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A TEST FOR COMPARING TWO POPULATIONS HAVING MORE INCREASING FAILURE RATE AVERAGE PROPERTY

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Abstract

In this paper, a test based on U-statistic for comparing two distributions possessing 'more increasing failure rate (IFRA)-ness' property of life distributions is proposed. The proposed test procedure rejects the hypothesis of one has more IFRA ness property than the other for large values of the statistic proposed. The distributional properties of the proposed test statistic are studied. The asymptotic relative efficiencies (ARE) of the proposed procedure are evaluated with respect to the tests available in the literature for this problem. It is observed that the members of proposed class perform well.

1. Introduction

In survival analysis, a problem of interest is selecting a population which possesses more positive ageing property among two populations possessing

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positive ageing property. The problem of testing exponentially against the alternative of NBU is studied by many researchers. This problem has been studied by Hollander and Proschan [6], Koul [7], Kumazawa [8], Ahmed [1, 2] among others. However, the problem of testing whether one distribution possess more 'positive ageing' property than the other distribution has received less attention. Hollander, Park and Proschan [5] is the first procedure developed specifically for detecting NBU-ness property. Pandit and Gudaganavar [11] proposed tests for detecting positive ageing (IFR and NBU ness) property of distributions. Pandit and Gudaganavar [12] proposed a test procedure for selecting a distribution possessing more new better than used at specified age $t_0(NBU\text{-}t_0)$ property between two $NBU - t_0$ populations Pandit and Gudaganavar [13] studied the problem of detecting more NBU-ness property of life distributions. In this paper, we consider a test procedure to test the hypothesis that the two life distributions are identical against the alternative that one possesses more 'increasing failure rate average (IFRA)' property than the other.

2. New Test Procedures

Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ denote two random samples from continuous life distributions F and G, respectively. Here, a class of tests is developed for testing $H_0 : F = G$ (the common distribution is not specified) against $H_1 : F$ is 'more IFRA' than G.

Consider the parameter,

$$\gamma(F, G) = \gamma(F) - \gamma(G),$$

where $\gamma(F) = 2\int_0^\infty F(x)\overline{F}^{2b}(x)dF(x)$ and $\gamma(G) = 2\int_0^\infty G(x)\overline{G}^{2b}(x)dG(x)$.

Here $\gamma(F)$ and $\gamma(G)$ can be considered as the measure of degree of the IFRA-ness. Astagi math et al. (2019) test used this measure as basis for their test statistic. If F, G belongs to IFRA, then $\gamma(F) > 0(\gamma(G) > 0)$ and $\gamma(F, G)$ can be taken as a measure by which F is 'more IFRA' than G.

Under H_0, $\gamma(F, G) = 0$ and it is strictly greater than zero under H_1.

An unbiased estimator for $\gamma(F, G)$, which is defined as

$$U_{m,n} = U_m - U_n,$$

where U_m and U_n are U-statistics with kernels of degree 4 which are defined as $h_{1b}(x_1, x_2, x_3, x_4) = \begin{cases} 1 & \text{if } \operatorname{Min}(x_1, x_2) > b\operatorname{Max}(x_3, x_4) \\ 0 & \text{Otherwise} \end{cases}$ and $h_{2b}(y_1, y_2, y_3, y_4) = \begin{cases} 1 & \text{if } \operatorname{Min}(y_1, y_2) > b\operatorname{Max}(y_3, y_4) \\ 0 & \text{Otherwise} \end{cases}$ respectively.

Asymptotic Normality of the Test $U_{m,n}$

In this subsection, we study the asymptotic distribution of $U_{m,n}$. For that define $\xi_1(F) = E[\psi_1(X_1)]^2 - [\gamma(F)]^2$, where $\psi_1(x) = E[h_{1b}(x, X_2, X_3, X_4)]$. Next, $\xi_1(G)$ is defined as $\xi_1(G) = E[\psi_1^*(Y_1)]^2 - [\gamma(G)]^2$, where $\psi_1^*(y) = E[h_{1b}(y, Y_2, Y_3, Y_4)]$.

The asymptotic normality of the test $U_{m,n}$ is presented in the following theorem.

Theorem. *The asymptotic distribution of* $\sqrt{N}[U_{m,n} - \gamma(F, G)]$ *is normal with mean zero and variance given by* $\sigma^2(U_{m,n}) = \sigma_1^2 + \sigma_2^2$, *where* $\sigma_1^2 = \frac{16\,\xi_1(F)}{\lambda}$ *and* $\sigma_2^2 = \frac{16\,\xi_1(G)}{1-\lambda}$, *where* $\xi_1(F)$ *and* $\xi_1(G)$ *are as defined above. Under then* $H_0: F = G = F_0$, *then* $\xi_1(F) = \xi_1(G) = \xi_1(F_0)$

Proof. Proof follows from Hoeffding [4].

The approximate α-level test rejects H_0 in favour of H_1, if $\frac{\sqrt{N}\ U_{m,n}}{\sigma^2(U_{m,n})} > Z_\alpha$, where Z_α is the upper α-percentile point of standard normal distribution. Since, $\gamma(F, G) > 0$ under H_1 and from the asymptotic normality of $U_{m,n}$, the test based on $U_{m,n}$ is consistent against the alternative F is 'more IFRA than' G.

3. Consistent Estimator of $\sigma^2(U_{m,n})$:

Here, we find a consistent estimator for $\sigma^2(U_{m,n})$, which is necessary to implement the test procedure developed here. For that let, $\hat{\xi}_1(F) = \frac{1}{m-1} \sum_{i=1}^{m} [\psi_1(X_i) - U_m]^2$, where $\psi_1(x_i) = \frac{1}{m-1} \sum_i h_1(X_i, X_j, X_k, X_l)$ and the summation Σ_i extends over all possible $i < j \neq k \neq l < m$.

We define $\hat{\xi}_1(G) = \frac{1}{n-1} \sum_{j=1}^{n} [\psi_1^*(Y_j) - U_n]^2$, where $\psi_1^*(Y_j) = \frac{1}{n-1} \sum_j h_1(Y_j, Y_i, Y_k, Y_l)$ and the summation is over all possible $j < i \neq k \neq l < n$.

Then, from Puri and Sen [14] $\hat{\xi}_1(F)$ and $\hat{\xi}_1(G)$ are consistent estimators of $\xi_1(F)$ and $\xi_1(G)$ respectively and hence the consistent estimator $\hat{\sigma}^2(U_{m,n})$ of $\sigma^2(U_{m,n})$ is obtained by replacing $\xi_1(F)$ and $\xi_1(G)$ by $\hat{\xi}_1(F)$ and $\hat{\xi}_1(G)$ respectively in the expression $\sigma^2(U_{m,n})$. That is,

$$\hat{\sigma}^2(U_{m,n}) = \frac{16N\hat{\xi}_1(F)}{\lambda} + \frac{16N\hat{\xi}_1(G)}{1-\lambda}, \quad N = m + n.$$

By Slutsky's theorem, we have $\frac{\sqrt{N}U_{m,n}}{\hat{\sigma}^2(U_{m,n})}$ is asymptotically $N(0, 1)$ under H_0.

4. Asymptotic Relative Efficiency

We study the asymptotic relative efficiency of $U_{m,n}$, relative to the $V_{k,n}$ test statistic given by Hollander, Park and Proschan [5] for the two pairs of distributions $(F_{i,\theta}, G)$. Here, we assume that G is an exponential distribution with mean one. We denote $F_{1,\theta}$ as survival function for Weibull distribution and $F_{2,\theta}$ as survival function for linear failure rate distribution and $F_{3,\theta}$ as survival function for Makeham distribution defined below:

1. Weibull Distribution:

$$F_{1,\theta}(x) = e^{-\theta x},\ x,\ \theta > 0.$$

2. Linear Failure Rate Distribution:

$$F_{2,\theta} = \exp\left(-\left(x + \theta\frac{x^2}{2}\right)\right),\ x > 0,\ \theta \geq 0.$$

3. Makeham Distribution:

$$F_{3,\theta} = \exp\{-x + \theta(x + e^{-x} - 1)\},\ x > 0,\ \theta \geq 0.$$

The Pitman asymptoticefficacy is

$$eff\ (U_{m,n} F_{i,\theta}, G) = \frac{1}{\sigma_0^2(U_{m,n})}\left\{\frac{d}{d\theta}\gamma(F_{i,\theta}, G)\right\}_{\theta\to\theta_0}.$$

The ARE's of the proposed tests with respect to the J_b test of Deshpande [3] and $V(b, k)$ test of Pandit et al. (2008) for Weibull distribution with parameter θ, Linear failure rate distribution and Makeham distribution are presented below in table 1 and table 2.

Table 1. AREs of $U_{m,n}$ with respect to J_b.

B	0.1	0.5	0.9
Weibull	1.0230	0.717	0.6579
LFR	1.1797	1.7295	2.0994
Makeham	1.4293	2.7322	9.4343

Table 2. AREs of $U_{m,n}$ with respect to $V(b, k)$.

B	0.1	0.5	0.9
Weibull	1.1755	1.6073	1.1625
LFR	1.8927	1.9428	3.4463
Makeham	1.3983	2.2888	2.0555

Next, we compute the efficiency of the two sample test based on $U_{m,n}$ proposed in section 2, by specifying the common null distribution in the null hypothesis as F_θ with $\theta \geq 1$ and considering sequence of alternatives $(F_{\theta\phi_N}, F_\theta)$, where $\phi = 1 + \frac{a}{\sqrt{N}}$, a being arbitrary positive constant. Note that as $N \to \infty$, the sequence of alternatives converges to the null hypothesis. The efficacy of the $U_{m,n}$ test is given by $eff(U_{m,n}) = \frac{[\gamma'(F_{\theta\phi_N}, F_\theta)]^2}{\sigma_0^2(U_{m,n})}$, where $\sigma_0^2(U_{m,n})$ is null asymptotic variance of $\sqrt{N}U_{m,,n}$ and $\gamma'(F, G) = \left[\frac{dy(F_{\theta\phi_N}, F_\theta)}{d\phi}\right]_{\phi=1}$.

The sequence of alternatives considered here are $(F_{1,\theta\phi_N}, F_{1,\theta})$, $(F_{2,\theta\phi_N}, F_{2,\theta})$ and $(F_{3,\theta\phi_N}, F_{3,\theta})$. The Asymptotic relative efficiencies of the proposed test $U_{m,n}$ relative to the test due to Hollander, Park and Proschan [5] $V_{k,n}$ for the various alternatives are presented in table 3, table 4 and table 5 respectively.

Table 3. ARE of $U_{m,n}$ w.r.t. $V_{k,n}$ for $(F_{1,\theta\phi_N}, F_{1,\theta})$.

$b \to$ $\theta \downarrow$	0.1	0.5	0.9
2	1.0684	1.1281	1.1628
3	1.6870	1.9111	2.1694
4	0.7868	0.9549	1.1967
5	0.8578	1.1092	1.5325
6	0.5295	0.7317	1.1065
7	0.4963	0.7134	1.1778
8	0.3531	0.5353	0.9605
9	0.3020	0.4943	0.8356
10	0.2278	0.3966	0.7911

Table 4. ARE of $U_{m,n}$ w.r.t. $V_{k,n}$ for $(F_{2,\theta\phi_N}, F_{2,\theta})$.

$b \rightarrow$ $\theta \downarrow$	0.1	0.5	0.9
1	1.2800	1.2879	1.2961
2	1.2707	1.2795	1.2900
3	1.2663	1.2759	1.2879
4	1.2642	1.2745	1.2877
5	1.2635	1.2742	1.2884
6	1.2634	1.2746	1.2897
7	1.2639	1.2755	1.2912
8	1.2646	1.2765	1.2929
9	1.2655	1.2778	1.2947

Table 5. ARE of $U_{m,n}$ w.r.t. $V_{k,n}$ for $(F_{3,\theta\phi_N}, F_{3,\theta})$.

$b \rightarrow$ $\theta \downarrow$	0.1	0.5	0.9
1	1.6055	1.6803	1.6956
2	1.0423	0.9356	0.8334
3	1.1842	0.9509	0.7682
4	1.3641	1.0363	0.7652
5	1.8216	1.1989	0.8125
6	1.5653	1.4680	0.9148
7	1.8594	1.8892	1.0840
8	2.2602	2.3305	1.1360
9	1.5929	1.1314	0.5396

5. Some Remarks

1. The Asymptotic relative efficiencies of the proposed test with respect to the test due to Deshpande [3] and Pandit et al. (2008) are computed for three pairs of distributions (F_θ, G) with G is exponential with mean one and F_θ as Weibull, Linear failure rate and Makeham distributions.

2. It is observed that the proposed test performs better for the alternatives considered F_θ is either Weibull, Linear failure rate and Makeham distributions when G is exponential.

3. The asymptotic efficacies of the test proposed are evaluated for three pairs of distributions $(F_{1,\,\theta\phi_N}, F_{1,\,\theta})$, $(F_{2,\,\theta\phi_N}, F_{2,\,\theta})$ and $(F_{3,\,\theta\phi_N}, F_{3,\,\theta})$ with F_1, F_2, F_3 as Weibull, linear failure rate and Makeham distributions respectively.

4. Hence, if the data under consideration is exactly IFRA, the new test proposed would be a better choice.

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