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# APPROXIMATION OF FUNCTION IN THE GENERALIZED ZYGMUND CLASS BY $(N, p_n)$ $(E, q)$ SUMMABILITY MEANS OF FOURIER SERIES

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## Abstract

In this paper, a theorem on degree of approximation of function in the generalized Zygmund class by $(N, p_n)$ $(E, q)$ summability means of Fourier series has been established.

## 1. Introduction

The degree of approximation of function belonging to different classes like $Lip\,\alpha$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$, $W(L_r, \xi(t))$ have been studied by many researchers using different summability means (see [2], [7], [8], [9], [10], [14]).

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The error estimation of function in Lipschitz and Zygmund class using different means of Fourier series and conjugate Fourier series have been great interest among the researcher. The generalized Zygmund class $Z_r^{(\omega)}(r \geq 1)$ has studied by Leindler [4], Moricz [5], Moricz and Nemeth [6] etc. Recently Singh et al. [16], Mishra et al. [11], Pradhan et al. [12] [15], Kim [3], Das et al. [1], find the results in Zygmund class by using different summability means. To the best of our knowledge, the degree of approximation of function in the generalized Zygmund class by $(N, p_n)$ $(E, q)$ summability means of Fourier series has not been studied so far. This motivated us to work in this direction.

## 2. Definition

Let $f$ be a periodic function of period $2\pi$ integrable in the sense of Lebesgue over $[\pi, -\pi]$. Then the Fourier series of $f$ given by

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

Let $C_{2\pi}$ denote the Banach space of all $2\pi$ periodic, continuous function $f$ on $R$ with norm $\| f \|_\infty = \max\{| f(t) | : | t | \leq \pi\}$ and modulus of continuity of $f$ is defined by

$$\omega(f, h) = \sup_{\substack{0 \leq t \leq h \\ x \in R}} | f(x + t) + f(x - t) - 2f(x) |.$$

For $0 < \alpha \leq 1$

$$Z_{(\alpha)} = \{f \in C_{2\pi} : | f(x + t) + f(x - t) - 2f(x) | = O(| t |^{\alpha})\} \tag{2.2}$$

is a Banach space under the norm $\| \cdot \|_\alpha$ defined by

$$\| f \|_\infty := \sup_{0 \leq x \leq 2\pi} | f(x) | + \sup_{x, t, l \neq 0} \frac{| f(x + t) + f(x - t) - 2f(x) |}{| t |^{\alpha}}.$$

For $\in L^p[0, 2\pi]$, $p \geq 1$

$$\omega_p(f, h) = \sup_{0<t\leq h}\left\{\frac{1}{2\pi}\int_0^{2\pi} | f(x+t) + f(x-t) - 2f(x) |^p dx\right\}^{\frac{1}{p}}. \tag{2.3}$$

For $\in L^p[0, 2\pi]$, where $1 \leq p \leq \infty$

$$\omega(f, h) = \omega_\infty(f, h) := \sup_{0<t\leq h} \max_x | f(x+t) + f(x-t) - 2f(x) |. \tag{2.4}$$

For $f \in C_{2\pi}$, $\omega_p(f, h) \to 0$ as $f \to 0$.

Now

$$Z_{(\alpha, p)} = \left\{ f \in L^p[0, 2\pi] : \left(\int_0^{2\pi} | f(x+t) + f(x-t) - 2f(x) |^p dx\right)^{\frac{1}{p}} = O(| t |^\alpha) \right\}.$$

The space $Z_{(\alpha, p)}$, $p \geq 1$, $0 < \alpha \leq 1$ is a Banach space under the norm $\| \cdot \|_{(\alpha, p)}$ which is given by

$$\| f \|_{(\alpha, p)} := \| f \|_p + \sup_{t\neq 0}\frac{\| f(\cdot + t) + f(\cdot - t) - 2f(\cdot) \|_p}{| t |^\alpha} \text{ and } \| f \|_{0, p} = \| f \|_p.$$

The class of function $Z^{(\omega)}$ is defined as

$$Z^{(\omega)} := \{f \in C_{2\pi} : | f(x+t) + f(x-t) - 2f(x) | = O(\omega(t))\}$$

where $\omega$ is a Zygmund modulus of continuity that is $\omega$ is a positive, non-decreasing continuous function with the property, $\omega(0) = 0$, $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$.

Let $\omega : [0, 2\pi] \to R$ be an arbitrary function with $\omega(t) > 0$ for $0 < t < 2\pi$ $\lim_{t\to 0} \omega(t) = \omega(0) = 0$ define

$$Z_p^{(w)} := \left\{ f \in L_p : 1 \leq p \leq \infty \sup \frac{\| f(\cdot + t) + f(\cdot - t) - 2f(\cdot) \|_p}{\omega(t)} < \infty \right\}$$

and

$$\| f \|_p^{(\omega)} := \| f \|_p + \sup \frac{\| f(\cdot + t) + f(\cdot - t) - 2f(\cdot) \|_p}{\omega(t)} \; p \geq 1. \tag{2.5}$$

Clearly $\| \cdot \|_p^{(\omega)}$ is a norm on $Z_p^{(w)}$. As we know that $L_r(r \geq 1)$ is complete so the space $Z_p^{(w)}$ is also complete. Hence the Zygmund space $Z_p^{(w)}$ is a Banach space under the norm $\| \cdot \|_p^{(\omega)}$.

We write through the paper

$$\emptyset_x(t) = f(x + t) - 2f(x) + f(x - t) \tag{2.6}$$

$$K_n(t) = \frac{1}{2\pi P_n} \sum_{k=0}^{n} \frac{p_{n-k}}{(1 + q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{\sin\left(v + \frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right\}. \tag{2.7}$$

## 3. Main Result

In this paper we prove the following theorem.

**Theorem.** *Let f be a* $2\pi$ *periodic function, Lebesgue integrable in* $[0, 2\pi]$ *and belonging to generalized Zygmund class* $Z_r^{(w)}(r \geq 1)$. *Then the degree of approximation of function f by* $(N, p_n)(E, q)$ *product mean of Fourier series is given by*

$$E_n(f) = \inf \| t_n^{NE} - f \|_r^{v} = 0\left( \int_{\frac{1}{n+1}}^{\pi} \frac{w(t)}{tv(t)} \, dt \right)$$

*where* $\omega(t)$ *and* $v(t)$ *denote the Zygmund modulai of continuity such that* $\frac{w(t)}{v(t)}$ *is positive and increasing.*

## 4. Lemma

To prove the theorem we need the following lemma.

**Lemma 4(a).** *For* $0 \leq t \leq \frac{\pi}{n + 1}$ *we have* $\sin nt = n \sin t$

$$|K_n(t)| = o(n) \tag{4.1}$$

**Proof.** For $0 \leq t \leq \frac{\pi}{n+1}$ and $\sin nt = n \sin t$ then

$$|K_n(t)| = \left| \frac{1}{2\pi P_n} \sum_{k=o}^{n} \frac{p_{n-k}}{(1+q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right\} \right|$$

$$\leq \frac{1}{2\pi P_n} \left| \sum_{k=o}^{n} \frac{p_{n-k}}{(1+q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{(2v+1)\sin\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)} \right\} \right|$$

$$\leq \frac{1}{2\pi P_n} \left| \sum_{k=o}^{n} \frac{p_{n-k}}{(1+q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{(2v+1)\sin\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)} \right\} \right|$$

$$\leq \frac{1}{2\pi P_n} \left| \sum_{k=o}^{n} \frac{p_{n-k}}{(1+q)^k} (2k+1) \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \right\} \right|$$

$$\leq \frac{1}{2\pi P_n} \left| \sum_{k=0}^{n} p_{n-k} (2k+1) \right|$$

$$= \frac{(2n+1)}{2\pi P_n} \left| \sum_{k=0}^{n} p_{n-k} \right|$$

$$= o(n).$$

**Lemma 4(b).** *For* $\frac{\pi}{n+1} \leq t \leq \pi$, $\sin \frac{t}{2} \geq \frac{t}{\pi}$ *and* $\sin nt \leq 1$, *we have*

$$|K_n(t)| = o\left(\frac{1}{t}\right). \tag{4.2}$$

**Proof.** For $\frac{\pi}{n} \leq t \leq \pi$, $\sin \frac{t}{2} \geq \frac{t}{\pi}$ and $\sin nt \leq 1$

$$|K_n(t)| = \left| \frac{1}{2\pi P_n} \sum_{k=o}^{n} \frac{p_{n-k}}{(1+q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right\} \right|$$

$$\leq \frac{1}{2\pi P_n}\left|\sum_{k=o}^{n} \frac{p_{n-k}}{(1+q)^k}\left\{\sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{\pi}{t}\right\}\right|$$

$$\leq \frac{1}{2tP_n}\left|\sum_{k=o}^{n} p_{n-k}\right|$$

$$= o\left(\frac{1}{t}\right).$$

**Lemma 4(c).** *Let* $f \in Z_p^{(w)}$ *then for* $0 < t \leq \pi$

(i) $\| \phi(\cdot, t) \|_p = o(w(t))$

(ii) $\| \phi(. + y, t) + \phi(. - y, t) - 2\phi(., t) \|_p = \begin{cases} o(w(t)) \\ o(w(y)) \end{cases}$

(iii) *If* $\omega(t)$ *and* $v(t)$ *are defined as in theorem then*

$$\| \phi(. + y, t) + \phi(. - y, t) - 2\phi(., t) \|_p = \left\{ v(y) \frac{\omega(t)}{v(t)} \right.$$

*where* $\phi(x, t) = f(x+t) + f(x-t) - 2f(x)$.

## 5. Proof

**Proof of Theorem.** Let $S_n(x)$ denotes the partial sum of Fourier series given in (2.1) then we have

$$S_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \emptyset(t) \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dt \tag{5.1}$$

The $(E, q)$ transform $E_n^q$ of $S_n$ is given by

$$E_n^q - f(x) = \frac{1}{2\pi(1+q)^n}\int_0^{\pi} \emptyset(t) \left\{ \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right\} dt \tag{5.2}$$

The $(N, P_n)(E, q)$ transform of $S_n(x)$ is given by

$$t_n^{NE}(f) - f(x)$$

$$= \frac{1}{2\pi P_n} \sum_{k=0}^{n} \left[ \frac{p_{n-k}}{(1+q)^k} \int_0^{\pi} \emptyset(t) \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right\} dt \right] \tag{5.3}$$

$$= \int_0^{\pi} \emptyset(t) k_n(t). \tag{5.4}$$

Let $l_n(x) = t_n^{NE} - f(x) = \int_0^{\pi} \emptyset(x, t) k_n(t) dt$ then

$$l_n(x+y) + l_n(x-y) - 2l_n(x) = \int_0^{\pi} [\phi(x+y, t) + \phi(x-y, t) - 2\phi(x, t)] k_n(t) dt.$$

Using the generalized Minkowaski's inequality we get

$$\| \phi(.+y, t) + \phi(.-y, t) - 2\phi(., t) \|_p$$

$$= \left\{ \frac{1}{2\pi} \int_0^{\pi} | l_n(x+y) + l_n(x-y) - 2l_n(x) |^p dx \right\}^{\frac{1}{p}}$$

$$= \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^{\pi} [\phi(x+y, t) + \phi(x-y, t) - 2\phi(x, t)] k_n(t) dt \right|^p dx \right\}^{\frac{1}{p}}$$

$$\leq \int_0^{\pi} \left\{ \frac{1}{2\pi} \int_0^{2\pi} | [\phi(x+y, t) + \phi(x-y, t) - 2\phi(x, t)] k_n(t) |^p dx \right\}^{\frac{1}{p}} dt$$

$$= \int_0^{\pi} (| k_n(t) |^p)^{\frac{1}{p}} \left\{ \frac{1}{2\pi} \int_0^{2\pi} | [\phi(x, +y, t) + \phi(x-y, t) - 2\phi(x, t)] |^p dx \right\}^{\frac{1}{p}} dt$$

$$= \int_0^{\pi} \| \phi(.+y, t) + \phi(.-y, t) - 2\phi(., t) \|_p | k_n(t) | dt$$

$$= \int_0^{\frac{1}{n+1}} \| \phi(.+y, t) + \phi(.-y, t) - 2\phi(., t) \|_p | k_n(t) | dt$$

$$+\int_{\frac{1}{n+1}}^{\pi} \| \phi(.+y, t)+\phi(.-y, t)-2\phi(., t)\|_p | k_n(t) | dt$$

$$= I_1 + I_2 \cdot (say) \tag{5.5}$$

Using Lemma 4(a) and 4(c) and the monotonically of $\frac{\omega(t)}{v(t)}$ with respect to $t$, we have

$$I_1 = \int_0^{\frac{1}{n+1}} \| \phi(.+y, t)+\phi(.-y, t)-2\phi(., t)\|_p | k_n(t) | dt$$

$$= \int_0^{\frac{1}{n+1}} o\left(v(y)\frac{\omega(t)}{v(t)}\right) o(n) dt$$

$$= o\left(nv(y)\int_0^{\frac{1}{n+1}} \frac{\omega(t)}{v(t)} dt\right).$$

Using second mean value theorem of integral, we have

$$I_1 \leq o\left(nv(y)\int_0^{\frac{1}{n+1}} \frac{\omega(t)}{v(t)} dt\right)$$

$$= o\left(\frac{n}{n+1} v(y) \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right)$$

$$= o\left(v(y) \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right). \tag{5.6}$$

For $I_2$ using Lemma 4(b) and 4(c), we have

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} \| \phi(.+y, t)+\phi(.-y, t)-2\phi(., t)\|_p | k_n(t) | dt$$

$$= o\left(\int_{\frac{1}{n+1}}^{\pi}\left(v(y)\frac{\omega(t)}{v(t)}\right)\frac{1}{t}\,dt\right)$$

$$= o\left(v(y)\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{tv(t)}\right)dt\right). \tag{5.7}$$

From (5.5) (5.6) and (5.7), we get

$$\| l_n(.+y) + l_n(.-y) - 2l_n(.)\|_p = o\left(v(y)\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) + o\left(v(y)\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{tv(t)}\right)dt\right)$$

$$\sup_{y\neq 0}\frac{\| l_n(.+y) + l_n(.-y) - 2l_n(.)\|_p}{v(y)} = o\left(\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) + o\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{tv(t)}\right)\right). \tag{5.8}$$

Again using Lemma we have

$$\| l_n(.)\|_p \leq \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\cdot\right)\| \phi(.,t)\| |K_n(t)|dt$$

$$= o\left(n\int_0^{\frac{1}{n+1}}\omega(t)dt\right) + o\left(\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t}\,dt\right)$$

$$= o\left(\frac{n}{n+1}\,\omega\left(\frac{1}{n+1}\right)\right) + o\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t}\,dt\right)$$

$$= o\left(\omega\left(\frac{1}{n+1}\right)\right) + o\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t}\,dt\right). \tag{5.9}$$

From (5.8) and (5.9), we have

$$\| l_n(.)\|_p^v = \| l_n(.)\|_p + \sup_{y\neq 0}\frac{\| l_n(.+y) + l_n(.-y) - 2l_n(\cdot)\|_p}{v(y)}$$

$$= o\left(\omega\left(\frac{1}{n+1}\right)\right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t}\, dt\right) + o\left(\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{tv(t)}\right)\right)$$

$$= \sum_{i=1}^{4} J_i.$$

Now we write $J_1$ in terms of $J_3$ and $J_2$, $J_3$ in term of $J_4$.

In view of the monotonicity of $v(t)$ we have

$$\omega(t) = \left(\frac{\omega(t)}{v(t)}\right), v(t) \le v(\pi)\left(\frac{\omega(t)}{v(t)}\right) = o\left(\frac{\omega(t)}{v(t)}\right) \text{ for } 0 < t \le \pi$$

therefore we can write

$$J_1 = o(J_3).$$

Again using monotonicity of $v(t)$

$$J_2 = \int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t}\, dt = \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{tv(t)}\right) dt \le v(\pi) \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{tv(t)}\right) dt = o(J_4). \tag{5.10}$$

Using the fact $\frac{\omega(t)}{v(t)}$ is positive and non decreasing, we have

$$J_4 = \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{tv(t)}\right) dt = \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} \int_{\frac{1}{n+1}}^{\pi} \left(\frac{1}{t}\right) dt \ge \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}$$

therefore we can write

$$J_3 = o(J_4).$$

So we have

$$\| l_n(.) \|_p^v = o(J_4) = o\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{tv(t)}\right) dt\right).$$

Hence

$$E_n(f) = \inf \| l_n(.) \|_p^v = o\left( \int_{\frac{1}{n+1}}^{\pi} \left( \frac{\omega(t)}{tv(t)} \right) dt \right).$$

This completes the proof.

## 6. Corollaries

Following corollaries can be derived from our main theorem.

**Corollary 1.** *The degree of approximation of function* $Z_r^{(w)} (r \geq 1)$ *by* $(C, 1)(E, 1)$ *means*

$$(CE)_n^1 = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} s_v$$

*of Fourier series is given by*

$$E_n(f) = o\left( \int_{\frac{1}{n+1}}^{\pi} \left( \frac{\omega(t)}{tv(t)} \right) dt \right).$$

**Corollary 2.** *The degree of approximation of function* $Z_r^{(w)} (r \geq 1)$ *by* $(N, p_n)(E, 1)$ *means*

$$t_n^{NE} = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \left\{ \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} s_v \right\}$$

*of Fourier series is given by*

$$E_n(f) = O\left( \int_{\frac{1}{n+1}}^{\pi} \left( \frac{\omega(t)}{tv(t)} \right) dt \right).$$

## 7. Conclusion

In this study, different types of results on the degree of approximation of periodic function belonging to the Lipschitz classes and Zygmund classes of function are reviewed. The established theorem in this paper, on degree of approximation of function in the generalized Zygmund class by $(N, p_n)$

$(E, q)$ summability means of Fourier series, which generalizes the several known results. Moreover, the result can be extended for other functions belonging to weighted Zygmund class.

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