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UNIT GRAPHS DERIVED FROM GROUP RINGS

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Abstract

Let RG be the group ring of the group G over a ring R and let $\mathcal{U}(RG)$ be the collection of all unit elements in a finite group ring RG. The unit graph of RG is defined as the graph $\mathcal{G}_{RG}(\mathcal{U})$, whose vertex set is the elements of RG and the distinct vertices v_x and v_y are adjacent if and only if $x + y \in \mathcal{U}(RG)$. In this paper we try to characterize the properties of unit graphs in Group rings. Also we analyse various graph theoretical parameters such as diameter, girth etc.

1. Introduction

In algebraic graph theory we have many algebraic methods to solve the problems by using graphs. Also there are so many articles in which graphs arising from rings and groups [6]. We consider the concept from group rings [1] [2] [5]. It is a very interesting algebraic structure. The central role of group ring in group representation theory was established by E. Noether and R. Brauer in the period of 1927 to 1929.

Let R be a ring and G be a group, then the group ring RG is the free R-module with basis G. That is it contains the elements of the form $\sum_{g \in G} a_g g$, where $a_g \in R$ and $g \in G$, with addition defined as $\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g$ and a product that extends the

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products of both R and G. That is, $(\sum_{g \epsilon G} a_g g)(\sum_{h \epsilon G} b_h h) = (\sum_{g, h \epsilon G} a_g b_h gh)$. RG be the set of all linear combinations of elements in G with coefficients in R. If R is commutative then RG is called group algebra. $\mathcal{U}(RG)$ denote the group of units in RG.

Ralph P Grimaldi defined a graph in Z_n, based on elements and units in Z_n. Also in 2010, N. Ashrafi defined a unit graph in rings. Based on her concept, we construct the unit graphs in finite group rings and study its characteristics.

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. If $\{u, v\}$ is a member of E then we say u and v are adjacent and this edge is denoted as uv. Two edges are said to be adjacent if they have a common vertex. If the edge set is empty it is called null (empty or void) graph [3]. The distance between two vertices u and v is the length of a shortest path joining them, and is denoted by $d(u, v)$. The diameter of a graph G, denoted by $d(G)$ is defined by $d(G) = \max_{u, v \in V(G)} d(u, v)$. The number of edges incident on a vertex v is called degree of that vertex, $d(v)$ and all the vertices of a graph are of same degree, (say) r then the graph is said to be r - regular graph. The girth of a graph G denoted by $g(G)$ is the length of a shortest cycle in G. Every two distinct pair of vertices are joined by an edge then the graph is said to be complete and is denoted by K_n with n vertices [4]. Also we make use of the following results and definitions in this paper.

Theorem 1. [3] *A non-trivial graph is bipartite if and only if it has no odd cycles.*

Definition 1.1. The homomorphism $\varepsilon : RG \to R$ given by $\varepsilon(\sum_{g \epsilon G} a_g g) = \sum_{g \epsilon G} a_g$ is called the augmentation mapping of RG and its kernel, de□noted by $\Delta(G)$, is called the augmentation ideal of RG.

Definition 1.2. For any ring R let $\mathcal{U}(R)$ denote the group of units of the ring R. Thus, we use $\mathcal{U}(RG)$ be the group of units of the group ring RG. Since the augmentation map ε is a ring homomorphism, for $u \in \mathcal{U}(RG)$, $\varepsilon(u) \in \mathcal{U}(R)$.

Let $\mathcal{U}_1(RG)$ be the subgroup of units of augmentation 1 in $\mathcal{U}(RG)$. That is $\mathcal{U}_1(RG) = \{u \in \mathcal{U}(RG)/\phi(u) = 1\}$. Hence, if u is a unit of the group ring ZG, then $\phi(u) = \pm 1$.

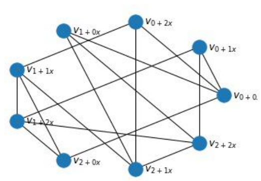
Therefore, $\mathcal{U}(ZG) = \pm \mathcal{U}_1(ZG)$. Thus for an arbitrary ring R, we have $\mathcal{U}(RG) = \mathcal{U}(R) \times \mathcal{U}_1(RG)$. This is a classical way in which we might construct units in a group ring.

2. Characterization of Unit Graphs in Group Rings

In this section we characterize the properties of unit graphs in Group Rings. Throughout this section we use $Z_n C_m$ in terms of RG, where Z_n is the ring of integers modulo n and $C_m = \langle x/x^m = 1 \rangle$.

Definition 2.1. Let RG be a group ring. We construct a graph associated with $\mathcal{U}(RG)$, the set of all unit elements of RG denoted by $\mathcal{G}_{RG}(\mathcal{U})$, is called the unit graph of RG. The set of all elements of RG are considered to be the vertices of the graph and the vertices v_x and v_y are adjacent if and only if $x + y \in \mathcal{U}(RG)$ for $x \neq y$; where v_x represents the vertices corresponding to the element $x \in RG$.

Example 1. Consider the group ring Z_3C_2; where $Z_3 = \{0, 1, 2\}$ and $C_2 = \{1, x\}$. Let $\mathcal{U}$ be the set of all unit elements in RG. Then the following figure represents the corresponding unit graph of Z_3C_2.



Theorem 2. *Let RG be a finite group ring with $R = Z_n$. Then*

(a) *If $2 \notin \mathcal{U}(RG)$, then the unit graph $\mathcal{G}_{RG}(\mathcal{U})$ is a $|\mathcal{U}(RG)|$-regular graph.*

(b) *If $2 \in \mathcal{U}(RG)$, then*

$$\deg(v_x) = \begin{cases} |\mathcal{U}(RG)| & for\, x \notin \mathcal{U}(RG) \\ |\mathcal{U}(RG)| - 1 & for\, x \in \mathcal{U}(RG) \end{cases}$$

Proof. (a) Consider RG be a finite group ring and let $\mathcal{U}(RG)$ be the set of all unit elements in RG. Let us assume that $2 \notin \mathcal{U}(RG)$. Also we know that for any $u \in \mathcal{U}(RG)$, there exist an element $x \in \mathcal{U}(RG)$, such that $x + y = u$ for some $y \in \mathcal{U}(RG); x \neq y$. Therefore v_x and v_y are adjacent in $\mathcal{G}_{RG}(\mathcal{U})$.

Now define a function $f : \mathcal{U}(RG) \to \mathcal{N}_{\mathcal{G}}(v_x)$ by $f(u) = v_y$; where $\mathcal{N}_{\mathcal{G}}(v_x)$ is the set of all adjacent vertices of v_x. That is $\mathcal{N}\mathcal{G}(v_x) = \{v_x / y \in RG$ with $x + y = u\}$. Clearly this is a bijective function. Therefore, the cardinality of $\mathcal{N}_{\mathcal{G}}(v_x) = |\mathcal{U}(RG)|$.

That is, the number of vertices adjacent to v_x is same as the total number of elements in $\mathcal{U}(RG)$ for all $x \in RG$. Hence for $2 \notin \mathcal{U}(RG); \mathcal{G}_{RG}(\mathcal{U})$ is a $|\mathcal{U}(RG)|$-regular graph.

(b) Next assume that $2 \in \mathcal{U}(RG)$ and $x \notin \mathcal{U}(RG)$. For $x \neq y, v_x$ is adjacent to v_y if and only if $x + y \in \mathcal{U}(RG)$. Therefore, from part (a), we can say that $\deg(v_x) = |\mathcal{U}(RG)|$.

Now suppose that $x \in \mathcal{U}(RG)$.

From the definition of $\mathcal{G}_{RG}(\mathcal{U})$, we have v_x is adjacent to v_y if and only if $x + y \in \mathcal{U}(RG)$. for $x \neq y$. In a group ring, for each $x \in RG$, there exist some $y \in RG$ such that $x + y = u; u \in \mathcal{U}(RG)$. Define a function $f : \mathcal{U}(RG) \to \mathcal{N}_{\mathcal{G}}(v_x)$ by $f(u) = v_x$.

Since $2 \in \mathcal{U}(RG)$, for each $x \in \mathcal{U}(RG)$, $2x \in \mathcal{U}(RG) \Rightarrow x + x \in \mathcal{U}(RG)$. But v_x is not adjacent to v_x itself in $\mathcal{G}_{RG}(\mathcal{U})$. Again since the given function is bijective, $|\mathcal{N}_{\mathcal{G}}(v_y)| = |\mathcal{U}(RG)| - 1$ for all $j \in \mathcal{U}(RG)$. That is, $\deg(v_x) = |\mathcal{U}(RG)| - 1; x \in \mathcal{U}(RG)$. Hence the proof.

Theorem 3. *Let R be a commutative ring with identity and G be a finite abelian group with $Char(R) = 2n$. Then $\mathcal{G}_{RG}(\mathcal{U})$ is a bipartite graph.*

Proof. Let R be a commutative ring with identity and G be a finite abelian group with $Char(R) = 2n$. To show that $\mathcal{G}_{RG}(\mathcal{U})$ is a bipartite graph. That is, to show that the vertex set of $\mathcal{G}_{RG}(\mathcal{U})$, that is $V[\mathcal{G}_{RG}(\mathcal{U})]$ can be divided into two disjoint set with no two elements in the same set is adjacent. Consider V_1 and V_2 be the two sets such that $V_1 \cup V_2 = V[\mathcal{G}_{RG}(\mathcal{U})]$ and $V_1 \cap V_2 = \phi$ Let us assume that $v_0 \in V_1$. For any $u_i \neq 0 \in \mathcal{U}(RG)$, $0 + u_i = u_i \in \mathcal{U}(RG)$. Therefore from the definition of unit graph, it is clear that there exist no $v_{u_i} \in V_1$ corresponding to $u_i \in \mathcal{U}(RG)$ for all $u_i \in \mathcal{U}(RG)$.

Next we have to show that for all $u_i \in \mathcal{U}(RG)$, $v_{u_i} \in V_2$. For it is enough to show that for any two elements u_{i_1} and $u_{i_2} \in \mathcal{U}(RG)$ then $u_{i_1} + u_{i_2} \notin \mathcal{U}(RG)$. Define an augmentation map $\varepsilon : RG \to R$ by $\varepsilon[\mathcal{U}(RG)] \subset \mathcal{U}(R)$. That is, ε maps unit elements of RG to unit elements of R. It follows that, for all $u \in \mathcal{U}(RG)$ has the property of elements in $\mathcal{U}(R)$. Again, since $char(R) = 2n$, is an even number, we have the sum of two unit elements is again an even number, that does not belongs to $\mathcal{U}(R)$; since $\mathcal{U}(R)$ contains the elements relatively prime to 2_n. Hence in $\mathcal{U}(R)$ also. Therefore for u_{i_1} and $u_{i_2} \in \mathcal{U}(R)$ then $u_{i_1} + u_{i_2} \notin \mathcal{U}(RG)$. Therefore from the definition of unit graphs for all $u_{i_2} \in \mathcal{U}(R)$, the corresponding vertices $v_{u_i} \in V_2$. Again since RG is a group under addition, for some non unit element $x \notin \mathcal{U}(RG)$, there exist some $u_i \in \mathcal{U}(RG)$, such that $x + u_i = u_j \in \mathcal{U}(RG)$. That implies $v_x \in V_1$. Otherwise, $v_x \in V_2$. Hence all the non unit element can also be partitioned into two sets V_1 and V_2. Therefore, in all cases, the total elements in RG can be partitioned into two, with no two elements in the same set are adjacent. Hence the resultant graph is bipartite.

Theorem 4. *Let R be a ring with $char(R) = 2^n$ and $G \cong C_{2^m}$. $\mathcal{G}_{RG}(\mathcal{U})$ is a complete bipartite graph.*

Proof. Suppose that R is a ring with $char(R) = 2^n$ and $G \cong C_{2^m}$. Then by theorem 3, we get $\mathcal{G}_{RG}(\mathcal{U})$ is a bipartite graph. Therefore, to prove this theorem, it is enough to show that the bipartite graph is complete.

Consider $R = Z_{2^n}$, then $\mathcal{U}(R)$ contains the elements relatively prime to 2^n.

Therefore,

$$|\mathcal{U}(R)| = 2^n\left(1 - \frac{1}{2}\right)$$

$$= \frac{2^n}{2}$$

$$= \frac{|Z_{2^n}|}{2}$$

Also from the augmentation map we get ε maps $\mathcal{U}(RG)$ to $\mathcal{U}(R)$. Hence $|\mathcal{U}(R)| = \frac{1}{2}|RG|$. Hence, if $V[\mathcal{G}_{RG}(\mathcal{U})] = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$, then V_2 contains the vertices corresponding to the elements of $\mathcal{U}(RG) \cdot V_1$ contains all the non-unit elements of RG. Since $char(R) = 2^n$, we have $2 \notin \mathcal{U}(RG)$. Therefore, by theorem 2, we have $\mathcal{G}_{RG}(\mathcal{U})$ is a $|\mathcal{U}(RG)|$ - regular graph. Since no two vertices in the same partition is adjacent, and each partition contains $|\mathcal{U}(RG)|$ number of elements, we get one vertex in V_1 is adjacent to all other vertices in V_2. Hence the graph is a complete bipartite graph.

Corollary 2.1. *For a group ring RG where R be a field, the only complete bipartite graph is* $\mathcal{G}_{Z_2 C_{2m}}(\mathcal{U})$.

Proof. The proof follows from theorem 4. Since 2 is a prime number, Z_2 is a field. Therefore from theorem 4, the only complete bipartite unit graph of a group ring RG is $\mathcal{G}_{Z_2 C_{2m}}(\mathcal{U})$.

3. Graph measurements of Unit Graphs in Group Rings

In this section we use some graph theoretical parameters for studying the properties of unit graphs in Group rings.

Lemma 3.1. *Let* $Z_{2^n}C_{2^m}$ *be a group ring. Then the diameter,* $d[\mathcal{G}_{Z_{2^n}C_{2^m}}(\mathcal{U})] = 2$.

Proof. By theorem 4, $\mathcal{G}_{Z_{2^n}C_{2^m}}(\mathcal{U})$ is a complete bipartite graph. And the diameter of a complete bipartite graph is 2. Hence $d[\mathcal{G}_{Z_{2^n}C_{2^m}}(\mathcal{U})] = 2$.

Theorem 5. *Let RG be a group ring and* $\mathcal{U}(RG)$ *be the set of unit element of RG. Then*

$$gr[\mathcal{G}_{RG}(\mathcal{U})] = \begin{cases} 3 & for\ 2 \in \mathcal{U}(RG) \\ 4 & for\ 2 \notin \mathcal{U}(RG) \end{cases}$$

Proof. Consider RG be a group ring and $\mathcal{U}(RG)$ be the set of unit element of RG. First to prove that for $2 \notin \mathcal{U}(RG)$, $gr[\mathcal{G}_{RG}(\mathcal{U})] = 4$. Since $2 \notin \mathcal{U}(RG)$, we have $char(R) = 2^n$. In this case we consider 2 possibilities, either $char(R) = 2^n$ or $char(R) = 2^n \cdot p$; where p is not a multiple of 2.

Case (i). If $char(R) = 2^n$, in this case, from theorem 4, we have $gr[\mathcal{G}_{RG}(\mathcal{U})]$ is a complete bipartite graph. Also from theorem 2, $gr[\mathcal{G}_{RG}(\mathcal{U})]$ is a $|\mathcal{U}(RG)|$-regular graph for $2 \notin \mathcal{U}(RG)$. Hence we have the girth of a complete bipartite graph is 4.

Case (ii). If $char(R) = 2^n \cdot p$; where p is not a multiple of 2, then from theorem 3, we have $\mathcal{G}_{RG}(\mathcal{U})$ is a bipartite graph. Then by theorem 1, $\mathcal{G}_{RG}(\mathcal{U})$ has no odd cycles and $gr(\mathcal{G}_{RG}(\mathcal{U})) \geq 2n$. Therefore to prove the theorem, it is enough to show that the length of the shortest cycle in $\mathcal{G}_{RG}(\mathcal{U})$ is 4. In this case $\mathcal{G}_{RG}(\mathcal{U})$ is a regular bipartite graph. Since the graph is regular bipartite, all the vertices in $\mathcal{G}_{RG}(\mathcal{U})$ can be partitioned into two with one portion, say V_1 contain all the vertices corresponding to the unit elements in RG. Let us take one vertex v_0 corresponds to the element $0 \in RG$, then from the definition of unit graphs, v_0 is adjacent to all the vertices vu_i for all $u_i \in \mathcal{U}(RG)$. Therefore $v_0 \in V_2$, the second partition set. If we take some $x \notin \mathcal{U}(RG)$ then $0 + x \notin \mathcal{U}(RG)$. That implies v_0 and $v_x \in V_2$ are non-adjacent vertices. If $x + u_i = u_j$ then v_x and v_{u_i} are adjacent. Again v_x is adjacent to any other v_{u_j} for some j, this v_{u_j} is adjacent to v_0 also. Hence we get a cycle $v_0 v_{u_i} v_x v_{u_j} v_0$ which is of length 4.

Again $x + y = u_j$ for some $y \in RG$ and $y \notin \mathcal{U}(RG)$. Therefore, v_x is adjacent to v_y in $\mathcal{G}_{RG}(\mathcal{U})$. Hence $v_y \in V_1$. Proceeding like this we get v_y is adjacent to some $v_y \in V_2$. Continuing the above steps we get a cycle of length greater than 4. If $y \in \mathcal{U}(RG)$. Then from above we get a cycle $v_0 v_{u_i} v_x v_y v_0$ which is of length 4. Hence for $2 \notin \mathcal{U}(RG)$. the shortest length of a cycle in $\mathcal{G}_{RG}(\mathcal{U})$ is 4.

Now to prove that for $2 \in \mathcal{U}(RG)$, $gr[\mathcal{G}_{RG}(\mathcal{U})] = 3$. From theorem 2, we have for $2 \in \mathcal{U}(RG)$,

$$\deg(v_x) = \begin{cases} |\mathcal{U}(RG)| & \textit{for } x \notin \mathcal{U}(RG) \\ |\mathcal{U}(RG)| - 1 & \textit{for } x \in \mathcal{U}(RG) \end{cases}$$

In this case, $\mathcal{G}_{RG}(\mathcal{U})$ is not a bipartite graph and also partition of the vertex set increases. Again for $x \notin \mathcal{U}(RG)$ and $\deg(v_x) = |\mathcal{U}(RG)|$, v_x is adjacent to v_{u_j} for some $u_i \in \mathcal{U}(RG)$. Again from the above process we get a cycle $v_x v_{u_i} v_0 v_{u_j} v_x$ which is of length 4.

If $x \in \mathcal{U}(RG)$ then from the definition of unit graphs, v_x is adjacent to v_0. Also v_0 is adjacent to v_{u_j} for some $x \neq u_i \in \mathcal{U}(RG)$. Now this v_{u_j} is adjacent to v_x for some i. Therefore we get a cycle, $v_x v_0 v_{u_i} v_x$, of length 3. In this case length of the shortest cycle is 3. Therefore, $gr[\mathcal{G}_{RG}(\mathcal{U})] = 3$.

Hence the theorem.

4. Conclusion

We used the unit group of group rings to create a novel graph from finite group rings in this paper. For each m and n, we additionally investigate graph theoretical measurements and characterise unit graphs. Anyone can utilise this principle in infinite dimensional group rings.

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