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FIXED POINT THEOREMS IN ORTHOGONAL FUZZY METRIC SPACES USING ALTERING DISTANCE FUNCTION

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Abstract

This paper aims to ascertain the presence of fixed point for mapping delineated on a complete orthogonal fuzzy metric space gratify a contractive condition through altering distance function.

1. Introduction

Zadeh invented the theory of fuzzy sets in [8]. The theory of fuzzy set has been developed extensively by many authors in different fields such as control theory, engineering sciences neural networks, etc. The concept of fuzzy metric space was introduced initially by Kramosil and Michalek [6]. Later on, George and Veeramani [1] presented the modified notion of fuzzy metric spaces due to Kramosil and Michalek and analysed a Hausdorff topology of fuzzy metric spaces. Recently, Gregori et al. [4] gave many interesting examples of fuzzy metrics in the sense of George and Veeramani and have also applied these fuzzy metrics to color image processing. In Grabiec [3] proved an analog of the Banach contraction theorem in fuzzy metric spaces. In his proof, he used a fuzzy version of Cauchy sequence. M. S. Khan et al. propounded a new notion of Banach fixed point theorem in metric

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spaces by introducing a control function called an altering distance function in 1984 [5]. Recently, Shen et al. [8] introduced the notion of control function in fuzzy metric space $(X, M, *)$ is given and obtained fixed point result for self-mapping T. In 2017 M. Eshaghi et al. introduced the concept of orthogonal sets in metric spaces and proved the Banach's fixed point theorem.

In this paper, we prove some fixed point theorem in the sense of Orthogonal strong fuzzy metric spaces in the contraction using altering distance function, which are the generalization of some existing results in the literature.

2. Preliminaries

The basic definitions are recalled here.

Definition 2.1. A fuzzy set $\widetilde{A}$ is defined by $\widetilde{A} = \{(x, \mu A(x)) : x \in A, \mu A(x) \in [0, 1]\}$. In the pair $(x, \mu A(x))$, the first element x belongs to the classical set A, the second element $\mu A(x)$ belongs to the interval [0, 1], is called the membership function.

Definition 2.2. A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,

(2) $*$ is continuous,

(3) $a * 1 = a$ for all $a \in [0, 1]$,

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$

Examples 2.3.

i. Lukasievicz t-norm: $a * b = \max\{a + b - 1, 0\}$

ii. Product t-norm: $a * b = a \cdot b$

iii. Minimum t-norm: $a * b = \min(a, b)$

Definition 2.4. A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on

$X \times X \times (0, \infty) \to [0, 1]$ satisfies the following conditions: $\forall x, y, z \in X$ and $s, t > 0$

(KM1) $M(x, y, 0) = 0, \forall t = 0$

(KM2) $M(x, y, t) = 1$ if and only if $x = y, t > 0$

(KM3) $M(x, y, t) = M(y, x, t)$

(KM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$

(KM5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left-continuous.

Then M is called a fuzzy metric on X.

Definition 2.5. A fuzzy metric space is an ordered triple such that X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty) \to [0, 1]$ satisfies the following conditions:

$\forall x, y, z \in X$ and $s, t > 0$

(GV1) $M(x, y, t) > 0, \forall t > 0$

(GV2) $M(x, y, t) = 1$ if and only if $x = y, t > 0$

(GV3) $M(x, y, t) = M(y, x, t)$

(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$

(GV5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Then M is called a fuzzy metric on X.

Definition 2.6. Let $(X, M, *)$ be a fuzzy metric space. M is said to be strong if it satisfies the following additional axiom:

(GV4′) $M(x, z, t) \geq M(x, y, t) * m(y, z, t), x, y, z \in X, t > 0,$ then $(X, M, *)$ is called a strong fuzzy metric space.

Definition 2.7. Let $(X, M, *)$ be a fuzzy metric space, for $t > 0$ the open ball $B(x, r, t)$ with a centre $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is topology on X, called the topology induced by the fuzzy metric M.

Definition 2.8. Let $(X, M, *)$ be a fuzzy metric space

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point x in $(X, M, *)$ if $\lim_{n \to \infty} M(x, y, t) = 1$ for all $t > 0$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence in $(X, M, *)$, if for each $0 < \in < 1$ and $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \in$ for each $n, m \geq n_0$.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

(iv) A fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.

Definition 2.9. Assume $X \neq \varnothing$ and $\perp \in X \times X$ be an binary relation. Suppose there exists $u_0 \in X$ such that $u_0 \perp u$ or $u \perp u_0$ for all $x \in X$. Thus we say that X is an orthogonal set (briefly 0-set). Further, we denote orthogonal set by $(X, \perp)$. Also, suppose that $(X, \perp)$ be an 0-set. A sequence $\{u_n\}_{n \in N}$ is called orthogonal sequence (briefly 0-sequence) if $(\forall n; u_n \perp u_{n+1})$ or $(\forall n; u_{n+1} \perp u_n)$.

Definition 2.10. A metric space (X, d) is an orthogonal metric space if $(X, \perp)$ is a 0-set. Further, $T : X \to X$ is $\perp$-continuous in $x \in X_w$ if for each 0-sequence $\{u_n\}_{n \in N}$ in X if $\lim_{n \to \infty} d(u_n, u) = 0$, then, $\lim_{n \to \infty} d(Tu_n, Tu) = 0$. Furthermore, T is $\perp$-continuous if T is $\perp$-continuous in each $u \in X$. Also, T is $\perp$-preserving if $Tu \perp Ty$ whence $x \perp y$. Finally, X is orthogonally complete (in brief 0-complete) if every Cauchy 0-sequence is convergent.

Now we introduce the notion of orthogonal fuzzy metric spaces by the following methods.

Definition 2.11. Let $(X, M, *)$ be a fuzzy metric space and $\perp \in X \times X$ be a binary relation. Assume that there exists $u_0 \in X$ such that $u_0 \perp u$ for all $u \in X$. Then we say that X is an orthogonal orthogonal fuzzy metric space. We denote orthogonal fuzzy metric by $(X, M, * \perp)$.

Definition 2.12. Let $(X, M, *, \perp)$ be an orthogonal fuzzy metric. A sequence $\{u_n\}_{n \in N}$ is called 0-sequence if $u_n \perp u_{n+1}$ for all $n \in N$. Also, $T : X \to X$ is $\perp$-continuous in $u \in X$ if for each $0I$-sequence $\{u_n\}_{n \in N}$ in X if $\lim_{n\to\infty} M(u_n, u, t) = 1$ for all $t > 0$, then, $\lim_{n\to\infty} M(Tu_n, Tu, t) = 1$ for all $t > 0$. Furthermore, T is $\perp$-continuous if T is $\perp$-continuous in each $u \in X$. Also, we say T is $\perp$-preserving if $Tu \perp Ty$ whence $x \perp y$. Finally, X is orthogonally complete (in brief 0-complete) if every Cauchy 0-sequence is convergent.

Lemma 2.13. *Let $(X, M, *)$ be a fuzzy metric space. For all $u, v \in X$, $M(u, v, \cdot)$ is non-decreasing function.*

Proof. If $M(u, v, t) > M(u, v, s)$ for some $0 < t < s$. Then $M(u, v, t) * M(v, v, s-t) \leq M(u, v, s) < M(u, v, t)$, thus $M(u, v, t) < M(u, v, t) < M(u, v, t)$, (since $M(v, v, s-t) = 1$) which is a contradiction.

Definition 2.14. A function $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function if

(i) ψ is monotone increasing and continuous

(ii) $\psi(t) = 0$ if and only if $t = 0$

Definition 2.15. A function $\varphi : [0, 1] \to [0, 1]$ is called control function or an altering distance function if it satisfies the following properties:

(AD1) φ is strictly decreasing and continuous;

(AD2) $\varphi(\lambda) \geq 0, \forall \lambda \neq 1$ if $\varphi(\lambda) = 0$ if and only if $\lambda = 1$.

It is obvious that $\lim_{\lambda\to 1^-} \varphi(\lambda) = \varphi(1) = 0$.

Definition 2.16. A function $\varphi : [0, 1] \to [0, 1]$ is called control function or an altering distance function if it satisfies the following properties:

(CF1) φ is strictly decreasing and continuous;

(CF2) $\varphi(\lambda) \geq 0, \forall \lambda \neq 1$ if $\varphi(\lambda) = 0$ if and only if $\lambda = 1$.

It is obvious that $\lim_{\lambda \to 1^-} \varphi(\lambda) = \varphi(1) = 0$.

(CF3) $\varphi(\lambda * \mu) \leq \varphi(\lambda) + \varphi(\mu), \lambda, \mu \in \{M(u, Tu, t) : u \in X, t > 0\}$

3. Main Results

Theorem 3.1. *Let* $(X, M, *, \perp)$ *be an orthogonal strong complete fuzzy metric space with continuous t-norm* $*$ *and let* T *is a self-mapping in* X. *Let there exist a function* φ *and if there exists a control function* φ *and* $(\lambda_1, 2\lambda_2 + \lambda_3)(t) < 1$ *such that*

$$\varphi(M(Tu, Tv, t)) \leq \lambda_1 \varphi(M(u, v, t)) + \lambda_2[\varphi(M(u, Tu, t)) + \varphi(M(v, Tv, t))]$$

$$+ \lambda_3 \varphi\{\min(M(u, Tu, t), M(v, Tv, t))\}. \quad (3.1.1)$$

For all $u, v \in X$ *where* $u \perp v$ *and* $u \neq v$. *Here there exist a point* $u \in X$ *and* T *is* $\perp$*-continuous at* u *then* T *has a unique fixed point in* X.

Proof. Let $(X, M, *, \perp)$ is an orthogonal strong fuzzy metric space, thus there exists $u_0 \in X$, such that $u_0 \perp v$ for all $v \in X$. Which is implies $u_0 \perp Tu_0$.

Since T is self-mapping on X for any orthogonal element u_0 such that

$$u_1 = Tu_0, u_2 = T^2 u_0, \ldots, u_n = T^n u_0 = Tu_{n-1} \forall n \in N.$$

Now T is an $\perp$-contraction then we get.

Assume that $u_{n+1} = Tu_n = u_n$ for some $n \in \mathbb{N}$, then u_n is a fixed point of T.

Suppose $u_{n+1} \neq u_n$, put $u = u_{n-1}$ and $v = u_n$ in equation (3.1.1) we get

Since T is $\perp$-preserving, hence $\{u_n\}$ is an 0-sequence and using the contraction (3.1.1)

$$\varphi(M(u_{n+1}, u_n, t)) = \varphi(M(Tu_n, Tu_{n-1}, t))$$
$$\leq \lambda_1 \varphi(M(u_n, u_{n-1}, t)) + \lambda_2[\varphi(M(u_n, Tu_n, t))$$
$$* \varphi(M(u_{n-1}, Tu_{n-1}, t))] + \lambda_3 \varphi\{\min(M(u_n, Tu_n, t)),$$
$$(M(u_{n-1}, Tu_{n-1}, t))\}$$
$$\leq \lambda_1 \varphi(M(u_n, u_{n-1}, t)) + \lambda_2[\varphi(M(u_n, u_{n+1}, t))$$
$$* \varphi(M(u_{n-1}, u_n, t))]$$
$$+ \lambda_3 \varphi\{\min(M(u_n, u_{n+1}, t)), (M(u_{n-1}, u_n, t))\}. \quad (3.1.2)$$

If

$$\min(M(u_{n-1}, u_n, t), M(u_n, u_{n+1}, t)) = M(u_n, u_{n-1}, t). \quad (3.1.3)$$

Substitute in (3.1.2)

$$\varphi(M(u_{n+1}, u_n, t)) \leq \lambda_1 \varphi(M(u_n, u_{n-1}, t)) + \lambda_2 \varphi(M(u_n, u_{n+1}, t))$$
$$+ \lambda_2 \varphi(M(u_{n-1}, u_n, t)) + \lambda_3 \varphi(M(u_{n-1}, u_n, t))$$
$$\varphi(M(u_{n+1}, u_n, t)) - \lambda_2 \varphi(M(u_n, u_{n+1}, t)) \leq \lambda_1 \varphi(M(u_n, u_{n-1}, t))$$
$$+ \lambda_2 \varphi(M(u_{n-1}, u_n, t)) + \lambda_3 \varphi M(u_{n-1}, u_n, t)$$
$$(1 - \lambda_2)\varphi(M(u_{n+1}, u_n, t)) \leq (\lambda_1 + \lambda_2 + \lambda_3)\varphi(M(u_n, u_{n-1}, t)).$$

We obtained

$$\varphi(M(u_n, u_{n+1}, t)) \leq \frac{\lambda_1 + \lambda_2 + \lambda_3}{(1 - \lambda_2)} \varphi(M(u_{n-1}, u_n, t)) < \varphi(M(u_{n-1}, u_n, t)). \quad (3.1.4)$$

Similarly,

If

$$\min(M(u_{n-1}, u_n, t)), (M(u_n, u_{n+1}, t)) = M(u_n, u_{n+1}, t). \quad (3.1.5)$$

Then the inequality (3.1.2) becomes

$$\varphi(M(u_n, u_{n+1}, t)) \leq \frac{\lambda_1 + \lambda_2}{(1 - (\lambda_2 + \lambda_3))} \varphi(M(u_{n-1}, u_n, t))$$
$$< \varphi(M(u_{n-1}, u_n, t)). \quad (3.1.6)$$

Hence

$$\varphi(M(u_n, u_{n+1}, t)) < \varphi(M(u_{n-1}, u_n, t)).$$

This gives $(M(u_n, u_{n+1}, t)) > (M(u_{n-1}, u_n, t))$.

Since the sequence $\{M(u_n, u_{n+1}, t)\}$ is non decreasing

Taking $\lim it\ n \to \infty$, we get

$$\lim_{n\to\infty} M(u_n, u_{n+1}, t) = q(r), \text{ for } q : (0, \infty) \to [0, 1]. \tag{3.1.7}$$

Suppose that $q(r) \neq 1$ for some $r > 0$ as $n \to \infty$,

Now (3.1.4) becomes,

$$\varphi(q(r)) \leq \frac{\lambda_1 + \lambda_2 + \lambda_3}{(1 - \lambda_2)} \varphi(q(r)) < \varphi(q(r)) \tag{3.1.8}$$

which is a contradiction.

Hence $\lim\limits_{n\to\infty} M(u_n, u_{n+1}, t) = 1, t > 0$

Next we prove that the sequence $\{u_n\}$ is a Cauchy's sequence.

Assume that $\{u_n\}$ is not a Cauchy 0-sequence then for any $0 < \epsilon < 1, t > 0$ then there exists sequence $\{u_{n_k}\}$ and $\{u_{m_k}\}$ where $n_k, m_k \geq n$ and $n_k, m_k \in \mathbb{N}(n_k > m_k)$ such that

$$M(u_{n_k}, u_{m_k}, t) \leq 1 - \varepsilon. \tag{3.1.9}$$

Let n_k be least integer exceeding m_k satisfying the above property.

That is $M(u_{n_k-1}, u_{m_k}, t) > 1 - \epsilon, n_k, m_k \in \mathbb{N}$ and $t > 0$

Put $u = u_{n_k-1}$ and $v = u_{m_k-1}$

$$\begin{aligned}\varphi(M(Tu_{n_k-1}, Tu_{m_k-1}, t) \leq{} & \lambda_1 \varphi(M(u_{n_k-1}, u_{m_k-1}, t)) + \lambda_2 \varphi(M(u_{m_k-1}, Tu_{m_k-1}, t)) \\ & + \lambda_2 \varphi(M(Tu_{n_k-1}, u_{n_k-1}, t)) \\ & + \lambda_3 \varphi\{\min (M(u_{n_k-1}, Tu_{n_k-1}, t), \\ & M(u_{m_k-1}, Tu_{m_k-1}, t))\}\end{aligned} \tag{3.1.10}$$

$$\varphi(M(u_{n_k}, u_{m_k}, t)) \leq \lambda_1 \varphi(M(u_{n_k-1}, u_{m_k-1}, t)) + \lambda_2 \varphi(M(u_{m_k-1}, u_{m_k-1}, t))$$

$$+ \lambda_2 \varphi(M(u_{n_k}, u_{n_k-1}, t)) + \lambda_3 \varphi\{\min (M(u_{n_k-1}, u_{n_k}, t), M(u_{m_k-1}, u_{m_k}, t))\}. \quad (3.1.11)$$

If $(M(u_{n_k-1}, u_{n_k}, t), M(u_{m_k-1}, u_{m_k}, t)) = M(u_{m_k-1}, u_{m_k}, t)$

$$\varphi(M(u_{n_k}, u_{m_k}, t)) \leq \lambda_1 \varphi(M(u_{n_k-1}, u_{m_k}, t)) + \lambda_2 \varphi(M(u_{m_k-1}, u_{m_k}, t)) + \lambda_2 \varphi(M(u_{n_k}, u_{n_k-1}, t)) + \lambda_3 \varphi(M(u_{m_k-1}, u_{m_k}, t))$$

$$\varphi(M(u_{n_k}, u_{m_k}, t)) \leq \lambda_1 \varphi(M(u_{n_k-1}, u_{m_{k-1}}, t)) + (\lambda_2 + \lambda_3)\varphi(M(u_{m_k-1}, u_{m_k}, t)) + \lambda_2 \varphi(M(u_{n_k}, u_{n_k-1}, t)). \quad (3.1.12)$$

Also (3.1.10) and (CF1) we get

$$\varphi(M(u_{n_k-1}, u_{m_k}, t)) \leq \varphi(1 - \varepsilon). \quad (3.1.13)$$

Substituting in (3.1.12) we have

$$\varphi(M(u_{n_k}, u_{m_k}, t)) \leq \lambda_1 \varphi(M(u_{n_k-1}, u_{m_k-1}, t)) + (\lambda_2 + \lambda_3) \varphi(1 - \varepsilon) + \lambda_2 \varphi(M(u_{n_k}, u_{n_k-1}, t))$$

$$\varphi(1 - \varepsilon) \leq \lambda_1 \varphi(M(u_{n_k-1}, u_{m_k-1}, t)) + (\lambda_2 + \lambda_3)\varphi(1 - \varepsilon) + \lambda_2 \varphi(M(u_{n_k}, u_{n_k-1}, t)). \quad (3.1.14)$$

Using (3.1.9) we obtain,

$$\varphi(M(u_{n_k}, u_{m_k}, t)) > \varphi(1 - \varepsilon) \quad (3.1.15)$$

Taking $k \to \infty$ in above inequality we obtain

$$(1 - \lambda_2 - \lambda_3)\varphi(1 - \varepsilon) \leq 0.$$

That is

$$(1 - \lambda_2 - \lambda_3)\varphi(1 - \varepsilon) \leq 0. \quad (3.1.16)$$

which implies that $\varepsilon = 0$ and we get a contradiction.

Hence $\{u_n\}$ is a Cauchy's sequence.

Since X is complete and there exist $z \in X$ such that $\lim_{n\to\infty} u_n = z$. Next we show that z is a fixed point of T when T is $\perp$-continuous.

That is $M(u_n, z, t) = 1$ as $n \to \infty$. Put $u = u_{n-1}$ and $v = z$ in equation (3.1.1) we get

$$\varphi(M(u_n, Tz, t)) \leq \lambda_1 \varphi(M(u_{n-1}, z, t)) + \lambda_2 \varphi(M(u_{n-1}, Tu_{n-1}, t))$$

$$+ \lambda_2 \varphi(M(z, Tz, t)) + \lambda_3 \varphi\{\min(M(u_{n-1}, u_n, t), M(z, Tz, t))\} \quad (3.1.17)$$

$$\varphi(M(u_n, z, t)) \leq \lambda_1 \varphi(M(u_{n-1}, z, t)) + \lambda_2 \varphi(M(u_{n-1}, u_n, t))$$

$$+ \lambda_2 \varphi(M(z, Tz, t)) + \lambda_3 \varphi\{\min(M(u_{n-1}, u_n, t), M(z, Tz, t))\}.$$

Taking $n \to \infty$ in (3.1.17) $(1 - \lambda_2 - \lambda_3)\varphi(M(z, Tz, t)) \leq 0, t > 0$.

Therefore $M(z, Tz, t) = 1$, and $z = Tz$. To prove uniqueness, suppose that w is another fixed point of T, that is $Tw = w$ where $q \neq z$

$$\varphi(M(Tz, Tw, t)) \leq \lambda_1 \varphi(M(z, w, t)) + \lambda_2 (M(z, Tz, t))$$

$$+ \lambda_2 \varphi(M(w, Tw, t)) + \lambda_3 \varphi\{\min(M(w, Tw, t), M(z, Tz, t))\}$$

$$\varphi(M(z, w, t)) \leq \lambda_1 \varphi(M(z, w, t)) + \lambda_2 \varphi(M(z, z, t))$$

$$+ \lambda_2 \varphi(M(w, w, t)) + \lambda_3 \varphi\{\min(M(w, w, t), M(z, z, t))\}$$

$$(1 - \lambda_1)\varphi(M(z, w, t)) \leq 0.$$

Hence $z = w$ is the unique fixed point of T.

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