These properties will certainly be helpful in studying the fuzzy graphs in detail.

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