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AN INTRODUCTION OF Iso-CYCLE PACKING GRAPH

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Abstract

Several tools are employed in graph theoretic research, such as covering, matching and packing. The generic concepts of packing arose by abstracting properties of isomorphism between sub graphs of a given graph. In this paper, a new packing problem called iso-cycle packing (ICP) is introduced and some of the basic properties are studied.

1. Introduction

The concepts of a packing and a covering are related to extremal problems involving the search for packings and coverings (for any given triple (V, E, Γ) that provide an extremum for some functional. Vertex independent sets and vertex coverings as also edge-independent sets and edge coverings of graphs occur very naturally in many practical situations and hence have several potential applications. Matching in bipartite graphs have varied applications in operations research. The concepts of covering and independent sets of a graph arise very naturally in practical problems. Suppose we want to store a set of chemicals in different rooms. Naturally, we would like to store incompatible chemicals, that is, chemicals that are likely to react violently when brought together, in distinct rooms. Let G be a graph whose vertex set represents the set of chemicals and let two vertices be made adjacent in G if and only if the corresponding chemicals are incompatible.

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Then any set of vertices representing compatible chemicals forms an independent set of G. Now consider the graph G whose vertices represent the various locations in a factory and whose edges represent the pathways between pairs of such locations. A light source placed at a location supplies light to all the pathways incident to that location. A set of light sources that supplies light to all the pathways in the factory forms a covering of G [3].

A generalization of the matching problem is to find in a given graph G as many disjoint sub graphs as possible that are each isomorphic to an element of a given class H of graphs. This is known as the packing problem. It is related to the covering problem, which asks how few covering vertices of G suffice to meet all its sub graphs isomorphic to a graph in H: clearly, we need at least as many vertices for such a cover as the maximum number k of H-graphs that we can pack disjointly into G. If there is no cover by just k vertices, perhaps there is always a cover by at most $f(k)$ vertices, where $f(k)$ may depend on H but not on G [11].

In geometry, circle packing is the study of the arrangement of circles (of equal or varying sizes) on a given surface such that no overlapping occurs and so that no circle can be enlarged without creating an overlap. Based on this motivation we introduced a Fibonacci type of iso cycle packing graph and studied its properties.

2. Preliminaries

Let us recall the basic definition and results. Let $G = (V, E)$ be a graph. A matching of G is a subset $M \subset E$, no two edges of which are adjacent. If $e = uv \in M$, then u and v are matched vertices, covered by M. If $o(M) = r$, then M is an r matching. The matching number, $\mu(G)$, is the largest value of r for any r-matching M of G. A subset of $V(G)$ is independent if no two of its vertices share on edge. Let G be a graph on n vertices. A perfect matching is an r-matching for which $2r = n$. A vertex is saturated by a matching M if it is incident to an edge of the matching; otherwise, x is called unsaturated. A subset $B \subset V(G)$ is a covering of G if every edge of G is incident with a vertex of B. The covering number, $\beta(G)$ is the smallest number of vertices that cover $G = (V, E)$ [12].

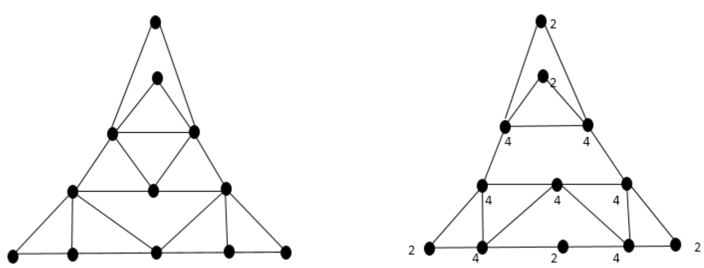
Fleury's Algorithm is used to display the Euler path or Euler circuit from a given graph. In this algorithm, starting from one edge, it tries to move other adjacent vertices by removing the previous vertices. Using this method, the graph becomes simpler in each step to find the Euler path or circuit [1, 9].

There is a function $f : N \to R$ such that, given any $k \in N$, every graph contains either k disjoint cycles or a set of at most $f(k)$ vertices meeting all its cycles (Erdos and Posa 1965) [11].

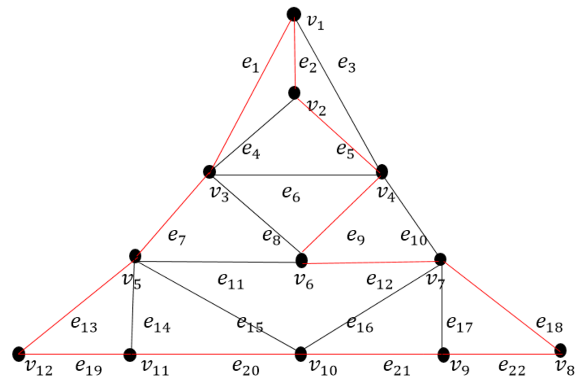
The Fibonacci numbers, commonly denoted f_n form a sequence, called the Fibonacci sequence, such that each number is the sum of the preceding ones. That is $f_0 = 1$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$.

3. A New Type of Fibonacci Sequence Based Graph (FSG)

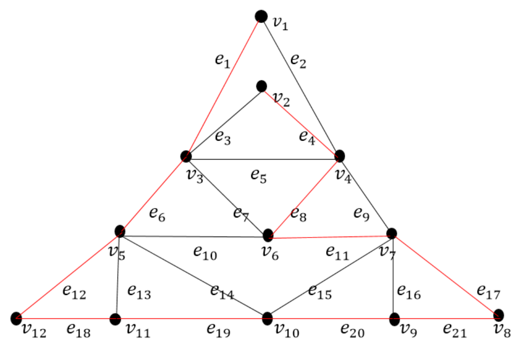
A graph $G = (V, E)$ based on Fibonacci Sequence is introduced. Here the Graph G contains the vertex set is $V = \bigcup_{i=1}^{\infty} V_i$, where V_i consists of f_i number of vertices, f_i is i^{th} term of the Fibonacci Sequence. Here, we consider still f_5. Therefore, $V = \bigcup_{i=1}^{5} V_i$. Hence the vertex set V contains 12 vertices. Therefore for 12 vertices in graph G then there exists at most $12C_2$ edges. Further in this graph has minimum 19 edges and maximum 22 edges.



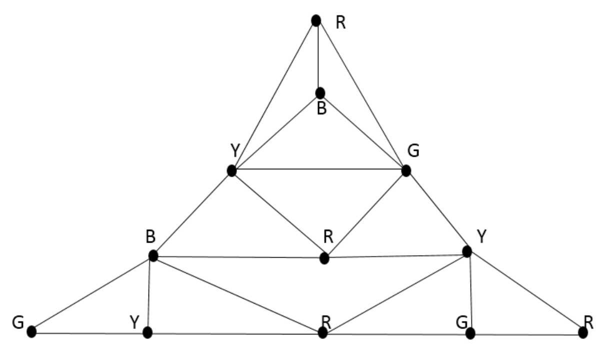
(a) Connected Graph G_1. (b) Eulerian G_2.



(c) Hamiltonian Circuit G_3.



(d) Hamiltonian Path G_4.



(e) Four Colourable Graph G_5.

Figure 1. Different Types of Fibonacci Sequence Graphs (a, b, c, d, e) with 12 vertices.

Table 1. Types of Graphs with 12 vertices.

Graphs	No. of Vertices	No. of Edges	Description
G_1	12	21	G_1 is connected graph. Since every pair of vertices contains at least one path.
G_2	12	19	G_2 is Eulerian graph. Because each vertex has even degree.
G_3	12	22	G_3 is Hamiltonian circuit. Since the walk in G_3 $v_1e_1v_3e_7\,v_5\,e_{13}v_{12}e_{22}v_{11}e_{21}v_{10}e_{20}v_9e_{19}$ $v_8e_{18}v_7\,e_{11}v_6e_9v_4e_5v_2e_3v_1$ is a closed walk, every vertex of G_3 exactly once.
G_4	12	21	G_4 is Hamiltonian path. Since the path $v_1e_1v_3e_6\,v_5\,e_{12}v_{12}e_{21}v_{11}e_{20}v_{10}e_{19}v_9e_{18}$ $v_8e_{17}v_7\,e_{10}v_6e_8v_4e_4v_2$. This path contains every vertex of G_4 exactly once.
G_5	12	22	G_5 is 4-colourable graph. Since colouring is done by using 4 colours.

3.1. Fleury's Algorithm

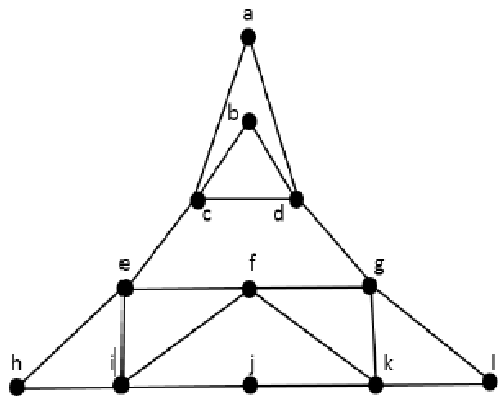


Figure 2. Eulerian Circuit G_6.

Table 2. Fleury's Algorithm 1 to G_6.

Edge Number	Current Path	Next Edge	Reasoning
1	A	(a, c)	No edge from a is a bridge choose anyone.
2	a, c	(c, b)	No edge from c is a bridge choose anyone.
3	a, c, b	(b, d)	No edge from b is a bridge choose anyone.
4	a, c, b, d	(d, c)	No edge from d is a bridge choose anyone.
5	a, c, b, d, c	(c, e)	No edge from c is a bridge choose anyone.
6	a, c, b, d, c, e	(e, f)	No edge from e is a bridge choose anyone.
7	a, c, b, d, c, e, f	(f, g)	No edge from f is a bridge choose anyone.
8	a, c, b, d, c, e, f, g	(g, k)	No edge from g is a bridge choose anyone.
9	a, c, b, d, c, e, f, g, k	(k, f)	(k, f) is the only edge.
10	a, c, b, d, c, e, f, g, k, f	(f, i)	(f, i) is the only edge.
11	a, c, b, d, c, e, f, g, k, f, i	(i, e)	(i, e) is the only edge.
12	a, c, b, d, c, e, f, g, k, f, i, e	(e, h)	(e, h) is the only edge.
13	a, c, b, d, c, e, f, g, k, f, i, e, h	(h, i)	(h, i) is the only edge.

14	a, c, b, d, c, e, f, g, k, f, i, e, h, i	(i, j)	No edge from i is a bridge choose anyone.
15	a, c, b, d, c, e, f, g, k, f, i, e, h, i, j	(j, k)	(j, k) is the only edge.
16	a, c, b, d, c, e, f, g, k, f, i, e, h, i, j, k	(k, l)	(k, l) is the only edge.
17	a, c, b, d, c, e, f, g, k, f, i, e, h, i, j, k, l	(l, g)	(l, g) is the only edge.
18	a, c, b, d, c, e, f, g, k, f, i, e, h, i, j, k, l, g	(g, d)	(g, d) is the only edge.
19	a, c, b, d, c, e, f, g, k, f, i, e, h, i, j, k, l, g, d	(d, a)	(d, a) is the only edge.
20	a, c, b, d, c, e, f, g, k, f, i, e, h, i, j, k, l, g, d, a	No edge is remaining	Halt

Table 3. Fleury's Algorithm 2 to G_6.

Edge Number	Current Path	Next Edge	Reasoning
1	A	(a, c)	No edge from a is a bridge choose anyone.
2	a, c	(c, b)	No edge from c is a bridge choose anyone.
3	a, c, b	(b, d)	No edge from b is a bridge choose anyone.
4	a, c, b, d	(d, c)	No edge from d is a bridge choose anyone.
5	a, c, b, d, c	(c, e)	No edge from c is a bridge choose anyone.
6	a, c, b, d, c, e	(e, h)	(e, h) is the only edge.
7	a, c, b, d, c, e, h	(h, i)	(h, i) is the only edge.
8	a, c, b, d, c, e, h, i	(i, e)	No edge from i is a bridge choose anyone.
9	a, c, b, d, c, e, h, i, e	(e, f)	(e, f) is the only edge.
10	a, c, b, d, c, e, h, i, e, f	(f, i)	(f, i) is the only edge.
11	a, c, b, d, c, e, h, i, e, f, i	(i, j)	(i, j) is the only edge.
12	a, c, b, d, c, e, h, i, e, f, i, j	(j, k)	(j, k) is the only edge.
13	a, c, b, d, c, e, h, i, e, f, i, j, k	(k, f)	(k, f) is the only edge.
14	a, c, b, d, c, e, h, i, e, f, i, j, k, f	(f, g)	(f, g) is the only edge.
15	a, c, b, d, c, e, h, i, e, f, i, j, k, f, g	(g, k)	(g, k) is the only edge.
16	a, c, b, d, c, e, h, i, e, f, i, j, k, f, g, k	(k, l)	(k, l) is the only edge.

17	a, c, b, d, c, e, h, i, e, f, i, j, k, f, g, k, l	(l, g)	(l, g) is the only edge.
18	a, c, b, d, c, e, h, i, e, f, i, j, k, f, g, k, l, g	(g, d)	(g, d) is the only edge.
19	a, c, b, d, c, e, h, i, e, f, i, j, k, f, g, k, l, g, d	(d, a)	(d, a) is the only edge.
20	a, c, b, d, c, e, h, i, e, f, i, j, k, f, g, k, l, g, d, a	No edge is remaining	Halt.

3.2. Pyramid Fibonacci Graph (PFG)

Now we study graph parameters like cycle packing, the isomorphism between sub graphs, iso-cycle packing and packing number for PFG. These parameters in graph theory representing various physical system like the molecular structure of a chemical compound, the interconnections in an electrical network, a railway network or a road network.

Structure of PFG

In this section, we define a new graph called a Pyramid Fibonacci Graph (PFG). Pyramid Fibonacci Graph is an undirected graph. The Pyramid Fibonacci Graph (PFG) is constructed as follows. The vertex set is defined by $V = \bigcup_{i=1}^{5} V_i$ where V_i consists of f_{i+3} number of vertices and f_i is the i^{th} term of the Fibonacci Sequence. For $i \geq 2$ the vertex set V_i is split into two types. If V_i is odd then V_i split into 3 rows and if V_i is even then V_i split into the equal number of vertices in 2 rows. The edge set is obtained by the following process. Each vertex set V_i is connected by a cycle and it is interesting to note that the number of edges in each cycle follows the Fibonacci Sequence $\{3, 5, 8, \ldots\}$. The remaining edges in PFG are given by the following: Vertices are adjacent v_2 and v_4, v_3 and v_5, v_7 and v_9, v_8 and v_{13}, v_{13} and v_{17}, v_{16} and v_{22}, v_{24} and v_{30}, v_{29} and v_{39}.

Definition 3.3. Cycle Packing. A graph G with copies of cycle graphs is called the cycle graph packing. Let G be a family of graphs. Formally, a cycle packing of graph H is a set of vertex disjoint or edge disjoint.

Definition 3. 4. Iso-Cycle Packing Graph. Let G be a graph with n vertices. Every subgraph of G (vertex-disjoint or edge-disjoint) is isomorphic to only cycle graph is called iso-cycle packing graph. Moreover, the cycle of the length not necessarily the same because the different length of cycle packing in the graph.

Now we have introduction a new Pyramid Fibonacci Graph (PFG). In this graph, every vertex disjoint subgraph is isomorphic to the different length of the cycle graph. This packing is called an iso-cycle packing graph.

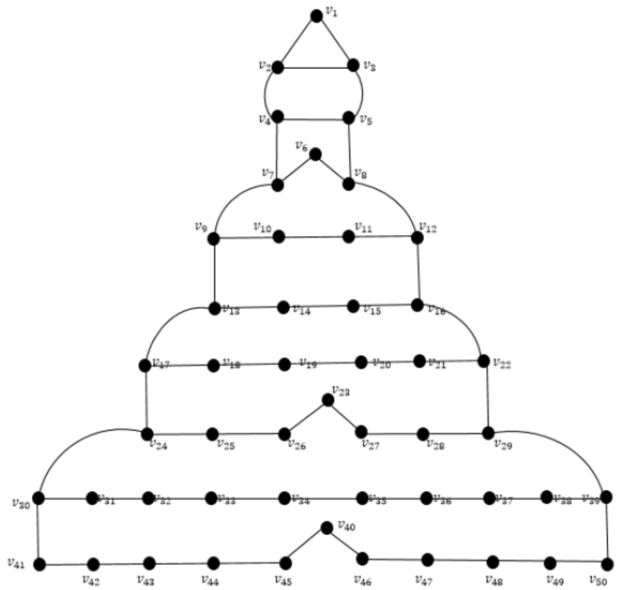


Figure 3. Pyramid Fibonacci Graph (PFG).

Theorem 3.5. Let G be a Pyramid Fibonacci Graph (PFG) whose vertex set is a Fibonacci Sequence. Let H be a subgraph of G. If H is isomorphic to C_{F_n}, where C_{F_n} is a cycle graph whose length n^{th} Fibonacci Sequence. Then the packing number is

$$M(G, H) = \sum_{n=4}^{k} \left(\frac{C_{F_n}}{F_n} \right) = k - 3.$$

Where F_n is the n^{th} Fibonacci Sequence.

Proof. We prove this theorem by induction on the level of the Pyramid Fibonacci Graph (PFG).

Base case: $k = 4$, that is level one contains the Fibonacci Sequence $F_4 = 3$.

H is isomorphic to $C_{F_4} = C_3$.

$$M(G, H) = 1 = \frac{C_{F_4}}{F_4} = \frac{C_3}{3} = \frac{3}{3} = 4 - 3.$$

Hence the packing is perfect packing.

Assume that the theorem is true for $k-1$, that is $M(G, H) = \sum_{n=4}^{k-1} \frac{C_{F_n}}{F_n} = k - 1 - 3 = k - 4.$

Hence

$$M(G, H) = \sum_{n=4}^{k} \frac{C_{F_n}}{F_n}$$

$$= \frac{C_{F_4}}{F_4} + \frac{C_{F_5}}{F_5} + \ldots + \frac{C_{F_{k-1}}}{F_{k-1}} + \frac{C_{F_k}}{F_k}$$

$$= \sum_{n=4}^{k-1} \frac{C_{F_n}}{F_n} + \frac{C_{F_k}}{F_k}$$

$$= k - 4 + 1$$

$\therefore M(G, H) = k - 3.$

Theorem 3.6. *The following statements are equivalent to the PFG.*

1. *PF G is a block.*

2. *Any two vertices of PFG lie on a common cycle.*

3. *Any point and any edge of PFG lie on a common cycle.*

4. *Any two edges of PFG lie on a common cycle.*

Proof. From the construction of PFG, it can be easily shown that the above statements are equivalent.

Definition 3.7. Let $G_1, G_2, \ldots, G_k$ be k copies of a graph PFG whose orders f_4, f_5, f_{i+3}, where f_i is the i^{th} term of the Fibonacci Sequence. We say that there is packing of $G_1, G_2, \ldots, G_k$ (into the complete graph K_n, where $n = | V(PFG_i) |$ if there exists k permutations $\sigma_i : V(PFG_i) \to V(K_n), i = 1, 2, \ldots, k,$ such that $\sigma_i^*(E(PFG_i)) \cap \sigma_j^*(E(PFG_j)) = \varnothing$ for $i \neq j$, where the mapping $\sigma_i^* : (E(PFG_i)) \to E(K_n)$ is the one induced by σ_i.

Observation 3.8.

1. Every permutation of a finite graph PFG can be written as a cycle or as a product of disjoint cycles.

2. If the pair of cycles of PFG $C_i = (v_1, v_2, \ldots, v_k)$ and $C_j = (w_1, w_2, \ldots, w_l)$ have no entries in common, than $C_i C_j = C_j C_i$.

3. The order of a permutation of a finite graph PFG written in disjoint cycle form is the least common multiple of the lengths of cycles (That is, $\operatorname{lcm}(f_4, f_5, \ldots, f_{i+3})$.

Theorem 3.9. *Let $G_1, G_2, \ldots, G_k$ be k copies of a graph PFG whose orders $f_4, f_5, \ldots, f_{i+3}$, where f_i is the i^{th} term of the Fibonacci Sequence. If $| E(PFG_i) | \leq | V(PFG_i) | - k$, then $G_1, G_2, \ldots, G_k$ are packable into k_n, where $n = | V(PFG_i) |$.*

Proof. Proof follows from Conjecture of Bollobas and Eldridge [4].

4. Conclusion

Different types of Fibonacci Sequence Graph (FSG) with 12 vertices and Pyramid Fibonacci Graph (PFG) are introduced. An iso-cycle packing theorem and equivalent relationship are proved for PFG. Future work aims to generalize PFG and to extend general graph theory results in PFG.

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