

LEVEL OPERATORS ON BIPOLE INTUITIONISTIC FUZZY α -IDEAL AND BIPOLE INTUITIONISTIC ANTI FUZZY α -IDEAL OF A BP-ALGEBRA

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Abstract

The concept of a bipolar intuitionistic fuzzy α -ideal and bipolar intuitionistic anti fuzzy α -ideal are a new algebraic structure of BP-algebra and to use level operators. The purpose of this study is to implement the fuzzy set theory and ideal theory of a BP-algebra. The relation between the operation of level operators on bipolar intuitionistic fuzzy α -ideal and bipolar intuitionistic anti fuzzy α -ideal are established.

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1. Introduction

The concept of fuzzy sets was initiated by L. A. Zadeh [14] then it has become a vigorous area of research in engineering, medical science, graph theory. S. S. Ahn [2] gave the idea of BP-algebra. Bipolar valued fuzzy sets was introduced by K. J. Lee [6] are an extension of fuzzy sets whose positive membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. In a bipolar valued fuzzy set, the membership degree 0 means that the elements are irrelevant to the corresponding property, the positive membership degree $(0, 1]$ indicates that elements somewhat satisfy the property and the negative membership degree $[-1, 0)$ indicates that elements somewhat satisfy the implicit counter property. The author W. R. Zhang [15] commenced the concept of bipolar fuzzy sets as a generalization of fuzzy sets in 1998. K. Chakrabarty and Biswas R. Nanda [3] investigated note on union and intersection of intuitionistic fuzzy sets. A. Rajeshkumar [13] was analyzed fuzzy groups and level subgroups. M. Palanivelrajan, K. Gunasekaran and S. Nandakumar [12] introduced the level operators on intuitionistic fuzzy primary ideal and semiprimary ideal. K. Gunasekaran, S. Nandakumar and S. Sivakaminathan [16] introduced the definition of bipolar intuitionistic fuzzy α -ideal of a BP-algebra.

2. Preliminaries

Definition 1. Let A and B be any two bipolar intuitionistic fuzzy set $A = (\mu_{\alpha_A}^P, \mu_{\alpha_A}^N, v_{\alpha_A}^P, v_{\alpha_A}^N)$ and $B = (\mu_{\alpha_B}^P, \mu_{\alpha_B}^N, v_{\alpha_B}^P, v_{\alpha_B}^N)$ in X , we define

$$(i) \quad A \cap B = \{(x, \min(\mu_{\alpha_A}^P(x), \mu_{\alpha_B}^P(x)), \max(\mu_{\alpha_A}^N(x), \mu_{\alpha_B}^N(x)),$$

$$\max(v_{\alpha_A}^P(x), v_{\alpha_B}^P(x)), \min(v_{\alpha_A}^N(x), v_{\alpha_B}^N(x))/x \in X\}$$

$$(ii) \quad A \cup B = \{(x, \max(\mu_{\alpha_A}^P(x), \mu_{\alpha_B}^P(x)), \min(\mu_{\alpha_A}^N(x), \mu_{\alpha_B}^N(x)),$$

$$\min(v_{\alpha_A}^P(x), v_{\alpha_B}^P(x)), \max(v_{\alpha_A}^N(x), v_{\alpha_B}^N(x))/x \in X\}$$

$$(iii) \quad \bar{A} = \{(x, v_{\alpha_A}^P(x), v_{\alpha_A}^N(x), \mu_{\alpha_A}^P(x), \mu_{\alpha_A}^N(x))/x \in X\}.$$

Definition 2. A bipolar intuitionistic fuzzy set $A = \{(x, \mu_{\alpha_A}^P(x), \mu_{\alpha_A}^N(x), v_{\alpha_A}^P(x), v_{\alpha_A}^N(x))/x \in X\}$, of BP-algebra X is called a bipolar intuitionistic fuzzy α -ideal of X if it satisfies the following conditions:

- (i) $\mu_{\alpha_A}^P(0) \geq \mu_{\alpha_A}^P(x)$ and $\mu_{\alpha_A}^N(0) \leq \mu_{\alpha_A}^N(x)$
- (ii) $\mu_{\alpha_A}^P(y * z) \geq \min\{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\}$
- (iii) $\mu_{\alpha_A}^N(y * z) \leq \max\{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * y)\}$
- (iv) $v_{\alpha_A}^P(0) \leq v_{\alpha_A}^P(x)$ and $v_{\alpha_A}^N(0) \geq v_{\alpha_A}^N(x)$
- (v) $v_{\alpha_A}^P(y * z) \leq \max\{v_{\alpha_A}^P(x * z), v_{\alpha_A}^P(x * y)\}$
- (vi) $v_{\alpha_A}^N(y * z) \geq \min\{v_{\alpha_A}^N(x * z), v_{\alpha_A}^N(x * y)\}$, for all $x, y, z \in X$.

Definition 3. A bipolar intuitionistic fuzzy set $A = \{(x, \mu_{\alpha_A}^P(x), \mu_{\alpha_A}^N(x), v_{\alpha_A}^P(x), v_{\alpha_A}^N(x))/x \in X\}$, of BP-algebra X is called a bipolar intuitionistic anti fuzzy α -ideal of X if it satisfies the following conditions:

- (i) $\mu_{\alpha_A}^P(0) \leq \mu_{\alpha_A}^P(x)$ and $\mu_{\alpha_A}^N(0) \geq \mu_{\alpha_A}^N(x)$
- (ii) $\mu_{\alpha_A}^P(y * z) \leq \max\{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\}$
- (iii) $\mu_{\alpha_A}^N(y * z) \geq \min\{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * y)\}$
- (iv) $v_{\alpha_A}^P(0) \geq v_{\alpha_A}^P(x)$ and $v_{\alpha_A}^N(0) \leq v_{\alpha_A}^N(x)$
- (v) $v_{\alpha_A}^P(y * z) \geq \min\{v_{\alpha_A}^P(x * z), v_{\alpha_A}^P(x * y)\}$
- (vi) $v_{\alpha_A}^N(y * z) \leq \max\{v_{\alpha_A}^N(x * z), v_{\alpha_A}^N(x * y)\}$, for all $x, y, z \in X$.

Definition 4. Let A is a bipolar intuitionistic fuzzy set of X , then the

level operator $!$ is defined by $!A = \left\{ \left(x, \max\left(\frac{1}{2}, \mu_{\alpha_A}^P(x)\right), \min\left(\frac{-1}{2}, \mu_{\alpha_A}^N(x)\right), \right. \right.$
 $\left. \left. \min\left(\frac{1}{2}, v_{\alpha_A}^P(x)\right), \max\left(\frac{-1}{2}, v_{\alpha_A}^N(x)\right) \right) / x \in X \right\} = L_1$.

Definition 5. Let A is a bipolar intuitionistic fuzzy set of X , then the level operator $?$ is defined by $?A = \left\{ \left(x, \min\left(\frac{1}{2}, \mu_{\alpha_A}^P(x)\right), \max\left(\frac{-1}{2}, \mu_{\alpha_A}^N(x)\right), \right. \right.$
 $\left. \left. \max\left(\frac{1}{2}, v_{\alpha_A}^P(x)\right), \min\left(\frac{-1}{2}, v_{\alpha_A}^N(x)\right) \right) / x \in X \right\} = L_2$.

3. Level Operators on Bipolar Intuitionistic Fuzzy α -Ideal

Theorem 1. If A is a bipolar intuitionistic fuzzy α -ideal of X , then $!A$ is a bipolar intuitionistic fuzzy α -ideal of X .

Proof. Given A is a bipolar intuitionistic fuzzy α -ideal of X .

Consider $0, x, y, z \in A$.

$$\begin{aligned} \text{(i) Now } \mu_{\alpha!A}^P(0) &= \max\left(\frac{1}{2}, \mu_{\alpha_A}^P(0)\right) \\ &\geq \max\left(\frac{1}{2}, \mu_{\alpha_A}^P(x)\right) \\ &= \mu_{\alpha!A}^P(x) \end{aligned}$$

Therefore $\mu_{\alpha!A}^P(0) \geq \mu_{\alpha!A}^P(x)$

$$\begin{aligned} \text{Now } \mu_{\alpha!A}^N(0) &= \min\left(\frac{-1}{2}, \mu_{\alpha_A}^N(0)\right) \\ &\leq \min\left(\frac{-1}{2}, \mu_{\alpha_A}^N(x)\right) \\ &= \mu_{\alpha!A}^N(x) \end{aligned}$$

Therefore $\mu_{\alpha!A}^N(0) \leq \mu_{\alpha!A}^N(x)$

$$\begin{aligned}
\text{(ii) Now } \mu_{\alpha!A}^P(y * z) &= \max\left(\frac{1}{2}, \mu_{\alpha_A}^P(y * z)\right) \\
&\geq \max\left(\frac{1}{2}, \min\{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\}\right) \\
&= \min\left\{\max\left(\frac{1}{2}, \mu_{\alpha_A}^P(x * z)\right), \max\left(\frac{1}{2}, \mu_{\alpha_A}^P(x * y)\right)\right\} \\
&= \min\{\mu_{\alpha!A}^P(x * z), \mu_{\alpha!A}^P(x * y)\}
\end{aligned}$$

Therefore $\mu_{\alpha!A}^P(y * z) \geq \min\{\mu_{\alpha!A}^P(x * z), \mu_{\alpha!A}^P(x * y)\}$

$$\begin{aligned}
\text{(iii) Now } \mu_{\alpha!A}^N(y * z) &= \min\left(\frac{-1}{2}, \mu_{\alpha_A}^N(y * z)\right) \\
&\leq \min\left(\frac{-1}{2}, \max\{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * y)\}\right) \\
&= \max\left\{\min\left(\frac{-1}{2}, \mu_{\alpha_A}^N(x * z)\right), \min\left(\frac{-1}{2}, \mu_{\alpha_A}^N(x * y)\right)\right\} \\
&= \max\{\mu_{\alpha!A}^N(x * z), \mu_{\alpha!A}^N(x * y)\}
\end{aligned}$$

Therefore $\mu_{\alpha!A}^N(y * z) \leq \max\{\mu_{\alpha!A}^N(x * z), \mu_{\alpha!A}^N(x * y)\}$

$$\begin{aligned}
\text{(iv) Now } v_{\alpha!A}^P(0) &= \min\left(\frac{1}{2}, v_{\alpha_A}^P(0)\right) \\
&\leq \min\left(\frac{1}{2}, v_{\alpha_A}^P(x)\right) \\
&= v_{\alpha!A}^P(x)
\end{aligned}$$

Therefore $v_{\alpha!A}^P(0) \leq v_{\alpha!A}^P(x)$

$$\begin{aligned}
\text{Now } v_{\alpha!A}^N(0) &= \max\left(\frac{-1}{2}, v_{\alpha_A}^N(0)\right) \\
&\geq \max\left(\frac{-1}{2}, v_{\alpha_A}^N(x)\right)
\end{aligned}$$

$$= v_{\alpha!A}^N(x)$$

Therefore $v_{\alpha!A}^N(0) \geq v_{\alpha!A}^N(x)$

$$\begin{aligned} \text{(v) Now } v_{\alpha!A}^P(y * z) &= \min\left(\frac{1}{2}, v_{\alpha_A}^P(y * z)\right) \\ &\leq \min\left(\frac{1}{2}, \max\{v_{\alpha_A}^P(x * z), v_{\alpha_A}^P(x * y)\}\right) \\ &= \max\left\{\min\left(\frac{1}{2}, v_{\alpha_A}^P(x * z)\right), \min\left(\frac{1}{2}, v_{\alpha_A}^P(x * y)\right)\right\} \\ &= \max\{v_{\alpha!A}^P(x * z), v_{\alpha!A}^P(x * y)\} \end{aligned}$$

Therefore $v_{\alpha!A}^P(y * z) \leq \max\{v_{\alpha!A}^P(x * z), v_{\alpha!A}^P(x * y)\}$

$$\begin{aligned} \text{(vi) Now } v_{\alpha!A}^N(y * z) &= \max\left(\frac{-1}{2}, v_{\alpha_A}^N(y * z)\right) \\ &\geq \max\left(\frac{-1}{2}, \min\{v_{\alpha_A}^N(x * z), v_{\alpha_A}^N(x * y)\}\right) \\ &= \min\left\{\max\left(\frac{-1}{2}, v_{\alpha_A}^N(x * z)\right), \max\left(\frac{-1}{2}, v_{\alpha_A}^N(x * y)\right)\right\} \\ &= \min\{v_{\alpha!A}^N(x * z), v_{\alpha!A}^N(x * y)\} \end{aligned}$$

Therefore $v_{\alpha!A}^N(y * z) \geq \min\{v_{\alpha!A}^N(x * z), v_{\alpha!A}^N(x * y)\}$

Therefore $!A$ is a bipolar intuitionistic fuzzy α -ideal of X .

Theorem 2. If A and B are bipolar intuitionistic fuzzy α -ideal of X , then $!(A \cap B) = !A \cap !B$ is also a bipolar intuitionistic fuzzy α -ideal of X .

Proof. Let A and B are bipolar intuitionistic fuzzy α -ideal of X .

Consider $0, x, y, z \in A \cap B$ then $0, x, y, z \in A$ and $0, x, y, z \in B$.

$$\text{(i) Now } \mu_{\alpha!(A \cap B)}^P(0) = \max\left(\frac{1}{2}, \mu_{\alpha_A \cap B}^P(0)\right)$$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \min\{\mu_{\alpha_A}^P(0), \mu_{\alpha_B}^P(0)\}\right) \\
&\geq \max\left(\frac{1}{2}, \min\{\mu_{\alpha_A}^P(x), \mu_{\alpha_B}^P(x)\}\right) \\
&= \min\left\{\max\left(\frac{1}{2}, \mu_{\alpha_A}^P(x)\right), \max\left(\frac{1}{2}, \mu_{\alpha_B}^P(x)\right)\right\} \\
&= \min\{\mu_{\alpha_{!A}}^P(x), \mu_{\alpha_{!B}}^P(x)\} \\
&= \mu_{\alpha_{!A \cap !B}}^P(x)
\end{aligned}$$

Therefore $\mu_{\alpha_{!(A \cap B)}}^P(0) \geq \mu_{\alpha_{!A \cap !B}}^P(x)$

$$\begin{aligned}
\text{Now } \mu_{\alpha_{!(A \cap B)}}^N(0) &= \min\left(\frac{-1}{2}, \mu_{\alpha_{A \cap B}}^N(0)\right) \\
&= \min\left(\frac{-1}{2}, \max\{\mu_{\alpha_A}^N(0), \mu_{\alpha_B}^N(0)\}\right) \\
&\leq \min\left(\frac{-1}{2}, \max\{\mu_{\alpha_A}^N(x), \mu_{\alpha_B}^N(x)\}\right) \\
&= \max\left\{\min\left(\frac{-1}{2}, \mu_{\alpha_A}^N(x)\right), \min\left(\frac{-1}{2}, \mu_{\alpha_B}^N(x)\right)\right\} \\
&= \max\{\mu_{\alpha_{!A}}^N(x), \mu_{\alpha_{!B}}^N(x)\} \\
&= \mu_{\alpha_{!A \cap !B}}^N(x)
\end{aligned}$$

Therefore $\mu_{\alpha_{!(A \cap B)}}^N(0) \leq \mu_{\alpha_{!A \cap !B}}^N(x)$

$$\begin{aligned}
\text{(ii) Now } \mu_{\alpha_{!(A \cap B)}}^P(y * z) &= \max\left(\frac{1}{2}, \mu_{\alpha_{A \cap B}}^P(y * z)\right) \\
&= \max\left(\frac{1}{2}, \min\{\mu_{\alpha_A}^P(y * z), \mu_{\alpha_B}^P(y * z)\}\right) \\
&\geq \max\left(\frac{1}{2}, \min\{\min\{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\},\right.
\end{aligned}$$

$$\begin{aligned}
& \min \{\mu_{\alpha_B}^P(x * z), \mu_{\alpha_B}^P(x * y)\}) \\
&= \max \left(\frac{1}{2}, \min \{\min \{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\}, \right. \\
&\quad \left. \min \{\mu_{\alpha_B}^P(x * z), \mu_{\alpha_B}^P(x * y)\}\} \right) \\
&= \max \left\{ \max \left(\frac{1}{2}, \min \{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\}, \right. \right. \\
&\quad \left. \left. \min \{\mu_{\alpha_B}^P(x * z), \mu_{\alpha_B}^P(x * y)\}\right) \right\} \\
&= \min \left\{ \min \left\{ \max \left(\frac{1}{2}, \mu_{\alpha_A}^P(x * z) \right), \max \left(\frac{1}{2}, \mu_{\alpha_B}^P(x * z) \right) \right\}, \right. \\
&\quad \left. \min \left\{ \max \left(\frac{1}{2}, \mu_{\alpha_A}^P(x * y) \right), \max \left(\frac{1}{2}, \mu_{\alpha_B}^P(x * y) \right) \right\} \right\} \\
&= \min \{\min \{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_B}^P(x * z)\}, \min \{\mu_{\alpha_A}^P(x * y), \mu_{\alpha_B}^P(x * y)\}\} \\
&= \min \{\mu_{\alpha_{A \cap B}}^P(x * z), \mu_{\alpha_{A \cap B}}^P(x * y)\}
\end{aligned}$$

Therefore $\mu_{\alpha_{(A \cap B)}}^P(y * z) \geq \min \{\mu_{\alpha_{A \cap B}}^P(x * z), \mu_{\alpha_{A \cap B}}^P(x * y)\}$

$$\begin{aligned}
\text{(iii) Now } \mu_{\alpha_{(A \cap B)}}^N(y * z) &= \min \left(\frac{-1}{2}, \mu_{\alpha_{A \cap B}}^N(y * z) \right) \\
&= \min \left(\frac{-1}{2}, \max \{\mu_{\alpha_A}^N(y * z), \mu_{\alpha_B}^N(y * z)\} \right) \\
&\leq \min \left(\frac{-1}{2}, \max \{\max \{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * y)\}, \right. \\
&\quad \left. \max \{\mu_{\alpha_B}^N(x * z), \mu_{\alpha_B}^N(x * y)\}\} \right) \\
&= \min \left(\frac{-1}{2}, \max \{\max \{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * y)\}, \right. \\
&\quad \left. \max \{\mu_{\alpha_B}^N(x * z), \mu_{\alpha_B}^N(x * y)\}\} \right)
\end{aligned}$$

$$\begin{aligned}
&= \max \left\{ \min \left(\frac{-1}{2}, \max \{ \mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * y) \} \right), \right. \\
&\quad \left. \min \left(\frac{-1}{2}, \max \{ \mu_{\alpha_B}^N(x * z), \mu_{\alpha_B}^N(x * y) \} \right) \right\} \\
&= \max \left\{ \max \left\{ \min \left(\frac{-1}{2}, \mu_{\alpha_A}^N(x * z) \right), \min \left(\frac{-1}{2}, \mu_{\alpha_B}^N(x * z) \right) \right\}, \right. \\
&\quad \left. \max \left\{ \min \left(\frac{-1}{2}, \mu_{\alpha_A}^N(x * y) \right), \min \left(\frac{-1}{2}, \mu_{\alpha_B}^N(x * y) \right) \right\} \right\} \\
&= \max \{ \max \{ \mu_{\alpha_{!A}}^N(x * z), \mu_{\alpha_{!B}}^N(x * z) \}, \max \{ \mu_{\alpha_{!A}}^N(x * y), \mu_{\alpha_{!B}}^N(x * y) \} \} \\
&= \max \{ \mu_{\alpha_{!A} \cap !B}^N(x * z), \mu_{\alpha_{!A} \cap !B}^N(x * y) \}
\end{aligned}$$

Therefore $\mu_{\alpha_{!(A \cap B)}}^N(y * z) \leq \max \{ \mu_{\alpha_{!A} \cap !B}^N(x * z), \mu_{\alpha_{!A} \cap !B}^N(x * y) \}$

$$\begin{aligned}
(\text{iv}) \text{ Now } v_{\alpha_{!(A \cap B)}}^P(0) &= \min \left(\frac{1}{2}, v_{\alpha_{A \cap B}}^P(0) \right) \\
&= \min \left(\frac{1}{2}, \max(v_{\alpha_A}^P(0), v_{\alpha_B}^P(0)) \right) \\
&\leq \min \left(\frac{1}{2}, \max(v_{\alpha_A}^P(x), v_{\alpha_B}^P(x)) \right) \\
&= \max \left\{ \min \left(\frac{1}{2}, v_{\alpha_A}^P(x) \right), \min \left(\frac{1}{2}, v_{\alpha_B}^P(x) \right) \right\} \\
&= \min \{ v_{\alpha_{!A}}^P(x), v_{\alpha_{!B}}^P(x) \} \\
&= v_{\alpha_{!A} \cap !B}^P(x)
\end{aligned}$$

Therefore $v_{\alpha_{!(A \cap B)}}^P(0) \leq v_{\alpha_{!A} \cap !B}^P(x)$

$$\text{Now } v_{\alpha_{!(A \cap B)}}^N(0) = \max \left(\frac{-1}{2}, v_{\alpha_{A \cap B}}^N(0) \right)$$

$$\begin{aligned}
&= \max\left(\frac{-1}{2}, \min(v_{\alpha_A}^N(0), v_{\alpha_B}^N(0))\right) \\
&\geq \max\left(\frac{-1}{2}, \min(v_{\alpha_A}^N(x), v_{\alpha_B}^N(x))\right) \\
&= \min\left\{\max\left(\frac{-1}{2}, v_{\alpha_A}^N(x)\right), \max\left(\frac{-1}{2}, v_{\alpha_B}^N(x)\right)\right\} \\
&= \min\{v_{\alpha_{A \cap !B}}^N(x), v_{\alpha_{!A \cap B}}^N(x)\} \\
&= v_{\alpha_{A \cap !B}}^N(x)
\end{aligned}$$

Therefore $v_{\alpha_{!(A \cap B)}}^N(0) \geq v_{\alpha_{!A \cap !B}}^N(x)$

$$\begin{aligned}
(v) \text{ Now } v_{\alpha_{!(A \cap B)}}^P(y * z) &= \min\left(\frac{1}{2}, v_{\alpha_{A \cap B}}^P(y * z)\right) \\
&= \min\left(\frac{1}{2}, \max(v_{\alpha_A}^P(y * z), v_{\alpha_B}^P(y * z))\right) \\
&\leq \min\left(\frac{1}{2}, \max\{\max\{v_{\alpha_A}^P(x * z), v_{\alpha_A}^P(x * y)\}, \right. \\
&\quad \left. \max\{v_{\alpha_B}^P(x * z), v_{\alpha_B}^P(x * y)\}\}\right) \\
&= \min\left(\frac{1}{2}, \max\{\max\{v_{\alpha_A}^P(x * z), v_{\alpha_A}^P(x * y)\}, \right. \\
&\quad \left. \max\{v_{\alpha_B}^P(x * z), v_{\alpha_B}^P(x * y)\}\}\right) \\
&= \max\left\{\min\left(\frac{1}{2}, \max\{v_{\alpha_A}^P(x * z), v_{\alpha_A}^P(x * y)\}\right), \right. \\
&\quad \left. \min\left(\frac{1}{2}, \max\{v_{\alpha_A}^P(x * y), v_{\alpha_A}^P(x * y)\}\right)\right\} \\
&= \max\left\{\max\left\{\min\left(\frac{1}{2}, v_{\alpha_A}^P(x * z)\right), \min\left(\frac{1}{2}, v_{\alpha_B}^P(x * z)\right)\right\}, \right. \\
&\quad \left. \max\left\{\min\left(\frac{1}{2}, v_{\alpha_A}^P(x * y)\right), \min\left(\frac{1}{2}, v_{\alpha_B}^P(x * y)\right)\right\}\right\}
\end{aligned}$$

$$= \max \{ \max \{ v_{\alpha!A}^P(x * z), v_{\alpha!B}^P(x * z) \}, \max \{ v_{\alpha!A}^P(x * y), v_{\alpha!B}^P(x * y) \} \}$$

$$= \max \{ v_{\alpha!A \cap !B}^P(x * z), v_{\alpha!A \cap !B}^P(x * y) \}$$

$$\text{Therefore } v_{\alpha!(A \cap B)}^P(y * z) \leq \max \{ v_{\alpha!A \cap !B}^P(x * z), v_{\alpha!A \cap !B}^P(x * y) \}$$

$$(vi) \text{ Now } v_{\alpha!(A \cap B)}^N(y * z) = \max \left(\frac{-1}{2}, v_{\alpha A \cap B}^N(y * z) \right)$$

$$= \max \left(\frac{-1}{2}, \min \{ v_{\alpha A}^N(y * z), v_{\alpha B}^N(y * z) \} \right)$$

$$\geq \max \left(\frac{-1}{2}, \min \{ \min \{ v_{\alpha A}^N(x * z), v_{\alpha A}^N(x * y) \}, \right.$$

$$\left. \min \{ v_{\alpha B}^N(x * z), v_{\alpha B}^N(x * y) \} \} \right)$$

$$= \max \left(\frac{-1}{2}, \min \{ \min \{ v_{\alpha A}^N(x * z), v_{\alpha A}^N(x * y) \}, \right.$$

$$\left. \min \{ v_{\alpha B}^N(x * y), v_{\alpha B}^N(x * y) \} \} \right)$$

$$= \min \left\{ \max \left(\frac{-1}{2}, \min \{ v_{\alpha A}^N(x * z), v_{\alpha A}^N(x * y) \} \right), \right.$$

$$\left. \max \left(\frac{-1}{2}, \min \{ v_{\alpha B}^N(x * z), v_{\alpha B}^N(x * y) \} \right) \right\}$$

$$= \min \left\{ \min \left\{ \max \left(\frac{-1}{2}, v_{\alpha A}^N(x * z) \right), \max \left(\frac{-1}{2}, v_{\alpha B}^N(x * z) \right) \right\}, \right.$$

$$\left. \min \left\{ \max \left(\frac{-1}{2}, v_{\alpha A}^N(x * y) \right), \max \left(\frac{-1}{2}, v_{\alpha B}^N(x * y) \right) \right\} \right\}$$

$$= \min \{ \min \{ v_{\alpha!A}^N(x * z), v_{\alpha!B}^N(x * z) \}, \min \{ v_{\alpha!A}^N(x * y), v_{\alpha!B}^N(x * y) \} \}$$

$$= \min \{ v_{\alpha!A \cap !B}^N(x * z), v_{\alpha!A \cap !B}^N(x * y) \}$$

$$\text{Therefore } v_{\alpha!(A \cap B)}^N(y * z) \geq \min \{ v_{\alpha!A \cap !B}^N(x * z), v_{\alpha!A \cap !B}^N(x * y) \}$$

Therefore $!(A \cap B) = !A \cap !B$ is a bipolar intuitionistic fuzzy α -ideal of X .

Theorem 3. *If A is a bipolar intuitionistic fuzzy α -ideal of X , then $?A$ is a bipolar intuitionistic fuzzy α -ideal of X .*

Proof. Given A is a bipolar intuitionistic fuzzy α -ideal of X .

Consider $0, x, y, z \in A$.

$$\begin{aligned} \text{(i) Now } \mu_{\alpha?A}^P(0) &= \min\left(\frac{1}{2}, \mu_{\alpha_A}^P(0)\right) \\ &\geq \min\left(\frac{1}{2}, \mu_{\alpha_A}^P(x)\right) \\ &= \mu_{\alpha?A}^P(x) \end{aligned}$$

$$\text{Therefore } \mu_{\alpha?A}^P(0) \geq \mu_{\alpha?A}^P(x)$$

$$\begin{aligned} \text{Now } \mu_{\alpha?A}^N(0) &= \max\left(\frac{-1}{2}, \mu_{\alpha_A}^N(0)\right) \\ &\leq \min\left(\frac{-1}{2}, \mu_{\alpha_A}^N(x)\right) \\ &= \mu_{\alpha?A}^N(x) \end{aligned}$$

$$\text{Therefore } \mu_{\alpha?A}^N(0) \leq \mu_{\alpha?A}^N(x)$$

$$\begin{aligned} \text{(ii) Now } \mu_{\alpha?A}^P(y * z) &= \min\left(\frac{1}{2}, \mu_{\alpha_A}^P(y * z)\right) \\ &\geq \min\left(\frac{1}{2}, \min\{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\}\right) \\ &= \min\left\{\min\left(\frac{1}{2}, \mu_{\alpha_A}^P(x * z)\right), \min\left(\frac{1}{2}, \mu_{\alpha_A}^P(x * y)\right)\right\} \\ &= \min\{\mu_{\alpha?A}^P(x * z), \mu_{\alpha?A}^P(x * y)\} \end{aligned}$$

Therefore $\mu_{\alpha?A}^P(y * z) \geq \min \{\mu_{\alpha?A}^P(x * z), \mu_{\alpha?A}^P(x * y)\}$

$$\begin{aligned}
 \text{(iii) Now } \mu_{\alpha?A}^N(y * z) &= \max\left(\frac{-1}{2}, \mu_{\alpha_A}^N(y * z)\right) \\
 &\leq \max\left(\frac{-1}{2}, \max \{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * y)\}\right) \\
 &= \max\left\{\max\left(\frac{-1}{2}, \mu_{\alpha_A}^N(x * z)\right), \max\left(\frac{-1}{2}, \mu_{\alpha_A}^N(x * y)\right)\right\} \\
 &= \max \{\mu_{\alpha?A}^N(x * z), \mu_{\alpha?A}^N(x * y)\}
 \end{aligned}$$

Therefore $\mu_{\alpha?A}^N(y * z) \leq \max \{\mu_{\alpha?A}^N(x * z), \mu_{\alpha?A}^N(x * y)\}$

$$\begin{aligned}
 \text{(iv) Now } v_{\alpha?A}^P(0) &= \max\left(\frac{1}{2}, v_{\alpha_A}^P(0)\right) \\
 &\leq \max\left(\frac{1}{2}, v_{\alpha_A}^P(x)\right) \\
 &= v_{\alpha?A}^P(x)
 \end{aligned}$$

Therefore $v_{\alpha?A}^P(0) \leq v_{\alpha?A}^P(x)$

$$\begin{aligned}
 \text{Now } v_{\alpha?A}^N(0) &= \min\left(\frac{-1}{2}, v_{\alpha_A}^N(0)\right) \\
 &\geq \max\left(\frac{-1}{2}, v_{\alpha_A}^N(x)\right) \\
 &= v_{\alpha?A}^N(x)
 \end{aligned}$$

Therefore $v_{\alpha?A}^N(0) \geq v_{\alpha?A}^N(x)$

$$\begin{aligned}
 \text{(v) Now } v_{\alpha?A}^P(y * z) &= \max\left(\frac{1}{2}, v_{\alpha_A}^P(y * z)\right) \\
 &\leq \max\left(\frac{1}{2}, \max \{v_{\alpha_A}^P(x * z), v_{\alpha_A}^P(x * y)\}\right)
 \end{aligned}$$

$$\begin{aligned}
&= \max \left\{ \max \left(\frac{1}{2}, v_{\alpha?A}^P(x * z) \right), \max \left(\frac{1}{2}, v_{\alpha?A}^P(x * y) \right) \right\} \\
&= \max \{v_{\alpha?A}^P(x * z), v_{\alpha?A}^P(x * y)\}
\end{aligned}$$

Therefore $v_{\alpha?A}^P(y * z) \leq \max \{v_{\alpha?A}^P(x * z), v_{\alpha?A}^P(x * y)\}$

$$\begin{aligned}
(\text{vi}) \text{ Now } v_{\alpha?A}^N(y * z) &= \min \left(\frac{-1}{2}, v_{\alpha?A}^N(y * z) \right) \\
&\geq \min \left(\frac{-1}{2}, \min \{v_{\alpha?A}^N(x * z), v_{\alpha?A}^N(x * y)\} \right) \\
&= \min \left\{ \min \left(\frac{-1}{2}, v_{\alpha?A}^N(x * z) \right), \min \left(\frac{-1}{2}, v_{\alpha?A}^N(x * y) \right) \right\} \\
&= \min \{v_{\alpha?A}^N(x * z), v_{\alpha?A}^N(x * y)\}
\end{aligned}$$

Therefore $v_{\alpha?A}^N(y * z) \geq \min \{v_{\alpha?A}^N(x * z), v_{\alpha?A}^N(x * y)\}$

Therefore $?A$ is a bipolar intuitionistic fuzzy α -ideal of X .

Theorem 4. If A and B are bipolar intuitionistic fuzzy α -ideal of X , then $?A \cap B = ?A \cap ?B$ is also a bipolar intuitionistic fuzzy α -ideal of X .

Proof. Let A and B are bipolar intuitionistic fuzzy α -ideal of X .

Consider $0, x, y, z \in A \cap B$ then $0, x, y, z \in A$ and $0, x, y, z \in B$.

$$\begin{aligned}
(\text{i}) \text{ Now } \mu_{\alpha?(A \cap B)}^P(0) &= \min \left(\frac{1}{2}, \mu_{\alpha?A \cap B}^P(0) \right) \\
&= \min \left(\frac{1}{2}, \min \{\mu_{\alpha?A}^P(0), \mu_{\alpha?B}^P(0)\} \right) \\
&\geq \min \left(\frac{1}{2}, \min \{\mu_{\alpha?A}^P(x), \mu_{\alpha?B}^P(x)\} \right) \\
&= \min \left\{ \min \left(\frac{1}{2}, \mu_{\alpha?A}^P(x) \right), \min \left(\frac{1}{2}, \mu_{\alpha?B}^P(x) \right) \right\}
\end{aligned}$$

$$= \min \{\mu_{\alpha?A}^P(x), \mu_{\alpha?B}^P(x)\}$$

$$= \mu_{\alpha?A \cap ?B}^P(x)$$

Therefore $\mu_{\alpha?(A \cap B)}^P(0) \geq \mu_{\alpha?A \cap ?B}^P(x)$

$$\begin{aligned} \text{Now } \mu_{\alpha?(A \cap B)}^N(0) &= \max\left(\frac{-1}{2}, \mu_{\alpha?A \cap ?B}^N(0)\right) \\ &= \max\left(\frac{-1}{2}, \max\{\mu_{\alpha_A}^N(0), \mu_{\alpha_B}^N(0)\}\right) \\ &\leq \max\left(\frac{-1}{2}, \max\{\mu_{\alpha_A}^N(x), \mu_{\alpha_B}^N(x)\}\right) \\ &= \max\left\{\max\left(\frac{-1}{2}, \mu_{\alpha_A}^N(x)\right), \max\left(\frac{-1}{2}, \mu_{\alpha_B}^N(x)\right)\right\} \\ &= \max\{\mu_{\alpha?A}^N(x), \mu_{\alpha?B}^N(x)\} \\ &= \mu_{\alpha?A \cap ?B}^N(x) \end{aligned}$$

Therefore $\mu_{\alpha?(A \cap B)}^N(0) \leq \mu_{\alpha?A \cap ?B}^N(x)$

$$\begin{aligned} \text{(ii) Now } \mu_{\alpha?(A \cap B)}^P(y * z) &= \min\left(\frac{1}{2}, \mu_{\alpha?A \cap ?B}^P(y * z)\right) \\ &= \min\left(\frac{1}{2}, \min\{\mu_{\alpha_A}^P(y * z), \mu_{\alpha_B}^P(y * z)\}\right) \\ &\geq \min\left(\frac{1}{2}, \min\{\min\{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\}, \right. \\ &\quad \left. \min\{\mu_{\alpha_B}^P(x * z), \mu_{\alpha_B}^P(x * y)\}\}\right) \\ &= \min\left(\frac{1}{2}, \min\{\min\{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\}, \right. \\ &\quad \left. \min\{\mu_{\alpha_B}^P(x * z), \mu_{\alpha_B}^P(x * y)\}\}\right) \end{aligned}$$

$$\begin{aligned}
&= \min \{\min(\frac{1}{2}, \min \{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\}, \\
&\quad \min \{\mu_{\alpha_B}^P(x * z), \mu_{\alpha_B}^P(x * y)\})\} \\
&= \min \left\{ \min \left\{ \min \left(\frac{1}{2}, \mu_{\alpha_A}^P(x * z) \right), \min \left(\frac{1}{2}, \mu_{\alpha_B}^P(x * z) \right) \right\}, \right. \\
&\quad \left. \min \left\{ \min \left(\frac{1}{2}, \mu_{\alpha_A}^P(x * y) \right), \min \left(\frac{1}{2}, \mu_{\alpha_B}^P(x * y) \right) \right\} \right\} \\
&= \min \{\min \{\mu_{\alpha_A \cap B}^P(x * z), \mu_{\alpha_B \cap A}^P(x * z)\}, \min \{\mu_{\alpha_A \cap B}^P(x * y), \mu_{\alpha_B \cap A}^P(x * y)\}\} \\
&= \min \{\mu_{\alpha_A \cap B}^P(x * z), \mu_{\alpha_B \cap A}^P(x * y)\}
\end{aligned}$$

Therefore $\mu_{\alpha_A \cap B}^P(y * z) \geq \min \{\mu_{\alpha_A \cap B}^P(x * z), \mu_{\alpha_B \cap A}^P(x * y)\}$

$$\begin{aligned}
\text{(iii) Now } \mu_{\alpha_A \cap B}^N(y * z) &= \max \left(\frac{-1}{2}, \mu_{\alpha_A \cap B}^N(y * z) \right) \\
&= \max \left(\frac{-1}{2}, \max \{\mu_{\alpha_A}^N(y * z), \mu_{\alpha_B}^N(y * z)\} \right) \\
&\leq \max \left(\frac{-1}{2}, \max \{\max \{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * y)\}, \right. \\
&\quad \left. \max \{\mu_{\alpha_B}^N(x * z), \mu_{\alpha_B}^N(x * y)\}\} \right) \\
&= \max \left(\frac{-1}{2}, \max \{\max \{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * z)\}, \right. \\
&\quad \left. \max \{\mu_{\alpha_B}^N(x * y), \mu_{\alpha_B}^N(x * y)\}\} \right) \\
&= \max \left\{ \max \left(\frac{-1}{2}, \max \{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * z)\} \right), \right. \\
&\quad \left. \max \left(\frac{-1}{2}, \max \{\mu_{\alpha_B}^N(x * y), \mu_{\alpha_B}^N(x * y)\} \right) \right\} \\
&= \max \left\{ \max \left\{ \max \left(\frac{-1}{2}, \mu_{\alpha_A}^N(x * z) \right), \max \left(\frac{-1}{2}, \mu_{\alpha_B}^N(x * z) \right) \right\}, \right.
\end{aligned}$$

$$\begin{aligned}
& \max \left\{ \max \left(\frac{-1}{2}, \mu_{\alpha_A}^N(x * y) \right), \max \left(\frac{-1}{2}, \mu_{\alpha_B}^N(x * y) \right) \right\} \\
&= \max \{ \max \{ \mu_{\alpha_A}^N(x * z), \mu_{\alpha_B}^N(x * z) \}, \max \{ \mu_{\alpha_A}^N(x * y), \mu_{\alpha_B}^N(x * y) \} \} \\
&= \max \{ \mu_{\alpha_A \cap ?B}^N(x * z), \mu_{\alpha_B \cap ?B}^N(x * y) \}
\end{aligned}$$

Therefore $\mu_{\alpha_2(A \cap B)}^N(y * z) \leq \max \{ \mu_{\alpha_2(A \cap ?B)}^N(x * z), \mu_{\alpha_2(A \cap ?B)}^N(x * y) \}$

$$\begin{aligned}
(\text{iv}) \text{ Now } v_{\alpha_2(A \cap B)}^P(0) &= \max \left(\frac{1}{2}, v_{\alpha_A \cap B}^P(0) \right) \\
&= \max \left(\frac{1}{2}, \max(v_{\alpha_A}^P(0), v_{\alpha_B}^P(0)) \right) \\
&\leq \max \left(\frac{1}{2}, \max(v_{\alpha_A}^P(x), v_{\alpha_B}^P(x)) \right) \\
&= \max \left\{ \max \left(\frac{1}{2}, v_{\alpha_A}^P(x) \right), \max \left(\frac{1}{2}, v_{\alpha_B}^P(x) \right) \right\} \\
&= \max \{ v_{\alpha_A \cap ?B}^P(x), v_{\alpha_B \cap ?B}^P(x) \} \\
&= v_{\alpha_2(A \cap ?B)}^P(x)
\end{aligned}$$

Therefore $v_{\alpha_2(A \cap B)}^P(0) \leq v_{\alpha_2(A \cap ?B)}^P(x)$

$$\begin{aligned}
\text{Now } v_{\alpha_2(A \cap B)}^N(0) &= \min \left(\frac{-1}{2}, v_{\alpha_A \cap B}^N(0) \right) \\
&= \min \left(\frac{-1}{2}, \min(v_{\alpha_A}^N(0), v_{\alpha_B}^N(0)) \right) \\
&\geq \min \left(\frac{-1}{2}, \min(v_{\alpha_A}^N(x), v_{\alpha_B}^N(x)) \right) \\
&= \min \left\{ \min \left(\frac{-1}{2}, v_{\alpha_A}^N(x) \right), \min \left(\frac{-1}{2}, v_{\alpha_B}^N(x) \right) \right\} \\
&= \min \{ v_{\alpha_A \cap ?B}^N(x), v_{\alpha_B \cap ?B}^N(x) \}
\end{aligned}$$

$$= v_{\alpha? A \cap ? B}^N(x)$$

Therefore $v_{\alpha?(A \cap B)}^N(0) \geq v_{\alpha? A \cap ? B}^N(x)$

$$\begin{aligned}
 \text{(v) Now } v_{\alpha?(A \cap B)}^P(y * z) &= \max\left(\frac{1}{2}, v_{\alpha A \cap B}^P(y * z)\right) \\
 &= \max\left(\frac{1}{2}, \max(v_{\alpha_A}^P(y * z), v_{\alpha_B}^P(y * z))\right) \\
 &\leq \max\left(\frac{1}{2}, \max\{\max\{v_{\alpha_A}^P(x * z), v_{\alpha_A}^P(x * y)\},\right. \\
 &\quad \left.\max\{v_{\alpha_B}^P(x * z), v_{\alpha_B}^P(x * y)\}\}\right) \\
 &= \max\left(\frac{1}{2}, \max\{\max\{v_{\alpha_A}^P(x * z), v_{\alpha_A}^P(x * z)\},\right. \\
 &\quad \left.\max\{v_{\alpha_B}^P(x * y), v_{\alpha_B}^P(x * y)\}\}\right) \\
 &= \max\left\{\max\left(\frac{1}{2}, \max\{v_{\alpha_A}^P(x * z), v_{\alpha_A}^P(x * z)\}\right),\right. \\
 &\quad \left.\max\left(\frac{1}{2}, \max\{v_{\alpha_A}^P(x * y), v_{\alpha_A}^P(x * y)\}\right)\right\} \\
 &= \max\left\{\max\left\{\max\left(\frac{1}{2}, v_{\alpha_A}^P(x * z)\right), \max\left(\frac{1}{2}, v_{\alpha_B}^P(x * z)\right)\right\},\right. \\
 &\quad \left.\max\left\{\max\left(\frac{1}{2}, v_{\alpha_A}^P(x * y)\right), \max\left(\frac{1}{2}, v_{\alpha_B}^P(x * y)\right)\right\}\right\} \\
 &= \max\{\max\{v_{\alpha? A}^P(x * z), v_{\alpha? B}^P(x * z)\}, \max\{v_{\alpha? A}^P(x * y), v_{\alpha? B}^P(x * y)\}\} \\
 &= \max\{v_{\alpha? A \cap ? B}^P(x * z), v_{\alpha? A \cap ? B}^P(x * y)\}
 \end{aligned}$$

Therefore $v_{\alpha?(A \cap B)}^P(y * z) \leq \max\{v_{\alpha? A \cap ? B}^P(x * z), v_{\alpha? A \cap ? B}^P(x * y)\}$

$$\text{(vi) Now } v_{\alpha?(A \cap B)}^N(y * z) = \min\left(\frac{-1}{2}, v_{\alpha A \cap B}^N(y * z)\right)$$

$$\begin{aligned}
&= \min\left(\frac{-1}{2}, \min(v_{\alpha_A}^N(y * z), v_{\alpha_B}^N(y * z))\right) \\
&\geq \min\left(\frac{-1}{2}, \min\{\min\{v_{\alpha_A}^N(x * z), v_{\alpha_A}^N(x * y)\}, \right. \\
&\quad \left. \min\{v_{\alpha_B}^N(x * z), v_{\alpha_B}^N(x * y)\}\}\right) \\
&= \min\left(\frac{-1}{2}, \min\{\min\{v_{\alpha_A}^N(x * z), v_{\alpha_A}^N(x * y)\}, \right. \\
&\quad \left. \min\{v_{\alpha_B}^N(x * y), v_{\alpha_B}^N(x * y)\}\}\right) \\
&= \min\left\{\min\left(\frac{-1}{2}, \min\{v_{\alpha_A}^N(x * z), v_{\alpha_A}^N(x * y)\}\right), \right. \\
&\quad \left. \min\left(\frac{-1}{2}, \min\{v_{\alpha_B}^N(x * z), v_{\alpha_B}^N(x * y)\}\right)\right\} \\
&= \min\left\{\min\left\{\min\left(\frac{-1}{2}, v_{\alpha_A}^N(x * z)\right), \min\left(\frac{-1}{2}, v_{\alpha_B}^N(x * z)\right)\right\}, \right. \\
&\quad \left. \min\left\{\min\left(\frac{-1}{2}, v_{\alpha_A}^N(x * y)\right), \min\left(\frac{-1}{2}, v_{\alpha_B}^N(x * y)\right)\right\}\right\} \\
&= \min\{\min\{v_{\alpha?A}^N(x * z), v_{\alpha?B}^N(x * z)\}, \min\{v_{\alpha?A}^N(x * y), v_{\alpha?B}^N(x * y)\}\} \\
&= \min\{v_{\alpha?A \cap ?B}^N(x * z), v_{\alpha?A \cap ?B}^N(x * y)\}
\end{aligned}$$

Therefore $v_{\alpha?(A \cap B)}^N(y * z) \geq \min\{v_{\alpha?A \cap ?B}^N(x * z), v_{\alpha?A \cap ?B}^N(x * y)\}$

Therefore $?A \cap B = ?A \cap ?B$ is also a bipolar intuitionistic fuzzy α -ideal of X .

Theorem 5. If A is a bipolar intuitionistic fuzzy α -ideal of X , then $\overline{?A} = !A$ is also a bipolar intuitionistic fuzzy α -ideal of X .

Proof. Given A is a bipolar intuitionistic fuzzy α -ideal of X .

Consider $0, x, y, z \in A$.

$$\begin{aligned}
(i) \text{ Now } \mu_{\alpha? \bar{A}}^P(0) &= v_{\alpha? \bar{A}}^P(0) \\
&= \max \left\{ \frac{1}{2}, v_{\alpha \bar{A}}^P(0) \right\} \\
&= \max \left\{ \frac{1}{2}, \mu_{\alpha A}^P(0) \right\} \\
&\geq \max \left\{ \frac{1}{2}, \mu_{\alpha A}^P(x) \right\} \\
&= \mu_{\alpha! A}^P(x)
\end{aligned}$$

Therefore $\mu_{\alpha? \bar{A}}^P(0) \geq \mu_{\alpha! A}^P(x)$

$$\begin{aligned}
\text{Now } \mu_{\alpha? \bar{A}}^N(0) &= v_{\alpha? \bar{A}}^N(0) \\
&= \min \left\{ \frac{-1}{2}, v_{\alpha \bar{A}}^N(0) \right\} \\
&= \min \left\{ \frac{-1}{2}, \mu_{\alpha A}^N(0) \right\} \\
&\leq \min \left\{ \frac{-1}{2}, \mu_{\alpha A}^N(x) \right\} \\
&= \mu_{\alpha! A}^N(x)
\end{aligned}$$

Therefore $\mu_{\alpha? \bar{A}}^N(0) \leq \mu_{\alpha! A}^N(x)$

$$\begin{aligned}
(ii) \text{ Now } \mu_{\alpha? \bar{A}}^P(y * z) &= v_{\alpha? \bar{A}}^P(y * z) \\
&= \max \left\{ \frac{1}{2}, v_{\alpha \bar{A}}^P(y * z) \right\} \\
&= \max \left\{ \frac{1}{2}, \mu_{\alpha A}^P(y * z) \right\}
\end{aligned}$$

$$\begin{aligned}
&\geq \max \left\{ \frac{1}{2}, \min \{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\} \right\} \\
&= \min \left\{ \max \left\{ \frac{1}{2}, \mu_{\alpha_A}^P(x * z) \right\}, \max \left\{ \frac{1}{2}, \mu_{\alpha_A}^P(x * y) \right\} \right\} \\
&= \min \{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\}
\end{aligned}$$

Therefore $\mu_{\alpha_A}^P(y * z) \geq \min \{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\}$

$$\begin{aligned}
\text{(iii) Now } \mu_{\alpha_A}^N(y * z) &= v_{\alpha_A}^N(y * z) \\
&= \min \left\{ \frac{-1}{2}, v_{\alpha_A}^N(y * z) \right\} \\
&= \min \left\{ \frac{-1}{2}, \mu_{\alpha_A}^N(y * z) \right\} \\
&\leq \min \left\{ \frac{-1}{2}, \max \{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * y)\} \right\} \\
&= \max \left\{ \min \left\{ \frac{-1}{2}, \mu_{\alpha_A}^N(x * z) \right\}, \min \left\{ \frac{-1}{2}, \mu_{\alpha_A}^N(x * y) \right\} \right\} \\
&= \max \{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * y)\}
\end{aligned}$$

Therefore $\mu_{\alpha_A}^N(y * z) \leq \max \{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * y)\}$

$$\begin{aligned}
\text{(iv) Now } v_{\alpha_A}^P(0) &= \mu_{\alpha_A}^P(0) \\
&= \min \left\{ \frac{1}{2}, \mu_{\alpha_A}^P(0) \right\} \\
&= \min \left\{ \frac{1}{2}, v_{\alpha_A}^P(0) \right\} \\
&\leq \min \left\{ \frac{1}{2}, v_{\alpha_A}^P(x) \right\}
\end{aligned}$$

$$= v_{\alpha!A}^P(x)$$

Therefore $v_{\alpha? \bar{A}}^P(0) \leq v_{\alpha!A}^P(x)$

$$\begin{aligned} \text{Now } v_{\alpha? \bar{A}}^N(0) &= \mu_{\alpha? \bar{A}}^N(0) \\ &= \max \left\{ \frac{-1}{2}, \mu_{\alpha \bar{A}}^N(0) \right\} \\ &= \max \left\{ \frac{-1}{2}, v_{\alpha A}^N(0) \right\} \\ &\geq \max \left\{ \frac{-1}{2}, v_{\alpha A}^N(x) \right\} \\ &= v_{\alpha!A}^N(x) \end{aligned}$$

Therefore $v_{\alpha? \bar{A}}^N(0) \geq v_{\alpha!A}^N(x)$

$$\begin{aligned} (\text{v}) \text{ Now } \mu_{\alpha? \bar{A}}^P(y * z) &= \mu_{\alpha? \bar{A}}^N(y * z) \\ &= \min \left\{ \frac{1}{2}, \mu_{\alpha \bar{A}}^P(y * z) \right\} \\ &= \min \left\{ \frac{1}{2}, v_{\alpha A}^P(y * z) \right\} \\ &\leq \min \left\{ \frac{1}{2}, \max \{v_{\alpha A}^P(x * z), v_{\alpha A}^P(x * y)\} \right\} \\ &= \max \left\{ \min \left\{ \frac{1}{2}, v_{\alpha A}^P(x * z) \right\}, \min \left\{ \frac{1}{2}, v_{\alpha A}^P(x * y) \right\} \right\} \\ &= \max \{v_{\alpha!A}^P(x * z), v_{\alpha!A}^P(x * y)\} \end{aligned}$$

Therefore $v_{\alpha? \bar{A}}^P(y * z) \leq \max \{v_{\alpha!A}^P(x * z), v_{\alpha!A}^P(x * y)\}$

(vi) Now $v_{\alpha? \bar{A}}^N(y * z) = \mu_{\alpha? \bar{A}}^N(y * z)$

$$\begin{aligned}
&= \max \left\{ \frac{-1}{2}, \mu_{\alpha\overline{A}}^N(y * z) \right\} \\
&= \max \left\{ \frac{-1}{2}, v_{\alpha_A}^N(y * z) \right\} \\
&\geq \max \left\{ \frac{-1}{2}, \min \{v_{\alpha_A}^N(x * z), v_{\alpha_A}^N(x * y)\} \right\} \\
&= \min \left\{ \max \left\{ \frac{-1}{2}, v_{\alpha_A}^N(x * z) \right\}, \max \left\{ \frac{-1}{2}, v_{\alpha_A}^N(x * y) \right\} \right\} \\
&= \min \{v_{\alpha_A}^N(x * z), v_{\alpha_A}^N(x * y)\}
\end{aligned}$$

Therefore $v_{\alpha\overline{A}}^N(y * z) \geq \min \{v_{\alpha_A}^N(x * z), v_{\alpha_A}^N(x * y)\}$

Therefore $\overline{?A} = !A$ is a bipolar intuitionistic fuzzy α -ideal of X .

Theorem 6. If A is a bipolar intuitionistic fuzzy α -ideal of X , then $!(?A) = ?(!A)$ is a bipolar intuitionistic fuzzy α -ideal of X .

Proof. Given A is a bipolar intuitionistic fuzzy α -ideal of X .

Consider $0, x, y, z \in A$.

$$\begin{aligned}
(i) \text{ Now } \mu_{\alpha_!(?A)}^P(0) &= \max \left\{ \frac{1}{2}, \mu_{\alpha_A}^P(0) \right\} \\
&= \max \left\{ \frac{1}{2}, \min \left\{ \frac{1}{2}, \mu_{\alpha_A}^P(0) \right\} \right\} \\
&= \min \left\{ \frac{1}{2}, \max \left\{ \frac{1}{2}, \mu_{\alpha_A}^P(0) \right\} \right\} \\
&\geq \min \left\{ \frac{1}{2}, \max \left\{ \frac{1}{2}, \mu_{\alpha_A}^P(x) \right\} \right\} \\
&= \min \left\{ \frac{1}{2}, \mu_{\alpha_A}^P(x) \right\} \\
&= \mu_{\alpha_!(?A)}^P(x)
\end{aligned}$$

Therefore $\mu_{\alpha!(?)A}^P(0) \geq \mu_{\alpha?(!A)}^P(x)$

$$\begin{aligned} \text{Now } \mu_{\alpha!(?)A}^N(0) &= \min\left\{\frac{-1}{2}, \mu_{\alpha?A}^N(0)\right\} \\ &= \min\left\{\frac{-1}{2}, \max\left\{\frac{-1}{2}, \mu_{\alpha_A}^N(0)\right\}\right\} \\ &= \max\left\{\frac{-1}{2}, \min\left\{\frac{-1}{2}, \mu_{\alpha_A}^N(0)\right\}\right\} \\ &\leq \max\left\{\frac{-1}{2}, \min\left\{\frac{-1}{2}, \mu_{\alpha_A}^N(x)\right\}\right\} \\ &= \max\left\{\frac{-1}{2}, \mu_{\alpha!A}^N(x)\right\} \\ &= \mu_{\alpha?(!A)}^N(x) \end{aligned}$$

Therefore $\mu_{\alpha!(?)A}^N(0) \leq \mu_{\alpha?(!A)}^N(x)$

$$\begin{aligned} \text{(ii) Now } \mu_{\alpha!(?)A}^P(y * z) &= \max\left\{\frac{1}{2}, \mu_{\alpha?A}^P(y * z)\right\} \\ &= \max\left\{\frac{1}{2}, \min\left\{\frac{1}{2}, \mu_{\alpha_A}^P(y * z)\right\}\right\} \\ &= \min\left\{\frac{1}{2}, \max\left\{\frac{1}{2}, \mu_{\alpha_A}^P(y * z)\right\}\right\} \\ &\geq \min\left\{\frac{1}{2}, \max\left\{\frac{1}{2}, \min\{\mu_{\alpha_A}^P(x * z), \mu_{\alpha_A}^P(x * y)\}\right\}\right\} \\ &= \min\left\{\frac{1}{2}, \min\left\{\max\left\{\frac{1}{2}, \mu_{\alpha_A}^P(x * z)\right\}, \max\left\{\frac{1}{2}, \mu_{\alpha_A}^P(x * y)\right\}\right\}\right\} \\ &= \min\left\{\frac{1}{2}, \min\{\mu_{\alpha!A}^P(x * z), \mu_{\alpha!A}^P(x * y)\}\right\} \\ &= \min\left\{\min\left\{\frac{1}{2}, \mu_{\alpha!A}^P(x * z)\right\}, \min\left\{\frac{1}{2}, \mu_{\alpha!A}^P(x * y)\right\}\right\} \end{aligned}$$

$$= \min \{\mu_{\alpha?(!A)}^P(x * z), \mu_{\alpha?(!A)}^P(x * y)\}$$

Therefore $\mu_{\alpha!(?A)}^P(y * z) \geq \min \{\mu_{\alpha?(!A)}^P(x * z), \mu_{\alpha?(!A)}^P(x * y)\}$

$$\begin{aligned} \text{(iii) Now } \mu_{\alpha!(?A)}^N(y * z) &= \min \left\{ \frac{-1}{2}, \mu_{\alpha?A}^N(y * z) \right\} \\ &= \min \left\{ \frac{-1}{2}, \max \left\{ \frac{-1}{2}, \mu_{\alpha_A}^N(y * z) \right\} \right\} \\ &= \max \left\{ \frac{-1}{2}, \min \left\{ \frac{-1}{2}, \mu_{\alpha_A}^N(y * z) \right\} \right\} \\ &\leq \max \left\{ \frac{-1}{2}, \min \left\{ \frac{-1}{2}, \max \{\mu_{\alpha_A}^N(x * z), \mu_{\alpha_A}^N(x * y)\} \right\} \right\} \\ &= \max \left\{ \frac{-1}{2}, \max \left\{ \min \left\{ \frac{-1}{2}, \mu_{\alpha_A}^N(x * z) \right\}, \min \left\{ \frac{-1}{2}, \mu_{\alpha_A}^N(x * y) \right\} \right\} \right\} \\ &= \max \left\{ \frac{-1}{2}, \max \{\mu_{\alpha!A}^N(x * z), \mu_{\alpha!A}^N(x * y)\} \right\} \\ &= \max \left\{ \max \left\{ \frac{-1}{2}, \mu_{\alpha!A}^N(x * z) \right\}, \max \left\{ \frac{-1}{2}, \mu_{\alpha!A}^N(x * y) \right\} \right\} \\ &= \max \{\mu_{\alpha?(!A)}^N(x * z), \mu_{\alpha?(!A)}^N(x * y)\} \end{aligned}$$

Therefore $\mu_{\alpha!(?A)}^N(y * z) \leq \max \{\mu_{\alpha?(!A)}^N(x * z), \mu_{\alpha?(!A)}^N(x * y)\}$

$$\begin{aligned} \text{(iv) Now } v_{\alpha!(?A)}^P(0) &= \min \left\{ \frac{1}{2}, v_{\alpha?A}^P(0) \right\} \\ &= \min \left\{ \frac{1}{2}, \max \left\{ \frac{1}{2}, v_{\alpha_A}^P(0) \right\} \right\} \\ &= \max \left\{ \frac{1}{2}, \min \left\{ \frac{1}{2}, v_{\alpha_A}^P(0) \right\} \right\} \\ &\leq \max \left\{ \frac{1}{2}, \min \left\{ \frac{1}{2}, v_{\alpha_A}^P(x) \right\} \right\} \end{aligned}$$

$$= \max \left\{ \frac{1}{2}, v_{\alpha_A}^P(x) \right\}$$

$$= v_{\alpha_{\neg(A)}}^P(x)$$

Therefore $v_{\alpha_{\neg(A)}}^P(0) \leq v_{\alpha_{\neg(A)}}^P(x)$

$$\begin{aligned} \text{Now } v_{\alpha_{\neg(A)}}^N(0) &= \max \left\{ \frac{-1}{2}, v_{\alpha_A}^N(0) \right\} \\ &= \max \left\{ \frac{-1}{2}, \min \left\{ \frac{-1}{2}, v_{\alpha_A}^N(0) \right\} \right\} \\ &= \min \left\{ \frac{-1}{2}, \max \left\{ \frac{-1}{2}, v_{\alpha_A}^N(0) \right\} \right\} \\ &\geq \min \left\{ \frac{-1}{2}, \max \left\{ \frac{-1}{2}, v_{\alpha_A}^N(x) \right\} \right\} \\ &= \min \left\{ \frac{-1}{2}, v_{\alpha_A}^N(x) \right\} \\ &= v_{\alpha_{\neg(A)}}^N(x) \end{aligned}$$

Therefore $v_{\alpha_{\neg(A)}}^N(0) \geq v_{\alpha_{\neg(A)}}^N(x)$

$$\begin{aligned} \text{(v) Now } v_{\alpha_{\neg(A)}}^P(y * z) &= \min \left\{ \frac{1}{2}, v_{\alpha_A}^P(y * z) \right\} \\ &= \min \left\{ \frac{1}{2}, \max \left\{ \frac{1}{2}, v_{\alpha_A}^P(y * z) \right\} \right\} \\ &= \max \left\{ \frac{1}{2}, \min \left\{ \frac{1}{2}, v_{\alpha_A}^P(y * z) \right\} \right\} \\ &\leq \max \left\{ \frac{1}{2}, \min \left\{ \frac{1}{2}, \max \{v_{\alpha_A}^P(x * z), v_{\alpha_A}^P(x * y)\} \right\} \right\} \\ &= \max \left\{ \frac{1}{2}, \max \left\{ \min \left\{ \frac{1}{2}, v_{\alpha_A}^P(x * z) \right\}, \min \left\{ \frac{1}{2}, v_{\alpha_A}^P(x * y) \right\} \right\} \right\} \end{aligned}$$

$$\begin{aligned}
&= \max \left\{ \frac{1}{2}, \max \{v_{\alpha!A}^P(x * z), v_{\alpha!A}^P(x * y)\} \right\} \\
&= \max \left\{ \max \left\{ \frac{1}{2}, v_{\alpha!A}^P(x * z) \right\}, \max \left\{ \frac{1}{2}, v_{\alpha!A}^P(x * y) \right\} \right\} \\
&= \max \{v_{\alpha?(!A)}^P(x * z), v_{\alpha?(!A)}^P(x * y)\}
\end{aligned}$$

Therefore $v_{\alpha!(!A)}^P(y * z) \leq \max \{v_{\alpha?(!A)}^P(x * z), v_{\alpha?(!A)}^P(x * y)\}$

$$\begin{aligned}
(\text{vi}) \quad &\text{Now } v_{\alpha!(!A)}^N(y * z) = \max \left\{ \frac{-1}{2}, v_{\alpha?A}^N(y * z) \right\} \\
&= \max \left\{ \frac{-1}{2}, \min \left\{ \frac{-1}{2}, v_{\alpha_A}^N(y * z) \right\} \right\} \\
&= \min \left\{ \frac{-1}{2}, \max \left\{ \frac{-1}{2}, v_{\alpha_A}^N(y * z) \right\} \right\} \\
&\geq \min \left\{ \frac{-1}{2}, \max \left\{ \frac{-1}{2}, \min \{v_{\alpha_A}^N(x * z), v_{\alpha_A}^N(x * y)\} \right\} \right\} \\
&= \min \left\{ \frac{-1}{2}, \min \left\{ \max \left\{ \frac{-1}{2}, v_{\alpha_A}^N(x * z) \right\}, \max \left\{ \frac{-1}{2}, v_{\alpha_A}^N(x * y) \right\} \right\} \right\} \\
&= \min \left\{ \frac{-1}{2}, \min \{v_{\alpha!A}^N(x * z), v_{\alpha!A}^N(x * y)\} \right\} \\
&= \min \left\{ \min \left\{ \frac{-1}{2}, v_{\alpha!A}^N(x * z) \right\}, \min \left\{ \frac{-1}{2}, v_{\alpha!A}^N(x * y) \right\} \right\} \\
&= \min \{v_{\alpha?(!A)}^N(x * z), v_{\alpha?(!A)}^N(x * y)\}
\end{aligned}$$

Therefore $v_{\alpha!(!A)}^N(y * z) \geq \min \{v_{\alpha?(!A)}^N(x * z), v_{\alpha?(!A)}^N(x * y)\}$

Therefore $!(?A) = ?(!A)$ is a bipolar intuitionistic fuzzy α -ideal of X .

Theorem 7. If A is a bipolar intuitionistic fuzzy α -ideal of X , then $!(\square A) = \square(!A)$ is a bipolar intuitionistic fuzzy α -ideal of X .

Theorem 8. If A is a bipolar intuitionistic fuzzy α -ideal of X , then $\square(?A) = ?(\square A)$ is a bipolar intuitionistic fuzzy α -ideal of X .

Theorem 9. If A is a bipolar intuitionistic fuzzy α -ideal of X , then $?(\Diamond A) = \Diamond(? A)$ is a bipolar intuitionistic fuzzy α -ideal of X .

Theorem 10. If A is a bipolar intuitionistic fuzzy α -ideal of X , then $!(\Diamond A) = \Diamond(! A)$ is a bipolar intuitionistic fuzzy α -ideal of X .

Theorem 11. If A is a bipolar intuitionistic anti fuzzy α -ideal of X , then $!A$ is a bipolar intuitionistic anti fuzzy α -ideal of X .

Theorem 12. If A and B are bipolar intuitionistic anti fuzzy α -ideal of X , then $!(A \cap B) = !A \cap !B$ is also a bipolar intuitionistic anti fuzzy α -ideal of X .

Theorem 13. If A is a bipolar intuitionistic anti fuzzy α -ideal of X , then $?A$ is a bipolar intuitionistic anti fuzzy α -ideal of X .

Theorem 14. If A and B are bipolar intuitionistic anti fuzzy α -ideal of X , then $?(A \cap B) = ?A \cap ?B$ is also a bipolar intuitionistic anti fuzzy α -ideal of X .

Theorem 15. If A is a bipolar intuitionistic anti fuzzy α -ideal of X , then $\overline{?A} = !A$ is also a bipolar intuitionistic anti fuzzy α -ideal of X .

Theorem 16. If A is a bipolar intuitionistic anti fuzzy α -ideal of X , then $?(!A) = ?(!A)$ is a bipolar intuitionistic anti fuzzy α -ideal of X .

Theorem 17. If A is a bipolar intuitionistic anti fuzzy α -ideal of X , then $!(\Box A) = \Box(!A)$ is a bipolar intuitionistic anti fuzzy α -ideal of X .

Theorem 18. If A is a bipolar intuitionistic anti fuzzy α -ideal of X , then $\Box(?A) = ?(\Box A)$ is a bipolar intuitionistic anti fuzzy α -ideal of X .

Theorem 19. If A is a bipolar intuitionistic anti fuzzy α -ideal of X , then $?(\Diamond A) = \Diamond(? A)$ is a bipolar intuitionistic anti fuzzy α -ideal of X .

Theorem 20. If A is a bipolar intuitionistic anti fuzzy α -ideal of X , then $!(\Diamond A) = \Diamond(!A)$ is a bipolar intuitionistic anti fuzzy α -ideal of X .

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