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ON CURVE PAIRS OF TZITZEICA TYPE

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Abstract

The most important curve pairs in differential geometry are involute-evolute, Bertrand, and Mannheim curve pairs. In this study, the condition of the conjugate of an original curve to be a Tzitzeica curve is formulated for each of these special curve pairs in Euclidean 3-space. Moreover, the particular states of the curvatures of the original curve of curve pairs are considered, and the conditions of the conjugate curves to be a Tzitzeica curve are interpreted. Especially, if a curve is a planar curve, circle, or helix, it is found whether its conjugate satisfies the condition of being a Tzitzeica curve.

1. Introduction

In 1911, Gheorghe Tzitzeica who is a Romanian mathematician, first expressed the Tzitzeica curves as the class of a space curve. For any curve α, τ is the torsion of the curve α and d^2 is the square of the distance between the origin and its osculating plane at an arbitrary point of the curve α. The curve α is called the Tzitzeica curve in Euclidean space if $\frac{\tau}{d^2}$ is a non-zero constant [1]. Since Gheorghe Tzitzeica researched affine invariants, the Tzitzeica curves and Tzitzeica surfaces are considered to be the first

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examples of affine invariants in differential geometry. Afterward, further studies have been produced on the Tzitzeica curve and Tzitzeica surface in Euclidean and Minkowski spaces, see [2, 3, 4, 5]. In [6, 7], depending on the solution of the harmonic equation, elliptic and hyperbolic cylindrical curves that satisfy the condition of being a Tzitzeica curve were investigated. In particular, the conditions of Salkowski and anti-Salkowski curves, rectifying curves, and spherical curves to be Tzitzeica curves were investigated in Euclidean space [8] and Minkowski space [9].

On the other hand, the well-known curve pairs in differential geometry are involute evolute, Bertrand, and Mannheim curve pairs. Recently, many research papers related to involute evolute curve [10, 11, 12, 13], Bertrand curve pair [14, 15, 16], and Mannheim curve pair [17, 18, 19, 20] have been treated in different spaces.

In this study, our aim is to formulate the condition of a pair of curves to be the Tzitzeica curve in Euclidean 3-space. In the first part of the study, a literature review is given. In the second part, definitions and theorems related to curve pairs are expressed. In the last part, according to the specific states of the curvatures of the curve, the condition of a curve to be a Tzitzeica curve has been investigated. Especially if the curve is planar, circle, and helix, it has been found whether this curve pair satisfies the Tzitzeica curve condition.

2. Preliminaries

Let $\alpha = \alpha(s)$ be a regular curve with speed $\nu = \| \alpha'(s) \| = dt/ds$ in Euclidean 3-space. If T, N and B denote the tangent, principal normal and binormal unit vectors at any point $\alpha(s)$ of the curve α, respectively, then the Frenet formulas are given

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \nu \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $\kappa = \dfrac{\| \alpha' \times \alpha'' \|}{\| \alpha' \|^3}$ is the curvature and $\tau = \dfrac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{\| \alpha' \times \alpha'' \|^2}$ is the torsion of the curve α.

Definition 2.1. Let $\alpha : I \to E^3$ and $\alpha_i : I \to E^3$ be regular curves. $\{T, N, B, \kappa, \tau\}$ and $\{T_i, N_i, B_i, \kappa_i, \tau_i\}$ denote the Frenet apparatus of the curves α and α_i at points $\alpha(s)$ and $\alpha_i(s)$, respectively, for all $s \in I$. The curve α_i is called involute of the curve α and the curve α is called evolute of the curve α_i, if α_i lie on the tangent surface $(\alpha_i(s_0)$ lies on the tangent line to α at $\alpha(s_0))$ and the tangents to α and α_i are perpendicular at $\alpha(s_0)$ and $\alpha_i(s_0)$, that is,

$$\langle T, T_i \rangle = 0.$$

Thus, it is evident that if α is a regular curve (not necessarily unit speed), then the involute curve of the curve α is given by

$$\alpha_i(s) = \alpha(s) - f(s)T(s), \tag{2.1}$$

where $f(s) = \int_{s_0}^{s} \| \alpha'(u) \| du$ is the arc-length function of the curve α, that is, $f'(s) = \| \alpha'(s) \| = \nu$ for all $s \in I$.

Especially, if α is a unit speed curve, then the formula of the curve α_i which is involute of the curve α is

$$\alpha_i(s) = \alpha(s) + (c - s)T(s),$$

where c is non-zero constant for all $s \in I$ [10, 11].

Definition 2.2. Let $\alpha : I \to E^3$ and $\alpha_b : I \to E^3$ be regular curves with arc-length parameters s and s_b in E^3. At points $\alpha(s)$ and $\alpha_b(s)$, the Frenet apparatus of the curvesαand α_b are $\{T, N, B, \kappa, \tau\}$ and $\{T_b, N_b, B_b, \kappa_b, \tau_b\}$, respectively. If the principal normal vectors of the curvesαand α_b are linearly dependent, the curve pair (α, α_b) is called the Bertrand curve pair, and the equation of the curve α_b is given by

$$\alpha_b = \alpha + \lambda N, \tag{2.2}$$

where λ is non-zero constant [16].

Definition 2.3. Let $\alpha : I \to E^3$ and $\alpha_m : I \to E^3$ be regular curves with arc-length parameters s and s_m in E^3. At points $\alpha(s)$ and $\alpha_m(s)$, the Frenet apparatus of the curves α and α_m are given as $\{T, N, B, \kappa, \tau\}$ and $\{T_m, N_m, B_m, \kappa_m, \tau_m\}$, respectively. If the principal normal vectors of the curve α and the binormal vector of the curve α_m are linearly dependent, then the curve α is called the Mannheim curve, the curve α_m is called the Mannheim pair of the curve α and the curve pair (α, α_m) is called the Mannheim curve pair. The equation of the curve α_m is

$$\alpha_m = \alpha - \varepsilon N, \tag{2.3}$$

where ε is non-zero constant [18].

Definition 2.4. Let $\alpha : I \to E^3$ be a regular curve with arc-length parameters and $\{T, N, B, \kappa, \tau\}$ be the Frenet apparatus in E^3. For the curve α, the following definitions are given

i. The curve α is a line if and only if $\kappa = 0$,

ii. The curve α is planar if and only if $\tau = 0$,

iii. The curve α is a circle if and only if $\kappa > 0$ is constant and $\tau = 0$,

iv. The curve α is a circular helix if and only if $\kappa > 0$ is constant and τ is constant,

v. The curve α is a cylindrical helix if and only if $\frac{\kappa}{\tau}$ is constant [21, 22].

3. On Curve Pairs of Tzitzeica Type

In this part of the study, we present conditions for the involute-evolute curve, Bertrand curve pair, and Mannheim curve pair to be Tzitzeica curve in E^3.

Theorem 3.1. *Let* $\alpha_i : I \to E^3$ *be an involute curve of a regular curve* $\alpha : I \to E^3$. *The involute curve* α_i *is a Tzitzeica curve if and only if*

$$\frac{5\nu\kappa'\tau + \kappa(6\nu'\tau + \nu\tau')}{f\nu^2\kappa(-\tau\langle\alpha_i, T\rangle + \kappa\langle\alpha_i, B\rangle)^2} = \rho_i,$$

such that ρ_i *is non-zero constant where* $\{T, N, B, \kappa, \tau\}$ *is Frenet apparatus of the curve* α *and* $f(s) = \int_{s_0}^{s} \| \alpha'(u) \| du$ *is the arc-length function of the curve* α.

Proof. If $\alpha_i : I \to E^3$ is the involute curve of a regular curve $\alpha : I \to E^3$, then we can write the equation of the involute curve of the curve α as

$$\alpha_i(s) = \alpha(s) - f(s)T(s).$$

For all $s \in I$, taking the derivatives of this curve equation, we find

$$\alpha_i' = -f\nu\kappa N,$$

$$\alpha_i'' : f\nu^2\kappa^2 T - (\kappa(\nu^2 + f\nu') + f\nu\kappa')N - f\nu^2\kappa\tau B,$$

$$\begin{aligned}\alpha_i''' = {} & \nu\kappa(2\nu^2\kappa + 3f\nu'\kappa + 3f\nu\kappa')T \\ & + (\nu(3\nu'\kappa + 2\nu\kappa') + f(\nu^3\kappa(\kappa^2 + \tau^2) - 2\nu'\kappa' - \nu''\kappa - \nu\kappa''))N \\ & + (2\nu^3\kappa\tau - f\nu(2\nu\kappa'\tau + \kappa(3\nu'\tau + \nu\tau')))B.\end{aligned}$$

Considering these last three equations, we find the torsion τ_i as

$$\tau_i = \frac{\langle \alpha_i' \times \alpha_i'' \times \alpha_i''' \rangle}{\| \alpha_i' \times \alpha_i'' \|^2} = -\frac{5\nu\kappa'\tau + \kappa(6\nu'\tau + \nu\tau')}{f\nu^2\kappa(\kappa^2 + \tau^2)}. \tag{3.1}$$

For curve α_i, the square of the distance between the origin and its osculating plane at an arbitrary point of the curve α_i is found by

$$d_i^2 = \frac{\langle \alpha_i, \alpha_i' \times \alpha_i'' \rangle^2}{\| \alpha_i' \times \alpha_i'' \|^2} = \frac{(-f^2\nu^3\kappa^2\tau\langle\alpha_i, T\rangle + f^2\nu^3\kappa^3\langle\alpha_i, B\rangle)^2}{f^4\nu^6\kappa^4(k^2 + \tau^2)}. \tag{3.2}$$

From the ratio of the equation (3.1) to the equation (3.2), we obtain

$$\frac{\tau_i}{d_i^2} = -\frac{5\nu\kappa'\tau + \kappa(6\nu'\tau + \nu\tau')}{f\nu^2\kappa(-\tau\langle\alpha_i, T\rangle + \kappa\langle\alpha_i, B\rangle)^2}.$$

The condition of being a Tzitzeica curve is $\frac{\tau_i}{d_i^2} = \frac{\langle\alpha_i' \times \alpha_i'' \times \alpha_i'''\rangle}{\langle\alpha_i, \alpha_i' \times \alpha_i''\rangle^2} = \text{constant}$ $\neq 0$, where τ_i is the torsion and d_i is the distance from the origin to the osculating plane at any point of the curve. This completes the proof.

Corollary 3.2. *Let $\alpha_i : I \to E^3$ be an involute curve of a regular curve $\alpha : I \to E^3$, then the involute curve α_i is a Tzitzeica curve if and only if*

$$\frac{\tau_i}{d_i^2} = -\frac{5\nu\kappa'\tau + \kappa(6\nu'\tau + \nu\tau')}{f\nu^2\kappa(-\tau\langle\alpha_i, T\rangle + \kappa\langle\alpha_i, B\rangle)^2} = \rho_i,$$

where ρ_i is a non-zero constant.

Corollary 3.3. *Let $\alpha_i : I \to E^3$ be an involute curve of a regular curve $\alpha : I \to E^3$ and the curveαbe a circle or any planar curve, then the involute curve α_i of the curve α cannot be a Tzitzeica curve.*

Corollary 3.4. *Let $\alpha_i : I \to E^3$ be an involute curve of a regular curve $\alpha : I \to E^3$ and the curveαbe a curve with constant curvatures or circular helix. The involute curve α_i of the curve$\alpha$is a Tzitzeica curve if and only if*

$$\frac{\tau_i}{d_i^2} = -\frac{6\nu'\tau}{f\nu^2(-\tau\langle\alpha_i, T\rangle + \kappa\langle\alpha_i, B\rangle)^2}.$$

Now, we express the condition of the Bertrand partner curve to be a Tzitzeica curve, and we interpret this condition in terms of the curvatures of the Bertrand curve pair.

Theorem 3.5. *Let $\alpha_b : I \to E^3$ be a Bertrand partner of a regular curve $\alpha : I \to E^3$, α_b satisfies the following equation*

$$\frac{\tau_b}{d_b^2} = \frac{\begin{array}{l}\nu^3\tau(\kappa(-1+\lambda\kappa)+\lambda\tau^2)^2 - \lambda^2\nu'(\kappa(-1+\lambda\kappa)+\lambda\tau^2)(-\lambda\tau\kappa'+\tau'(-1+\lambda\kappa)) \\ +\lambda\nu(-\lambda^2\tau^3\kappa'' + \lambda\tau((-1+3\lambda\kappa)\kappa'^2 - (-1+\lambda\kappa)(3\tau'^2+\kappa\kappa''))) \\ +\lambda\tau^2(3\lambda\kappa'\tau' + \tau''(-1+\lambda\kappa) + (-1+\lambda\kappa)(\kappa'\tau'(1-3\lambda\kappa)+\kappa\tau''(-1+\lambda\kappa)))\end{array}}{\nu\begin{pmatrix}\lambda\nu\tau(\kappa(-1+\lambda\kappa)+\lambda\tau^2)\langle\alpha_b, T\rangle - \lambda(\lambda\tau\kappa'+\tau'(1-\lambda\kappa))\langle\alpha_b, N\rangle \\ +\nu(-1+\lambda\kappa)(\kappa(-1+\lambda\kappa)+\lambda\tau^2)\langle\alpha_b, B\rangle\end{pmatrix}^2},$$

where τ_b *is the torsion of the curve* α_b *and* d_b *is the distance between the origin and its osculating plane at any point of* α_b.

Proof. If $\alpha_b : I \to E^3$ is a Bertrand partner of a regular curve $\alpha : I \to E^3$, then we can write

$$\alpha_b = \alpha + \lambda N,$$

where λ is constant. For all $s \in I$, taking the derivatives of this curve equation, we find

$$\alpha_b' = \nu(1-\lambda\kappa)T + \nu\lambda\tau B,$$

$$\alpha_b'' = ((1-\lambda)\nu' - \nu\lambda\kappa')T + \nu^2((1-\lambda\kappa)\kappa - \lambda\tau^2)N + \lambda(\nu'\tau + \nu\tau')B,$$

$$\begin{aligned}\alpha_b''' = {} & (\nu^3\kappa(\kappa(-1+\lambda\kappa)+\lambda\tau^2) - 2\lambda\nu'\kappa' + \nu''(1-\lambda\kappa) - \lambda\nu\kappa'')T \\ & + \nu(-3\lambda\nu'\kappa^2 + \nu\kappa' + 3\kappa(\nu' - \lambda\nu\kappa') - 3\lambda\tau(\tau\nu' + \nu\tau'))N \\ & + (\nu^3\tau(\kappa - \lambda\kappa^2 - \lambda\tau^2) + \lambda(2\nu'\tau' + \tau\nu'') + \lambda\nu\tau'')B.\end{aligned}$$

Considering these last three equations, we find the torsion of α_b as

$$\tau_b = \frac{\langle\alpha_b, \alpha_b' \times \alpha_b''\rangle}{\| \alpha_b' \times \alpha_b'' \|}$$

$$= \frac{\begin{array}{l}\nu^3\tau(\kappa(-1+\lambda\kappa)+\lambda\tau^2)^2 - \lambda\nu'(\kappa(-1+\lambda\kappa)+\lambda\tau^2)(-\lambda\tau\kappa'+\tau'(-1+\lambda\kappa)) \\ +\lambda\nu(-\lambda^2\tau^3\kappa'' + \lambda\tau((-1+3\lambda\kappa)\kappa'^2 - (-1+\lambda\kappa)(3\tau'^2+\kappa\kappa''))) \\ +\lambda\tau^2(3\lambda\kappa'\tau' + \tau''(-1+\lambda\kappa) + (-1+\lambda\kappa)(\kappa'\tau'(1-3\lambda\kappa)+\kappa\tau''(-1+\lambda\kappa)))\end{array}}{\nu(\nu^2(\kappa(-1+\lambda\kappa)+\lambda\tau^2) + (-1+\lambda\kappa)^2 + \lambda^2\tau^2 + \lambda^2(\lambda\tau\kappa' + \tau'(1-\lambda\kappa))^2)}$$

Also, the square of the distance from the origin to the osculating plane of the curve α_b is

$$d_b^2 = \frac{\langle \alpha_b, \alpha_b' \times \alpha_b'' \rangle^2}{\| \alpha_b' \times \alpha_b'' \|} = \frac{\begin{matrix}(\lambda\nu^3\tau(\kappa(-1+\lambda\kappa)+\lambda\tau^2)\langle \alpha_b, T\rangle \\ -\lambda\nu^2(\lambda\tau\kappa' + \tau'(1-\lambda\kappa))\langle \alpha_b, N\rangle \\ +\nu^3(-1+\lambda\kappa)(\kappa(-1+\lambda\kappa)+\lambda\tau^2)\langle \alpha_b, B\rangle)^2\end{matrix}}{\nu\begin{pmatrix}\nu^2(\kappa(-1+\lambda\kappa)+\lambda\tau^2)^2 \\ +(-1+\lambda\kappa)^2+\lambda^2\tau^2+\lambda^2(\lambda\tau\kappa'+\tau'(1-\lambda\kappa))^2\end{pmatrix}}. \quad (3.4)$$

Thus, the ratio of the equation (3.3) to the equation (3.4) gives us

$$\frac{\tau_b}{d_b^2} = \frac{\begin{matrix}\nu^3\tau(\kappa(-1+\lambda\kappa)+\lambda\tau^2)^2 - \lambda^2\nu'(\kappa(-1+\lambda\kappa)+\lambda\tau^2)(-\lambda\tau\kappa'+\tau'(-1+\lambda\kappa)) \\ +\lambda\nu(-\lambda^2\tau^3\kappa''+\lambda\tau((-1+3\lambda\kappa)\kappa'^2-(-1+\lambda\kappa)(3\tau'^2+\kappa\kappa''))) \\ +\lambda\tau^2(3\lambda\kappa'\tau'+\tau''(-1+\lambda\kappa)+(-1+\lambda\kappa)(\kappa'\tau'(1-3\lambda\kappa)+\kappa\tau''(-1+\lambda\kappa)))\end{matrix}}{\nu\begin{pmatrix}\lambda\nu\tau(\kappa(-1+\lambda\kappa)+\lambda\tau^2)\langle \alpha_b, T\rangle - \lambda(\lambda\tau\kappa'+\tau'(1-\lambda\kappa))\langle \alpha_b, N\rangle \\ +\nu(-1+\lambda\kappa)(\kappa(-1+\lambda\kappa)+\lambda\tau^2)\langle \alpha_b, B\rangle\end{pmatrix}^2}.$$

This completes the proof. The condition of being a Tzitzeica curve is $\frac{\tau_b}{d_b^2} = \frac{\langle \alpha_b' \times \alpha_b'', \alpha_b''' \rangle}{(\langle \alpha_b, \alpha_b' \times \alpha_b'' \rangle)^2}$, where τ_b is the torsion and d_b is the distance from the origin to the osculating plane at any point of the curve α_b. According to this condition and the last theorem, one can characterize the Tzitzeica curve for a Bertrand curve pair in the following corollary.

Corollary 3.6. *Let $\alpha_b : I \to E^3$ be a Bertrand partner of a regular curve $\alpha : I \to E^3$, then the Bertrand pair curve α_b is a Tzitzeica curve if and only if*

$$\frac{\begin{matrix}\nu^3\tau(\kappa(-1+\lambda\kappa)+\lambda\tau^2)^2 - \lambda\nu'(\kappa(-1+\lambda\kappa)+\lambda\tau^2)(-\lambda\tau\kappa'+\tau'(-1+\lambda\kappa)) \\ +\lambda\nu(-\lambda^2\tau^3\kappa''+\lambda\tau((-1+3\lambda\kappa)\kappa'^2-(-1+\lambda\kappa)(3\tau'^2+\kappa\kappa''))) \\ +\lambda\tau^2(3\lambda\kappa'\tau'+\tau''(-1+\lambda\kappa)+(-1+\lambda\kappa)(\kappa'\tau'(1-3\lambda\kappa)+\kappa\tau''(-1+\lambda\kappa)))\end{matrix}}{\nu\begin{pmatrix}\lambda\nu\tau(\kappa(-1+\lambda\kappa)+\lambda\tau^2)\langle \alpha_b, T\rangle - \lambda(\lambda\tau\kappa'+\tau'(1-\lambda\kappa))\langle \alpha_b, N\rangle \\ +\nu(-1+\lambda\kappa)(\kappa(-1+\lambda\kappa)+\lambda\tau^2)\langle \alpha_b, B\rangle\end{pmatrix}^2} = \rho_b,$$

where ρ_b is a non-zero constant.

Corollary 3.7. *Let* $\alpha_b : I \to E^3$ *be a Bertrand partner of a regular curve* $\alpha : I \to E^3$ *and the curve be a planar curve or a circle, then the Bertrand partner curve* α_b *is not a Tzitzeica curve.*

Corollary 3.8. *Let* $\alpha_b : I \to E^3$ *be a Bertrand partner of a regular curve* $\alpha : I \to E^3$ *and the curve* α *be a circular helix curve, then* α_b *is a Tzitzeica curve if and only if*

$$\frac{\tau_b}{d_b^2} = \frac{\begin{matrix} \nu^3\tau(\kappa(-1+\lambda\kappa)+\lambda\tau^2)^2 + \lambda^2\tau\kappa'\nu'(\kappa(-1+\lambda\kappa)+\lambda\tau^2) \\ +\lambda(-\lambda^2\tau^3\kappa'' + \lambda\tau((-1+3\lambda\kappa)\kappa'^2 - \kappa\kappa''(-1+\lambda\kappa))) \end{matrix}}{\begin{pmatrix} \lambda\nu\tau(\kappa(-1+\lambda\kappa)+\lambda\tau^2)\langle\alpha_b, T\rangle - \lambda^2\tau\kappa'\langle\alpha_b, N\rangle \\ +\nu(-1+\lambda\kappa)(\kappa(-1+\lambda\kappa)+\lambda\tau^2)\langle\alpha_b, B\rangle \end{pmatrix}^2}.$$

In this regard, the condition of the Mannheim pair curve to be a Tzitzeica curve and the interpretation of this condition in terms of the curvatures of the Mannheim pair curve are given in the following.

Theorem 3.9. *Let* $\alpha_m : I \to E^3$ *be a Mannheim partner of a regular curve* $\alpha : I \to E^3$, *then at any point of the curve* α_m, *the Mannheim partner curve* α_m *satisfies the following equation*

$$\frac{\tau_m}{d_m^2} = \frac{\begin{pmatrix} \nu^3\tau(\kappa(1+\varepsilon\kappa)+\varepsilon\tau^2)^2 + \varepsilon\nu'(\kappa(1+\varepsilon\kappa)+\varepsilon\tau^2)(-c\tau\kappa' + \tau'(1+\varepsilon\kappa)) \\ +\varepsilon\nu\begin{pmatrix} \varepsilon^2\tau^3\kappa'' + \varepsilon\tau(-\kappa'^2(1+3\varepsilon\kappa) + (1+\varepsilon\kappa)(3\tau'^2 + \kappa\kappa'')) \\ -\varepsilon\tau^2(3\varepsilon\kappa'\tau' + \tau''(1+\varepsilon\kappa)) + (1+\varepsilon\kappa)((1+3\varepsilon\kappa)\kappa'\tau' - \kappa\tau''(1+\varepsilon\kappa)) \end{pmatrix} \end{pmatrix}}{\nu\begin{pmatrix} \varepsilon\nu\tau((1++\varepsilon\kappa)\kappa + \varepsilon\tau^2)\langle\alpha, T\rangle + \varepsilon((1+\varepsilon\kappa)\tau' - \varepsilon\kappa'\tau)\langle\alpha, N\rangle \\ +\nu(1+\varepsilon\kappa)((1+\varepsilon\kappa)\kappa + \varepsilon\tau^2)\langle\alpha, B\rangle \end{pmatrix}^2},$$

where d_m *is the distance between the origin and its osculating plane at any point of* α_m.

Proof. If $\alpha_m''' : I \to E^3$ is a Mannheim partner of a regular curve $\alpha : I \to E^3$, then we can give the equation of the Mannheim pair curve as

$$\alpha_m = \alpha - \varepsilon N,$$

where ε is a non-zero constant. For all $s \in I$, taking the derivatives of this curve equation, we find as

$$\alpha'_m = \nu(1+\varepsilon\kappa)T - \varepsilon\nu\tau B,$$

$$\alpha''_m = (\nu'(1+\varepsilon\kappa) + \varepsilon\nu\kappa')T + \nu^2((1+\varepsilon\kappa)\kappa + \varepsilon\tau^2)N - \varepsilon(\tau\nu' + \nu\tau')B,$$

$$\alpha'''_m = (-\nu^3\kappa(\kappa(1+\varepsilon\kappa) + \varepsilon\tau^2) + 2\varepsilon\nu'\kappa' + (1+\varepsilon\kappa)\nu'' + \varepsilon\nu\kappa'')T$$

$$+ \nu(3\varepsilon\kappa^2\nu' + \nu\kappa' + 3\kappa(\nu' + 3\nu\kappa') + 3\varepsilon\tau(\tau\nu' + \nu\tau'))N$$

$$+ (\nu^3\tau(\kappa(1+\varepsilon\kappa) + \varepsilon\tau^2) - \varepsilon(2\nu'\tau' + \tau\nu'') - \varepsilon\nu\tau'')B.$$

Considering these last three equations, we find the torsion of α_m as

$$\tau_m = \frac{\langle \alpha, \alpha'_m \times \alpha''_m \rangle}{\| \alpha'_m \times \alpha''_m \|^2}$$

$$= \frac{\begin{array}{l} \nu^3\tau(\kappa(1+\varepsilon\kappa) + \varepsilon\tau^2)^2 + \varepsilon\nu'(\kappa(1+\varepsilon\kappa) + \varepsilon\tau^2)(-\varepsilon\tau\kappa' + \tau'(1+\varepsilon\kappa)) \\ + \varepsilon\nu\left(\begin{array}{l} \varepsilon^2\tau^3\kappa'' + \varepsilon\tau(-\kappa'^2(1+\varepsilon\kappa) + (1+\varepsilon\kappa)(3\tau'^2 + \kappa\kappa'')) \\ -\varepsilon\tau^2(3\varepsilon\kappa'\tau' + \tau''(1+\varepsilon\kappa)) \\ + (1+\varepsilon\kappa)(\kappa'\tau'(1+3\varepsilon\kappa) - \kappa\tau''(1+\varepsilon\kappa)) \end{array}\right) \end{array}}{\nu\left(\begin{array}{l} \nu^2(\kappa(1+\varepsilon\kappa) + \varepsilon\tau^2)^2((1+\varepsilon\kappa)^2 + \varepsilon^2\tau^2) \\ + \varepsilon^2(\varepsilon\tau\kappa' - (1+\varepsilon\kappa)\tau')^2 \end{array}\right)}. \tag{3.5}$$

Also, the square of the distance from the origin to the osculating plane of the curve α_m is

$$d_m^2 = \frac{(\langle \alpha, \alpha'_m \times \alpha''_m \rangle)}{\| \alpha'_m \times \alpha''_m \|} =$$

$$\frac{\left(\begin{array}{l} \varepsilon\nu\tau(\kappa(1+ +\varepsilon\kappa) + \varepsilon\tau^2)\langle \alpha, T\rangle + \varepsilon(-\varepsilon\tau\kappa' + \tau'(1+\varepsilon\kappa))\langle \alpha, N\rangle \\ + \nu(1+\varepsilon\kappa)(\kappa(1+\varepsilon\kappa) + \varepsilon\tau^2)\langle \alpha, B\rangle \end{array}\right)^2}{\nu^2(\kappa(1+ +\varepsilon\kappa) + \varepsilon\tau^2)^2((1+ +\varepsilon\kappa)^2 + \varepsilon^2\tau^2) + \varepsilon^2(\varepsilon\tau\kappa' - \tau'(1+ +\varepsilon\kappa))^2} \tag{3.6}$$

Thus, the ratio of the equation (3.5) to the equation (3.6) gives us

$$\frac{\tau_m}{d_m^2} = \frac{\begin{pmatrix} \nu^3\tau(\kappa(1+\varepsilon\kappa)+\varepsilon\tau^2)^2 + \varepsilon\nu'(\kappa(1+\varepsilon\kappa)+\varepsilon\tau^2)(-\varepsilon\tau\kappa'+\tau'(1+\varepsilon\kappa)) \\ +\varepsilon\nu\begin{pmatrix} \varepsilon^2\tau^3\kappa'' + \varepsilon\tau(-\kappa'^2(1+3\varepsilon\kappa)+(1+\varepsilon\kappa)(3\tau'^2+\kappa\kappa'')) \\ -\varepsilon\tau^2(3\varepsilon\kappa'\tau'+\tau''(1+\varepsilon\kappa))+(1+\varepsilon\kappa)(\kappa'\tau'(1+3\varepsilon\kappa)-\kappa\tau''(1+\varepsilon\kappa)) \end{pmatrix} \end{pmatrix}}{\nu\begin{pmatrix} \varepsilon\nu\tau((1++\varepsilon\kappa)\kappa+\varepsilon\tau^2)\langle\alpha, T\rangle + \varepsilon((1+\varepsilon\kappa)\tau'-\varepsilon\kappa'\tau)\langle\alpha, N\rangle \\ +\nu(1+\varepsilon\kappa)((1+\varepsilon\kappa)\kappa+\varepsilon\tau^2)\langle\alpha, B\rangle \end{pmatrix}}.$$

Corollary 3.10. *Let* $\alpha_m : I \to E^3$ *be a Mannheim partner of a regular curve* $\alpha : I \to E^3$, *then the Mannheim pair* α_m *is a Tzitzeica curve if and only if*

$$\frac{\begin{pmatrix} \nu^3\tau(\kappa(1+\varepsilon\kappa)+\varepsilon\tau^2)^2 + \varepsilon\nu'(\kappa(1+\varepsilon\kappa)+\varepsilon\tau^2)(-\varepsilon\tau\kappa'+\tau'(1+\varepsilon\kappa)) \\ +\varepsilon\nu\begin{pmatrix} \varepsilon^2\tau^3\kappa'' + \varepsilon\tau(-\kappa'^2(1+3\varepsilon\kappa)+(1+\varepsilon\kappa)(3\tau'^2+\kappa\kappa'')) \\ -\varepsilon\tau^2(3\varepsilon\kappa'\tau'+\tau''(1+\varepsilon\kappa))+(1+\varepsilon\kappa)(\kappa'\tau'(1+3\varepsilon\kappa)-\kappa\tau''(1+\varepsilon\kappa)) \end{pmatrix} \end{pmatrix}}{\nu\begin{pmatrix} \varepsilon\nu\tau((1++\varepsilon\kappa)\kappa+\varepsilon\tau^2)\langle\alpha, T\rangle + \varepsilon((1+\varepsilon\kappa)\tau'-\varepsilon\kappa'\tau)\langle\alpha, N\rangle \\ +\nu(1+\varepsilon\kappa)((1+\varepsilon\kappa)\kappa+\varepsilon\tau^2)\langle\alpha, B\rangle \end{pmatrix}} = \rho_m,$$

where ρ_m *is a non-zero constant.*

Corollary 3.11. *Let* $\alpha_m : I \to E^3$ *be a Mannheim partner of a regular curve* $\alpha : I \to E^3$ *and the Mannheim curve* α *be a planar curve or a circle, then the Mannheim pair is* α_m *not a Tzitzeica curve.*

Corollary 3.12. *Let* $\alpha_m : I \to E^3$ *be a Mannheim partner of a regular curve* $\alpha : I \to E^3$ *and* α *be a circular helix, then the Mannheim partner curve* α_m *is a Tzitzeica curve if and only if*

$$\frac{\begin{pmatrix} \nu^3\tau(\kappa(1+\varepsilon\kappa)+\varepsilon\tau^2)^2 - \varepsilon^2\nu'\tau\kappa'(\kappa(1+\varepsilon\kappa)+\varepsilon\tau^2 \\ +\varepsilon\nu(\varepsilon^2\tau^3\kappa''+\varepsilon\tau(-\kappa'^2(1+3\varepsilon\kappa)+(1+\varepsilon\kappa)(+\kappa\kappa''))) \end{pmatrix}}{\nu(\varepsilon\nu\tau((1+\varepsilon\kappa)\kappa+\varepsilon\tau^2)\langle\alpha, T\rangle - \varepsilon^3\kappa'\tau\langle\alpha, N\rangle + \nu(1+\varepsilon\kappa)((1+\varepsilon\kappa)\kappa+\varepsilon\tau^2)\langle\alpha, B\rangle)^2}$$

is a non-zero constant.

4. Conclusion

In this study, the conditions of involute-evolute, Bertrand, and Mannheim curve pairs to be a Tzitzeica curve are formulated for each of these special curve pairs in Euclidean 3-space. According to the specific states of the curvatures of the curve, the conditions of a curve pair to be a Tzitzeica curve have been investigated. Especially if the original curve is planar, circle, or helix, it is found whether its conjugate satisfies the Tzitzeica curve condition.

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