Advances and Applications in Mathematical Sciences
Volume 20, Issue 9, July 2021, Pages 1811-1821

DOMINATION INTEGRITY OF SNAKE AND SUNLET GRAPHS

D. BHUVANESWARI and S. MEENAKSHI

Vels Institute of Science
Technology and Advanced Studies
Chennai-600117, Tamil Nadu, India
E-mail: bhuvanamaths15@gmail.com
meenakshikarthikeyan@yahoo.co.in

Abstract

The stability of a network plays a significant role in a communication network for its designers and users. The domination integrity is the well-known parameter used to measure the vulnerability of a network and for the simple connected graph $G = (V(G), E(G))$ its defined as $DI(G) = \min \{| D | + m(G - D) : D$ is a dominating set$\}$ where $m(G - D)$ is the order of a maximum component of $G - D.$ Here we determine the domination integrity of snake graphs and sunlet graphs.

I. Introduction

We begin this with a connected, finite, undirected graph without loopless and multiple edges. The undefined notations and terminology are in [1, 13]. Any communication network can be modeled as a graph by considering the processor as a nodes and connection or links between them as edges. The vulnerability means that any type of damage caused on processor (nodes) or connection between them (edges) or any failure in transmission and also fault in hardware and software for a long duration will result in a loss of effectiveness. On constructing (or designing) the communication network, the network designers and its users give much more importance to its stability. Always the communication network requires the higher degree of stability or less vulnerable. A modeled graph of a network is highly vulnerable if any

2010 Mathematics Subject Classification: 05Cxx.
Keywords: Vulnerability, Domination integrity, Snake graph, Sunlet graph.
Received November 20, 2019; Accepted May 25, 2020

small destruction cause huge consequences and also the communication links exist only between the few members.

In terms of a graph, vulnerability means the weakness or lack of resistance in its structure after the deletion of nodes or edges or both. On measuring the vulnerability two quantities are important (i) the number of non-working members (ii) the order (the number of vertices) of maximum component within which a mutual communication exists still. Here the first one shows the vertices which are left free to attack by the attacker and the second one shows the reaction after a disruption. We have a variety of graph theoretic parameter used to measure the vulnerability of a network. Among that Domination Integrity is one of the parameters introduced by Sundareswaran and Swaminathan [2] and investigated it for various graphs, trees, power of cycles and also the middle graphs of some basic standard graphs [3-5].

Definition 1.1. Let G be a connected graph with the vertex set $V(G)$ and the edge set $E(G)$. A subset D of the vertex set $V(G)$ is a dominating set if $V(G) = N(D) \cup D$.

Definition 1.2. The domination integrity is denoted as $DI(G)$ and is defined by $DI(G) = \min \{| X | + m(G - X) : X$ is a dominating set$\}$, where $m(G - X)$ is the order (number of vertices) of a maximum component of $G - X$.

Vaidya and Kothari [7-9], Vaidya and Shah [10-12], Sulthan Seena and Veena [6], were investigated the domination integrity of various graph operation on some standard graphs.

Definition 1.3. The Sunlet graph is the graph of $2n$ vertices obtained by joining the pendant vertices of the cycle graph C_n and is denoted by S_n.

Definition 1.4. The Triangular Snake graph is obtained from a path having vertex $v_1, v_2, v_3, \ldots, v_n$ by joining v_j and v_{j+1} to a new vertex u_i for $1 \leq j \leq n-1$ and is denoted by $S_{3,n}$.

Example 1.1.

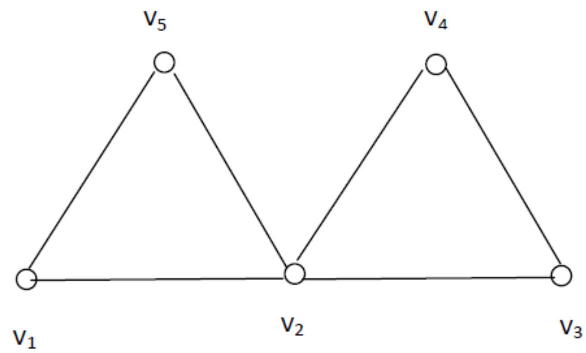

Figure 1. Labeling of a vertex in a Triangular Snake Graph.

Definition 1.5. An Alternate triangular snake graph $A(S_{3,n})$ is obtained from a path $v_1, v_2, v_3, \ldots, v_n$ by joining alternately v_j and v_{j+1} to a new vertex u_i for $1 \le j \le n-1$.

Definition 1.6. A Double Triangular Snake graph $D(S_{3,n})$ consist of two triangular snakes that have a common path.

II. Main Results

Theorem 2.1. $DI = S_n = n + 1$ *for* $n \ge 3$.

Proof. Let the vertex set of S_n be $\{v_1, v_2, \ldots, v_n, v_{1'}, v_{2'}, \ldots, v_{n'}\}$ and $|E(S_n)| = 2n$.

- If $n = 2i, i = N \setminus \{1\}$, (i.e., $n = 4, 6, 8, \ldots$). Let the set $S = \{v_{1+2k}, v'_{2+2k} \setminus k = 0$ to $i-1\}$ and $|S| = n$.

- If $n = 2i - i, i = N \setminus \{1\}$, (i.e., $n = 3, 5, 7, 9, 11, \ldots$). Consider the set $S = \{v_{1+2k}, v'_{2+2kt} \setminus k = 0$ to $i-1$ and $t = 0$ to $i-2\}$ and $|S| = n$.

For the above two cases, the set S will be a dominating set as $v_{2+2r}, v'_{1+2r} \in N(v_{1+2r})$. Further we have $m(S_n - S) = 1$. Therefore, $|S| + m(S_n - S) = n + 1$.

Now to analyze the minimum of $|S| + m(S_n - S)$.

Let S_1 be any other possible dominating set with $m(S_n - S) > 1$, then due to the structural nature of the graph S_n, we get $| S_1 | = | S |$. Therefore,

$$| S_1 | + m(S_n - S_1) > | S | + m(S_n - S).$$

From this we conclude that,

$$| S | + m(S_n - S) = \min \{| X | + m(S_n - X)\}$$

$$= DI(S_n).$$

Hence,

$$DI(S_n) = n + 1 \text{ for } n \geq 3.$$

Theorem 2.2.

$$DI(S_{3,n}) = \begin{cases} 3, & n = 2, 3 \\ n, & n = 4, 5. \end{cases}$$

Proof. Let the vertex set of $S_{3,n}$ be $\{v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n-1}\}$. To prove the result we need the following three cases.

Case (i) $n = 2, 3$.

For $n = 2$, consider $S = \{v_2\}$ is a dominating set $S_{3,2}$ and $m(S_{3,2} - S) = 2$. Therefore, $| S | + m(S_{3,2} - S) = 3$. Also for $S = \{v_1\}$ or $\{v_3\}$ we get $| S | + m(S_{3,2} - S) = 3$. If we consider $S = \{v_1, v_3\}$ or $S = \{v_1, v_2\}$ or $S = \{v_2, v_3\}$ and $m(S_{3,2} - S) = 1$, then also we get $| S | + m(S_{3,2} - S) = 3$. Hence $DI(S_{3,2}) = 3$.

For $n = 3$, consider $S = \{v_2\}$, the dominating set of $S_{3,3}$ and we have $m(S_{3,3} - S) = 2$. So $| S | + m(S_{3,3} - S) = 5$. If $S = \{v_1, v_3\}$ or $S = \{v_4, v_5\}$ and $m(S_{3,3} - S) = 3$, then for these S we get $| S | + m(S_{3,3} - S) = 5$. Hence $DI(S_{3,3}) = 3$.

Case (ii) $n = 4, 5$.

For $n = 4$, let us consider $S = \{v_2, v_3\}$ as a dominating set of $S_{3,4}$ and

$m(S_{3,4} - S) = 2$. Therefore, $|S| + m(S_{3,4} - S) = 4$. If for $S = \{v_2, v_4, v_5\}$ and $S = \{v_5, v_6, v_7\}$, we have $m(S_{3,4} - S)$ are 2 and 4 respectively. So $|S| + m(S_{3,4} - S)$ is equal to 5 and 6 respectively. Therefore, $|S| + m(S_{3,5} - S) = 4$ is minimum. Hence $DI(S_{3,4}) = 4$.

For $n = 5$. Now consider the set $S = \{v_2, v_4\}$ as a dominating set and $m(S_{3,5} - S) = 3$. So $|S| + m(S_{3,5} - S) = 5$. Also for $S = \{v_2, v_4, v_7\}$, $m(S_{3,5} - S) = 2$, then $|S| + m(S_{3,5} - S) = 5$. If $S = \{v_1, v_3, v_5\}$ we have $m(S_{3,5} - S) = 3$ and then $|S| + m(S_{3,5} - S) = 6$. Therefore $|S| + m(S_{3,5} - S) = 5$ is minimum. Hence $DI(S_{3,5}) = 5$. From this we conclude that,

$$DI(S_{3,n}) = \begin{cases} 3, n = 2, 3 \\ n, n = 4, 5. \end{cases}$$

Theorem 2.3. $DI(S_{3,n}) = \left\lfloor \frac{n}{2} \right\rfloor + 3$ *for* $n \geq 6$.

Proof. Let $S = \{v_{2+2t} \setminus t = 0$ to $\left\lfloor \frac{n}{2} \right\rfloor - 1\}$ for $n \geq 6$ and $|S| = \left\lfloor \frac{n}{2} \right\rfloor$. Further the set S will be a dominating set as $v_{1+2t}, v_{n+1+t}, v_{2n-1-t} \in N(v_{2+2t})$ and we have $m(S_{3,n} - S) = 3$. Therefore, $|S| + m(S_{3,n} - S) = \left\lfloor \frac{n}{2} \right\rfloor + 3$. Now to check the minimality of $|S| + m(S_{3,n} - S)$.

Let S_1 be any other dominating set with $m(S_{3,n} - S_1) = 1$, then each component of $S_{3,n} - S_1$ is an isolated vertex. Therefore, $|S_1| > |S|$. Hence,

$$|S_1| + m(S_{3,n} - S_1) > |S| + m(S_{3,n} - S). \tag{1}$$

Consider any other dominating set S_2 with $m(S_{3,n} - S_2) = 2$, then due to the structural nature of the graph $S_{3,n}$ we have $|S_2| > |S|$. Therefore,

$$|S_2| + m(S_{3,n} - S_2) > |S| + m(S_{3,n} - S). \tag{2}$$

For any other possible dominating set S_3 with $m(S_{3,n} - S_3) > 3$, then $| S_3 | > | S |$. So

$$| S_2 | + m(S_{3,n} - S_2) > | S | + m(S_{3,n} - S). \tag{3}$$

From (1), (2) and (3) we conclude that,

$$| S | + m(S_{3,n} - S) = \min \{| X | + m(S_{3,n} - X) : X \text{ is a dominating set}\}$$

$$= DI(S_{3,n}).$$

Hence,

$$DI(S_{3,n}) = \left\lfloor \frac{n}{2} \right\rfloor + 3 \text{ for } n \geq 6.$$

Theorem 2.4. *In an alternate triangular snake graph, the triangle starts from the first vertex of the path, then*

$$DI(A(S_{3,n})) = \left\lfloor \frac{n}{2} \right\rfloor + 2 \textit{ for } n \geq 2.$$

Proof. Let the vertex set of $| V(A(S_{3,n})) | = \left\lfloor \frac{3n}{2} \right\rfloor$.

Consider $S = \{v_{2+2t} \setminus t = 0 \text{ to } \left\lfloor \frac{n}{2} \right\rfloor - 1\}$ and it's a dominating set of $A(S_{3,n})$ as $v_{1+2t}, v_{\left\lfloor \frac{3n}{2} \right\rfloor - t} \in N(v_{2+2t})$ for $t = N \cup \{0\}$ and $| S | = \left\lfloor \frac{n}{2} \right\rfloor$.

Moreover, $m(A(S_{3,n}) - S) = 2$. Therefore, $| S | + m(A(S_{3,n}) - S) = \left\lfloor \frac{n}{2} \right\rfloor + 2$.

Now to check the minimality of $| S | + m(A(S_{3,n}) - S)$.

Let S_1 be any other possible dominating set of $A(S_{3,n})$ with $m(A(S_{3,n}) - S_1) < 2$, then each component of $A(S_{3,n}) - S_1$ is an isolated vertex. So $| S_1 | > | S |$. Therefore,

$$| S_1 | + m(A(S_{3,n}) - S_1) > | S | + m(A(S_{3,n}) - S). \tag{4}$$

Let S_2 be other possible dominating set with $m(A(S_{3,n})-S)=4,$ then due to the structure of the graph $A(S_{3,n})$ we get $|S_2|>|S|.$ Therefore,

$$|S_2|+m(A(S_{3,n})-S_2)>|S|+m(A(S_{3,n})-S). \tag{5}$$

From equations (4) and (5), we conclude that

$$|S|+m(S_{3,n}-S)=\min\{|X|+m(A(S_{3,n})-X): X \text{ is a dominating set}\}$$

$$=DI(A(S_{3,n})).$$

Hence,

$$DI(A(S_{3,n}))=\left\lfloor\frac{n}{2}\right\rfloor+2 \text{ for } n\geq 2.$$

Theorem 2.5. *In an alternate triangular snake graph, the triangle starts from the second vertex, then*

$$DI(A(S_{3,n}))=\begin{cases}3, & n=3,4\\ 4, & n=5\\ \left\lfloor\frac{n}{2}\right\rfloor+2, & n\geq 6.\end{cases}$$

Proof. Let the vertex set of $|V(A(S_{3,n}))|=n+\left\lceil\frac{n-2}{2}\right\rceil$. We need the following three cases to prove this theorem.

Case (i) $n=3,4.$

For $n=3,$ consider $S=\{v_2\}$ the dominating set of $A(S_{3,3}),$ then $m(A(S_{3,3})-S)=2.$ Thus, $|S|+m(A(S_{3,3})-S)=3.$ If $S=\{v_1, v_3\},$ then $m(A(S_{3,3})-S)=2.$ So $|S|+m(A(S_{3,3})-S)=4.$ Hence $|S|+m(A(S_{3,3})-S)=3$ is minimum.

For $n=4,$ consider the dominating set $S=\{v_2, v_3\}$ of $A(S_{3,4}),$ then $m(A(S_{3,4})-S)=1.$ Thus $|S|+m(A(S_{3,4})-S)=3.$ If $S=\{v_1, v_3\}$ or $S=\{v_2, v_4\},$ $m(A(S_{3,4})-S)=2,$ then we get $|S|+m(A(S_{3,4})-S)=4.$ Hence $|S|+m(A(S_{3,4})-S)=3$ is minimum.

Case (ii) $n = 5$.

For $n = 5$, consider the set $S = \{v_2, v_4\}$ which is a dominating set of $A(S_{3,5})$, then $m(A(S_{3,5}) - S) = 2$. So $|S| + m(A(S_{3,5}) - S) = 4$. If $S = \{v_1, v_3, v_5\}$, then $m(A(S_{3,5}) - S) = 2$. So $|S| + m(A(S_{3,5}) - S) = 5$. Hence $|S| + m(A(S_{3,5}) - S) = 4$ is minimum.

Case (iii) $n \geq 6$.

Let us consider $S = \{v_{2+2t} \setminus t = 0$ to $\left\lfloor \frac{n}{2} \right\rfloor - 1\}$ as a dominating set because, $v_{1+2t}, v_{3+2t}, v_{n+\left\lceil \frac{n}{2} \right\rceil - t} \in N(v_{2+2t})$ for $t = 0, 1, 2, \ldots$ and $|S| = \left\lfloor \frac{n}{2} \right\rfloor$. Further, $m(A(S_{3,n}) - S) = 2$. Therefore, $|S| + m(A(S_{3,n}) - S) = \left\lfloor \frac{n}{2} \right\rfloor + 2$.

Now to analyze the minimum of $|S| + m(A(S_{3,n}) - S)$.

Let S_1 be any other dominating set with $m(A(S_{3,n}) - S) < 2$, then each component in $A(S_{3,n}) - S$ will be an isolated vertex. So $|S_1| > |S|$. Therefore,

$$|S_1| + m(A(S_{3,n}) - S_1) > |S| + m(A(S_{3,n}) - S). \quad (6)$$

Now consider other possible dominating set S_2 with $m(A(S_{3,n}) - S) = 4$, then due to the construction of the graph $A(S_{3,n})$, we get $|S_2| \geq |S|$. Therefore,

$$|S_2| + m(A(S_{3,n}) - S_2) > |S| + m(A(S_{3,n}) - S). \quad (7)$$

From equations (6) and (7), we conclude that

$$|S| + m(A(S_{3,n}) - S) = \min\{|X| + m(A(S_{3,n}) - X) : X \text{ is a dominating set}\}$$

$$= DI(A(S_{3,n})).$$

Hence,

$$DI(A(S_{3,n})) = \begin{cases} 3, & n = 3, 4 \\ 4, & n = 5 \\ \left\lfloor \frac{n}{2} \right\rfloor + 2, & n \geq 6. \end{cases}$$

Theorem 2.6.

$$DI(D(S_{3,n})) = \begin{cases} n+1, \ for\ 2 \le n \le 7 \\ \left\lfloor \frac{n}{2} \right\rfloor + 5, \ n \ge 8. \end{cases}$$

Proof. Let the vertex set of $D(S_{3,n})$ be $|V(D(S_{3,n}))| = 3n - 2$. To prove this we need the following two cases.

Case (i) For $2 \le n \le 7$.

Consider $S = \{v_i \backslash i = 1 \text{ to } n\}$ as a dominating set and $|S| = n$. Moreover $m(D(S_{3,n}) - S) = 1$. Thus $|S| + m(D(S_{3,n}) - S) = n + 1$.

Now to check the minimality of $|S| + m(D(S_{3,n}) - S)$.

Let S_1 be any other dominating set with $m(D(S_{3,n}) - S) = 5$ or 3, then due to the nature of the graph $D(S_{3,n})$, we have $|S_1| \ge |S|$. So,

$$|S_1| + m(D(S_{3,n}) - S_1) > |S| + m(D(S_{3,n}) - S). \tag{8}$$

From the equation (8), we get

$$|S| + m(D(S_{3,n}) - S) = \min\{|X| + m(D(S_{3,n}) - X) : X \text{ is a dominating set}\}$$

$$= DI(D(S_{3,n})).$$

Case (ii) $n \ge 8$.

Consider the set $S = \{v_{2+2t} \backslash t = 0 \text{ to } \left\lfloor \frac{n}{2} \right\rfloor - 1\}$ will be a dominating set as $v_{1+2t}, v_{2n+2t}, v_{2n+1+2k}, v_{2n-1-2t}, v_{2n-2-2k} \in N(v_{2+2t})$ where $t = 0$ to $\left\lfloor \frac{n}{2} \right\rfloor - 1$ and $k = 0$ to $\left\lfloor \frac{n}{2} \right\rfloor - 2$ and $|S| = \left\lfloor \frac{n}{2} \right\rfloor$. Moreover, $m(D(S_{3,n}) - S) = 5$. So, $|S| + m(D(S_{3,n}) - S) = \left\lfloor \frac{n}{2} \right\rfloor + 5$. Let us consider the dominating set S_1 with $m(D(S_{3,n}) - S_1) < 5$, then due to the structure of the graph $D(S_{3,n})$, we get $|S_1| \ge |S|$. So,

$$|S_1| + m(D(S_{3,n}) - S_1) > |S| + m(A(S_{3,n}) - S). \tag{9}$$

Consider the other possible dominating set S_2 with $m(D(S_{3,n}) - S_2) > 5$, then at least one component of the graph $D(S_{3,n}) - S_2$ will have number of vertices greater than six. So $|S_2| \geq |S|$.

Thus we conclude that,

$$|S_2| + m(D(S_{3,n}) - S_2) > |S| + m(A(S_{3,n}) - S). \qquad (10)$$

From equations (9) and (10) we get

$$|S| + m(D(S_{3,n}) - S) = \min\{|X| + m(D(S_{3,n}) - X) : X \text{ is a dominating set}\}$$

$$= DI(D(S_{3,n})).$$

Hence,

$$DI(D(S_{3,n})) = \begin{cases} n+1, & for\ 2 \leq n \leq 7 \\ \left\lfloor \frac{n}{2} \right\rfloor + 5, & n \geq 8. \end{cases}$$

III. Conclusion

In modern days, we are in search of more secured and sustainable network. We have investigated the resistance of a network (i.e. vulnerability of a network) after disruption or failure of a nodes or links or both by the parameter known as Domination Integrity. We studied the domination integrity of sunlet and some Snake graphs and analyzed that the domination integrity value is less than the order of the graphs. The investigation of shadow, line, total, middle and central graph of sunlet and some snake graphs are an open area.

References

[1] T. Haynes, S. Hedetniemi and P. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, NY, USA, 1998.

[2] R. Sundareswaran and V. Swaminathan, Domination integrity in graphs, Proceedings of International Conference on Mathematical and Experimental Physics, Narosa Publishing House (2010), 46-57.

[3] R. Sundareswaran and V. Swaminathan, Domination Integrity of Middle Graphs,

Algebra, Graph Theory and their Applications, T. Tamizh Chelvam, S. Somasundaram and R. Kala eds., Narosa Publishing House (2010), 88-92.

[4] R. Sundareswaran, and V. Swaminathan, Domination Integrity of Powers of Cycles, International Journal of Mathematical Research 3(3) (2011), 257-265.

[5] R. Sundareswaran and V. Swaminathan, Domination Integrity in Trees, Bulletin of International Mathematical Virtual Institute, 2, pp. 153-161, 2012, ISSN 1840-4367.

[6] Sultan Senna Mahde and Veena Mathad, Domination Integrity of Line Splitting Graph and Central Graph of Path, Cycle and Star Graphs, AAM International Journal 11(1) (2016), 408-423.

[7] S. K. Vaidya and N. J. Kothari, Some new results on domination integrity of graphs, Open J. Discrete Math. 2 (2012), 96-98.

[8] S. K. Vaidya and N. J. Kothari, Domination integrity of Splitting graph of path and cycle, ISRN Combin. 2013, Article ID 795427, 7 pp., 2013.

[9] S. K. Vaidya and N. J. Kothari, Domination integrity of splitting and degree splitting graphs of some graphs, Advances and Applications in Discrete Mathematics 17(2) (2016), 185-199.

[10] S. K. Vaidya and N. H. Shah, Domination integrity of shadow graphs, (2013), In: Advances in Domination Theory II, (edited by V. R. Kulli), Vishwa International Publications, pp. 19-31.

[11] S. K. Vaidya and N. H. Shah, Domination integrity of total graphs, TWMS J. Appl. Engin. Math. 4(1) (2014), 117-126.

[12] S. K. Vaidya and N. H. Shah, Domination integrity of some path related graphs, Appl. Appl. Math.: An Inter. J. 9(2) (2014), 780-794.

[13] D. B. West, Introduction to Graph Theory, 2nd ed., Prentice-Hall of India, New Delhi, 2003.