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AN ITERATIVE METHOD WITH TWELFTH ORDER CONVERGENCE FOR SOLVING NON-LINEAR EQUATIONS

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Abstract

The aim of this paper is to introduce a new twelfth order iterative method for solving non-linear equations. Modification of Newton's method with higher-order convergence is presented. Analysis of convergence finalized that the order of convergence is 12. Some numerical examples illustrate that the algorithm is more efficient and performs better than classical Newton's method and other methods with the same order.

1. Introduction

Nowadays a lot of attention has been made on solving non-linear scalar equations in engineering and science. In this literature of finding a root of

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non-linear Newton's method (NR) [2] is one of the well known optimal methods to obtain the zero of a non-linear equation

$$h(t) = 0 \tag{1.1}$$

and is given by

$$t_{n+1} = t_n - (h(t_n) / h'(t_n))$$

$$n = 0, 1, 2, \ldots \tag{1.2}$$

and the NR method converges quadratically and its efficiency index is $\sqrt{2} = 1.414$.

New predictor-corrector iterative method (JA) with twelfth order convergence for solving non-linear equations proposed by Yasir, Hassan [5] is given by

$$y_n = t_n - (2h(t_n) / 3h'(t_n))$$

$$z_n = x_n - J_{f(x_n)}(h(t_n) / h'(t_n)),$$

where $J_{f(x_n)} = [(3h'(y_n) + h'(t_n)) / (6h'(y_n) - 3h'(t_n))]$

$$t_{n+1} = z_n - [(h(z_n)h'(z_n)(2 + 2h'(z_n)^2$$

$$+ h(z_n)h''(z_n))) / (2h'(z_n)^2(1 + h'(z_n)^2) - h(z_n)h''(z_n))]. \tag{1.3}$$

Twelfth order iterative method (HL) for non-linear equations proposed by Hou, Li [3] is given by

$$y_n = t_n - (h(t_n)/h'(t_n))$$

$$z_n = y_n - [(2h(t_n) - h(y_n)) / (2h(t_n) - 5h(y_n))](h(y_n) / h'(t_n))$$

$$\omega_n = z_n - [(2h(t_n) - h(z_n)) / (2h(t_n) - 5h(z_n))](h(z_n) / F(t_n)),$$

where, $F(t_n) = [(h(z_n) - h(y_n)) / (z_n - y_n)] + [(((h(z_n) - h(t_n)) / (z_n - t_n)$

$$-h'(t_n)) / (z_n - t_n)](z_n - y_n)$$

$$t_{n+1} = \omega_n - [(h(t_n) + 2h(z_n)) / (h(t_n) + 2h(z_n))](h(\omega_n) / F(t_n)). \tag{1.4}$$

A quadrature based three-step twelfth order iterative method (SK) proposed by Khattri [7] is given by

$$y_n = t_n - (h(t_n)/h'(t_n))$$

$$z_n = y_n - [((t_n - y_n)h(y_n))/(h(t_n) - 2h'(y_n))]$$

$$t_{n+1} = z_n - [h(z_n)/(h'(z_n) - h(z_n)(h(z_n - (h(z_n)/h'(z_n)))/h'(z_n)))]. \quad (1.5)$$

An iterative method (WA) for solving the twelfth order convergence proposed by Liu, Wang [9] is given by

$$y_n = t_n - (h(t_n)/h'(t_n))$$

$$z_n = y_n - [2h(t_n)/(h'(t_n) + h'(y_n))]$$

$$\omega_n = z_n - (h(z_n)/h'(z_n))$$

$$t_{n+1} = \omega_n - [(h(z_n) + 2h(\omega_n))/h(z_n)](h(\omega_n)/h'(z_n)). \quad (1.6)$$

A New twelfth order J-Halley method (JH) for solving non-linear equations proposed by Ahmad, Hussain [1] is given by

$$y_n = t_n - (2h(t_n)/3h'(t_n))$$

$$z_n = x_n - J_{f(x_n)}(h(t_n)/h'(t_n)),$$

where $J_{f(x_n)} = [(3h'(y_n) + h'(t_n))/(6h'(y_n) - 3h'(t_n))]$

$$t_{n+1} = z_n - [(2h(z_n)h'(z_n))/(2h'(z_n)^2 - h(z_n)h''(z_n))]. \quad (1.7)$$

In section 2, we described the new three-step iterative method, and section 3, we provided its convergence analysis. Finally in section 4, a comparison of our new method with other schemes using some defined examples.

2. Twelfth Order Convergent (SR) Method

Consider t^* is an exact root of (1.1) where $h(t)$ is continuous and has well defined first derivatives. Let t_n be the root of n^{th} approximation of (1.1) and is

$$t^* = t_n + \varepsilon_n, \tag{2.1}$$

where ε_n is the error. Thus, we get

$$h(t^*) = 0. \tag{2.2}$$

Writing $h(t^*)$ by Taylor's series about t_n, we have

$$h(t^*) = h(t_n) + (t^* - t_n)h'(t_n) + [(t^* - t_n)^2 \, /2!]h''(t_n) +, \ldots$$

$$h(t^*) = h(t_n) + \varepsilon_n h'(t_n) + [\varepsilon_n^2 \, /2!]h''(t_n) +, \ldots. \tag{2.3}$$

Here higher powers of ε_n are neglected that to from ε_n^3 onwards. Using (2.2) and (2.3), we have

$$\varepsilon_n^2 h''(t_n) + 2\varepsilon_n h'(t_n) + 2h(t_n) = o$$

$$\varepsilon_n = [- 2h'(t_n) \pm \sqrt{4h'(t_n) - 8h(t_n)h''(t_n)}] \, /2h''(t_n). \tag{2.4}$$

On Substituting t^* by t_{n+1} in (2.1) and from (2.4), we get

$$t_{n+1} = t_n - (2h(t_n) \, /h'(t_n))(1 + \sqrt{1 - 2\rho n})^{-1}. \tag{2.5}$$

where,

$$\rho_n = h(t_n)h''(t_n)[h'(t_n)]^{-2}$$

The second derivative in ρ_n was considered by Solaiman [6] as

$$h''(t_n) = [2/(t_{n-1} - t_n)][3[(h(t_{n-1}) - h(t_n)) \, /(t_{n-1} - t_n)] - 2h'(t_n) - h'(t_{n-1})].$$

We develop the algorithm by taking (1.2) as the first step and (2.5) as the second step and (1.2) as the third step.

Algorithm: The iterative scheme is computed by x_{n+1} as

$$z_n = t_n - (h(t_n) \, /h'(t_n))$$

$$y_n = z_n - (2h(z_n) \, /h'(z_n))(1 + \sqrt{1 - 2\rho n})^{-1}, \rho_n = h(z_n)h''(z_n)(h'(z_n))^{-2} \tag{2.6}$$

and $h'(z_n) = [2/(t_n - z_n)][3[(h(t_n) - h(z_n)) \, /(t_n - z_n)] - 2h'(z_n) - h'(t_n)]$

$$t_{n+1} = y_n - (h(y_n) \, /h'(y_n)).$$

The method (2.6) is called a twelfth order convergent method (SR), which requires three functional evaluations and three of its first derivatives.

3. Convergence Criteria

Theorem. *Let* $t_0 \in D$ *be a single zero of a sufficiently differentiable function h for an open interval D. If* t_0 *is in the neighborhood of* t^*. *Then the algorithm* (2.6) *has twelfth order convergence.*

Proof. Let the single zero of (1.1) be t^* and $t^* = t_n + \varepsilon_n$ then $h(t^*) = 0$. By Taylor's series, writing $h(t^*)$ about t_n, we obtain

$$h(t_n) = h'(t^*)(\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + c_4\varepsilon_n^4 +, \ldots \tag{3.1}$$

$$h'(t_n) = h'(t^*)[1 + 2c_2\varepsilon_n + c_3\varepsilon_n^2 + 4c_4\varepsilon_n^3 +, \ldots. \tag{3.2}$$

From the first step of (2.6), we get

$$z_n = t^* + c_2\varepsilon_n^2 + (2c_3 - 2c_2^2)\varepsilon_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)\varepsilon_n^4 + \ldots.$$

Now, we obtain

$$h''(z_n) = h'(t^*)(2c_2 + 2(3c_2c_3 - c_4)\varepsilon_n^2 - 4(3c_2^2c_3 - 3c_3^2 - c_2c_4 + c_5)\varepsilon_n^3 +, \ldots) \tag{3.4}$$

and

$$h(z_n)(h'(z_n))^{-1} = c_2\varepsilon_n^2 + (2c_3 - 2c_2^2)\varepsilon_n^3 + (3c_2^3 - 7c_2c_3 + 3c_4)\varepsilon_n^4 +, \ldots. \tag{3.5}$$

From $\rho_n = h(z_n)h''(z_n)[h'(z_n)]^{-2}$, we get

$$2\rho_n = 4c_2^2\varepsilon_n^2 + 4(6c_2c_3^2 - 2c_3c_4 - 6c_2^3c_3 + 2c_2^2c_4)\varepsilon_n^3$$

$$+ 2(-8c_2^2c_3 + 4c_2c_4 + 2c_2^4)\varepsilon_n^4 +, \ldots. \tag{3.6}$$

From (3.6), on simplification

$$(1 + \sqrt{1 - 2\rho_n})^{-1} = 2[1 + c_2^2\varepsilon_n^2 + (6c_2c_3^2 - 2c_3c_4 - 6c_2^3c_3)\varepsilon_n^3$$

$$+ (-4c_2^2c_3 + 2c_2c_4 + 6c_2^4)\varepsilon_n^4 +, \ldots]. \tag{3.7}$$

Using (3.4) and (3.7), we get

$$(2h(z_n)/h'(z_n))(1+\sqrt{1-2\rho_n})^{-1} = c_2\varepsilon_n^2 + (2c_3 - 2c_2^2)\varepsilon_n^3 + (3c_2^3 - 7c_2c_3 + 3c_4)\varepsilon_n^4$$

$$+ (16c_2^2c_3 - 10c_2c_4 - 4c_2^4 - 6c_3^2 + 4c_5)\varepsilon_n^5 + (11c_2^3c_3 - 5c_2^2c_4 - 9c_2^5 - 12c_2c_3^3$$

$$+ 4c_3^2c_4 + 12c_2^3c_3^2 - 8c_2^2c_3c_4 + 12c_2^3c_3^2 - 12c_2^5c_3 + 4c_2^4c_4)\varepsilon_n^6 + o(\varepsilon_n^7). \quad (3.8)$$

From the second step of (2.6), we get

$y_n = t^* + y$, where $y = (11c_2^3c_3 - 5c_2^2c_4 - 9c_2^5 - 12c_2c_3^3 + 4c_3^2c_4 + 12c_2^3c_2^2 - 8c_2^2c_3c_4 + 12c_2^3c_3^2 - 12c_2^5c_3 + 4c_2^4c_4)\varepsilon_n^6 + o(\varepsilon_n^7)$

$$h(y_n) = h'(t^*)(y + c_2y^2 + c_3y^3 + 4c_4y^4 +, \ldots) \quad (3.9)$$

$$h'(y_n) = h'(t^*)(1 + 2c_2y + c_3y^2 + 4c_4y^3 +, \ldots). \quad (3.10)$$

Using (3.9) and (3.10) in the third step of (2.6), we get

$$\varepsilon_{n+1} = (11c_2^4c_3 - 5c_2^3c_4 - 9c_2^6 - 12c_2^2c_3^3 + 4c_2c_3^2c_4 + 12c_2^4c_3^2 - 8c_2^3c_3c_4 + 12c_2^4c_3^2$$

$$- 12c_2^6c_3 + 4c_2^5c_4)\varepsilon_n^{12} + o(\varepsilon_n^{13}).$$

Thus, we proved the convergence of this new method which is of twelfth order and its efficiency index is $\sqrt[6]{12} = 1.513$.

4. Numerical Examples

We consider some examples considered by Vatti [8] and Mylapalli [4], and compared our method with NR, SK, WA, JA, JH, and HL methods. The computations are carried out by using mpmath-PYTHON and the number of iterations for these methods is obtained for comparisons such that $|x_{n+1} - x_n| < 10^{-201}$ and $|h(x_{n+1})| < 10^{-201}$

The test functions and simple zeros are given below:

$h_1(x) = \sin(2\cos x) - 1 - x^2 + e^{\sin(x^3)}$, $t^* = -0.7848959876612125$

$h_2(x) = \sin x + \cos x + x$, $t^* = -0.4566247045676308$

$h_3(x) = (x+2)e^x - 1$, $t^* = -0.442854010023885$

$$h_4(x) = x^2 + \sin((x/5)) - (1/4),\ t^* = 0.4099920179891371$$

$$h_5(x) = \cos x - x,\ t^* = 0.7390851332151606$$

$$h_6(x) = x^3 - 10,\ t^* = 2.1544346900318837$$

$$h_7(x) = e^{-x} + \cos x,\ t^* = 1.7461395304080124$$

$$h_8(x) = e^{\sin x} - x + 1, t^* = 2.6306641479279036.$$

A chemical equilibrium problem: Consider the equation from [6] describing the fraction of the nitrogen hydrogen feed that gets converted to ammonia (this fraction is called fractional conversion) in polynomial form as

$$h_9(x) = x^4 - 7.79075x^3 + 2.511x - 1.674,\ t^* = 0.2777595428417206.$$

Volume from van der Waals equation: One has to find out the volume from Van der Waals' equation [6] in polynomial form as $h_{10}(x) = 40x^3 - 95.26535116x^2 + 35.28x - 5.6998368, t^* = 1.9707842194070294.$

Table 4(a). Analogy of efficiency.

Methods	P	N	EI
NR	2	2	1.414
SK	12	6	1.513
WA	12	6	1.513
JA	12	6	1.513
JH	12	6	1.513
HL	12	6	1.513
SR	12	6	1.513

Where P is the convergence order, N is the number of functional values per iteration and EI is the efficiency index.

Table 4(b). Analogy of different methods.

h	Methods	x_0	n	er	fv	x_0	n	er	fv
h_1	NR	0.8	8	8.9(201)	2.4(200)	-0.5	10	8.9(201)	2.4(200)
	SK		4	6.5(201)	2.4(200)		4	6.5(201)	2.4(200)
	WA		5	8.9(201)	4.1(201)		6	4.8(201)	4.8(200)
	JA		8	3.2(201)	2.4(201)		9	4.1(201)	4.1(201)
	JH		8	4.1(201)	4.1(201)		9	4.1(201)	4.1(201)
	HL		5	2.4(201)	4.1(201)		4	4.1(201)	4.1(201)
	SR		3	4.1(201)	4.1(201)		4	4.1(201)	4.1(201)
h_2	NR	-1	8	2.4(201)	5.3(201)	-0.2	9	2.4(201)	5.3(201)
	SK		4	2.4(201)	5.3(201)		4	2.4(201)	5.3(201)
	WA		6	5.3(200)	5.3(201)		5	2(201)	5.3(201)
	JA		6	6.5(201)	5.3(201)		6	2.6(201)	5.3(201)
	JH		6	6.5(201)	5.3(201)		6	6.5(201)	5.3(201)
	HL		5	4.3(200)	1.8(200)		6	1.3(200)	5.3(201)
	SR		3	3.2(201)	1.8(200)		3	3.6(201)	5.3(201)
h_3	NR	0.7	11	6.9(201)	1.1(200)	-1.2	11	2.4(201)	4.1(201)
	SK		5	3.6(201)	4.1(201)		5	2.8(201)	1.1(200)
	WA		6	3.9(200)	4.1(201)		7	1.2(201)	4.1(201)
	JA		7	2.8(201)	4.1(201)		7	2.8(201	4.1(200)
	JH		6	8.9(201)	1.1(200)		7	6.5(201)	4.1(201)
	HL		5	1.5(200)	4.1(201)		5	2.7(200)	1.1(200)
	SR		4	6.1(201)	4.1(201)		4	6.1(201)	4.1(201)
h_4	NR	0.3	9	2.0(201)	2.2(201)	0.5	9	2.0(201)	2.2(201)
	SK		4	2.0(201)	2.2(201)		4	3.2(201)	2.2(201)
	WA		6	2.8(201)	2.2(201)		5	1.2(200)	2.2(201)
	JA		6	4.1(201)	7.7(201)		6	4.1(201)	2.2(201)
	JH		6	5.7(201)	2.2(201)		6	5.7(201)	2.2(201)
	HL		4	4.7(200)	7.7(201)		5	9.7(201)	2.2(201)
	SR		3	6.9(201)	2.2(201)		3	3.6(201)	7.7(201)

h_5	NR	0.9	9	1.6(201)	2.4(201)	-0.9	12	1.6(201)	2.4(201)
	SK		4	1.6(201)	2.4(201)		5	7.3(201)	2.4(201)
	WA		5	4.8(201)	4.1(201)		16	8.9(201)	2.4(201)
	JA		6	2.4(201)	1.3(201)		11	4.1(201)	2.4(201)
	JH		6	5.7(201)	2.4(201)		DIVERGENT		
	HL		5	4.1(201)	2.4(201)		8	2.4(201)	2.4(201)
	SR		3	4.1(201)	2.4(201)		4	4.1(201)	2.4(201)
h_6	NR	2.2	8	1.6(200)	2.0(199)	1.2	11	1.6(201)	2.0(201)
	SK		4	2.6(200)	2.0(199)		5	5.8(200)	1.2(198)
	WA		5	9.4(200)	2.0(199)		6	9.4(201)	2.0(201)
	JA		6	4.2(200)	2.0(199)		7	4.2(200)	2.0(199)
	JH		6	4.2(200)	2.0(199)		7	4.2(200)	2.0(199)
	HL		3	3.5(200)	2.0(199)		8	7.1(200)	2.0(201)
	SR		3	3.9(200)	2.0(199)		4	2.2(200)	2.0(199)
h_7	NR	1.5	9	4.8(201)	6.5(201)	1.9	8	4.8(201)	6.5(201)
	SK		4	4.8(201)	6.5(201)		4	4.8(201)	6.5(201)
	WA		5	3.1(200)	6.5(201)		5	3.1(200)	6.5(201)
	JA		6	1.3(200)	6.5(201)		6	1.3(200)	6.5(201)
	JH		6	1.3(200)	6.5(201)		6	1.3(200)	6.5(201)
	HL			Divergent			3	3.4(200)	6.5(201)
	SR		3	1.1(200)	6.5(201)		3	6.5(201)	6.5(201)
h_8	NR	1.4	13	3.5(200)	8.8(200)	2.4	8	3.5(200)	8.8(200)
	SK		5	3.5(200)	8.8(201)		4	9.7(200)	1.4(199)
	WA		DIVERGENT				5	3.2(201)	8.8(200)
	JA		DIVERGENT				6	3.2(201)	8.8(200)
	JH		10	3.2(201)	8.8(200)		6	3.2(201)	8.8(200)
	HL		5	6.5(200)	1.4(199)		3	6.5(200)	1.4(199)
	SR		4	4.2(200)	8.8(200)		3	4.2(200)	8.8(200)
	NR	0	11	8.1(201)	8.1(201)	0.3	8	8.1(201)	8.1(201)
	SK		5	8.9(201)	8.1(201)		4	7.7(201)	8.3(200)
	WA		6	5.3(201)	8.1(201)		5	6.9(201)	8.3(200)

h_9	JA		7	4.8(201)	8.3(200)		6	2.0(201)	8.1(201)
	JH		7	2.0(201)	8.1(201)		6	2.0(201)	8.1(201)
	HL		5	2.8(201)	8.1(201)			divergent	
	SR		4	2.0(201)	8.1(201)		3	2.0(201)	8.1(201)
h_{10}	NR	2.5	26	1.6(200)	2.1(198)	1.6	27	1.6(200)	2.1(198)
	SK		10	5.5(200)	1.0(197)		10	1.6(200)	2.1(198)
	WA		18	9.9(200)	2.8(198)		14	4.1(200)	2.1(198)
	JA		24	2.1(200)	1.0(197)		24	4.4(200)	2.1(198)
	JH		23	5.5(201)	2.1(201)		24	4.4(200)	2.1(198)
	HL		10	9.7(200)	1.0(197)		14	9.7(200)	1.0(197)
	SR		9	7.5(200)	2.1(201)		10	3.9(200)	2.1(198)

Where x_0 is the initial approximation, n is the number of iterations, *er* is the error and *fv* is the functional value.

5. Conclusion

In this method, we introduced the new twelfth order convergent iterative method with efficiency index 1.513. Table 4(a) compares the efficiency of different methods and the computational results in table 4(b) show the dominance of SR over NR, SK, WA, JA, and JH methods in terms of the number of iterations. Moreover, our method SR requires an either fewer or equal number of iterations when compared to the method HL, even though its efficiency index is more.

References

[1] T. Ahmad and S. Hussain, New twelfth order J-Halley method for solving non-linear equations, Open Science Journal of Mathematics and Applications 1(1) (2013), 1-4.

[2] J. F. Traub, Iterative Methods for the Solution of Equations, Chelsea Publishing Company, New York, 1977.

[3] Linke Hou and Xiaowu Li, Twelfth order iterative method for non-linear equations, IJRRAS 3(1) (2010) 30-36.

[4] M. S. K. Mylapalli, R. K. Palli and R. Sri, An optimal three-step method for solving non-linear equations, Journal of Critical Reviews 7(6) (2020), 100-103.

[5] Noori Yasir and Abdul-Hassan, New predictor-corrector iterative method with twelfth order convergence for solving non-linear equations, American Journal of App. Math. 4(4)

(2016), 175-180.

[6] Obadah S Solaiman and H. Ishak, Efficacy of optimal methods for nonlinear equations with chemical engineering applications, Mathematical Problems in Engineering (2019), 1-11.

[7] Sanjay Kumar Khattri, Another note on some quadrature based three-step iterative methods, Numerical Algebra 3(3) (2013), 549-555.

[8] V. B. K. Vatti, M. S. K. Mylapalli, S. Ramadevi and S. Deb, Two-step extrapolated Newton's method with high-efficiency index, J. Adv. Res. Dynamical and Control Systems 9(5) (2017), 8-15.

[9] Xilan Liu and Xiaoreu Wang, A family of iterative methods for solving twelfth order, Appl. Math. 4 (2013), 326-329.