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EXTENSION OF WEAKLY COMPATIBLE MAPS IN *G*-METRIC SPACE

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Abstract

The main incentive of this paper is to associate some results in the literature by conferring the occurrence and distinctiveness of fixed points for new classes of mappings define on G-metric space. Precisely, we use the notion of the property E.A, to prove fixed point theorems for weakly compatible maps in G-metric space.

1. Introduction

The fixed point theorems ins metric spaces are playing a major role to construct methods in mathematics to solve problems in applied mathematics and sciences. So the attraction of metric spaces to a large numbers of mathematicians is understandable. Some generalizations of the notion of a metric space have been proposed by some authors. After Gahler [4, 5] introduced the concept of 2-metric space and proved that a 2-metric is a generalization of usual notion of the metric. But different authors proved that

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there is no relation between these two functions. For instance, Ha et al. [8] showed that a 2-metric space need not be a continuous function of its variables, whereas an ordinary metric is continuous function of its variables these considerations led Dhage [2] in 1992, to introduce a new class of generalized metric called as D-metric. Dhage claimed in his doctoral research work that D-metric provides a generalization of ordinary metric functions. Further in 2004, Mustafa and Sims [12] demonstrated that most of the claims concerning the fundamental topological properties of D-metric spaces are incorrect and proved that a D-metric need not be a continuous function of its variables. These consideration led researchers to seek a more appropriate notion of generalized metric space. In 2006, Mustafa and Sims [13] introduced the concept of G-metric space to overcome fundamental flaws in Dhage's theory of generalized metric spaces. Afterwards, Mustafa et al. [15], Shatanawi [18], Manro and Bhatia [11], Saadati et al. [16] and Mustafa and Sims [14] obtained several fixed point theorems for mappings satisfying different contractive conditions in G-metric spaces. In fact Shatanawi et al. [17] and others studied many fixed point results for a self mapping in G-metric space under certain conditions [10, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28 and 29].

Considering the contemplations given by different researchers, the main incentive of this paper is to associate some results in the literature by conferring the occurrence and distinctiveness of fixed points for new classes of mappings define on G-metric space. Precisely, we use the notion of the property E.A, to prove fixed point theorems for weakly compatible maps in G-metric space. Here, we give preliminaries and basic definitions which are helpful in the sequel.

2. Preliminaries

Definition 2.1. Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ (where R^+ is the set of all non negative real numbers) be a function satisfying the following axioms:

(G_1) $G(x, y, z) = 0$ if $x = y = z$,

(G_2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,

(G_3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (Symmetry in all three variables)

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (Rectangle inequality)

Then the function G is called a generalized metric or more specifically a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 2.2. Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points in X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m, n \to \infty} G(x, x_n, x_m) = 0$ and one says that sequence $\{x_n\}$ is G-convergent to x. So, that if $x_n \to x$ or $\lim_{n \to \infty} x_n \to x$ as $n \to \infty$ in a G-metric space (X, G) then for each $\epsilon > 0$, there exists $k \in N$ such that $G(x, x_n, x_m) < \epsilon$ for all $m, n \geq k$.

Proposition 2.1. *Let (X, G) be a G-metric space. Then the following are equivalent*:

(1) $\{x_n\}$ *is G-convergent to x*,

(2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$,

(3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$,

(4) $G(x_m, x_n, x) \to 0$ *as* $m, n \to \infty$.

Definition 2.3. Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\epsilon > 0$ there exists $k \in N$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq k$ that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.2. *If (X, G) be a G-metric space. Then the following are equivalent*:

(1) *The sequence $\{x_n\}$ is G-Cauchy*,

(2) *For each* $\epsilon > 0$, *there exists* $k \in N$ *such that* $G(x_n, x_m, x_m) < \epsilon$ *for all* $n, m \geq k$.

Proposition 2.3. *Let* (X, G) *be a G-metric space, then the function* $G(x, y, z)$ *is jointly continuous in all three of its variables.*

Definition 2.4. A G-metric space (X, G) is called a symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2.5. A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.

Proposition 2.4. *Let* (X, G) *be a G-metric space, then for any* $x, y, z, a \in X$ *it follows that*

(1) *If* $G(x, y, z) = 0$, *then* $x = y = z$,

(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(3) $G(x, y, y) \leq 2G(y, x, x)$,

(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(5) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,

(6) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$

An interesting observation is that any G-metric space (X, G) induces a metric d_G on X given by $d_G(x, y) = G(x, y, y) + G(y, x, x)$ for all $x, y \in X$. Moreover, (X, G) is G-complete if and only if (X, d_G) is complete. It was observed that in the symmetric case ((X, G) is symmetric), many fixed point theorems on G-metric spaces are particular cases of the existing fixed point theorems in metric spaces. This allows us to readily transport many results from the metric spaces into the G-metric spaces. On the other hand, by reasoning on the properties of the mappings, the practice of coining weaker forms of commutativity to ensure the existence of a common fixed point for self-mappings on metric spaces is still on. To read more in this direction, we refer to [3] and the references therein. Here, for our further use, we recall only the two fundamental notions of weakly compatible mappings and property E.A, [6, 7].

In 1976, Jungck [9] introduced the notion of weakly compatible mappings as follows.

Definition 2.6. Let S and T be two self-mappings of a metric space (X, d). Then the pair (S, T) is said to be weakly compatible or coincidently commuting if they commute at their coincidence points that is if $Su = Tu$ for some $u \in X$ then $STu = TSu$.

In 2002, Amari and El Moutawakil [1] introduced a new concept of the property E.A, in metric spaces to generalize the concept of non-compatible mappings. Then, they proved some common fixed point theorems.

Definition 2.7. Let S and T be two self-mappings of a metric space (X, d). Then the pair (S, T) is said to satisfy the property E.A if there exists a sequence $\{x_n\}$ is in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, where $t \in X$.

3. The Main Result

Theorem 3.1. *Let* (X, G) *be a G-metric space and* $P, Q, S, T, A, B : X \to X$ *be six self-mappings such that*:

(1) $P(X) \subseteq ST(X)$ *and* $Q(X) \subseteq AB(X)$;

(2) *One of the pairs* (P, AB) *and* (Q, ST) *satisfies the property E.A*;

(3) $G(Px, Qy, Qy)$

$$\leq h \max \left\{ \begin{array}{c} G(ABx, STy, STy), G(Qy, ABx, STy), G(Px, STy, STy), \\ G(Qy, STy, STy), G(Qy, Px, STy), G(Px, ABx, ABx), \\ G(STy, ABx, ABx), G(Qy, ABx, ABx), G(Px, Qy, ABx), \\ G(Px, ABx, STy), G(ABx, Qy, Qy), G(Qy, Qy, STy), \\ \frac{1}{2}(G(Px, STy, STy) + G(Qy, ABx, ABx)), \\ \frac{1}{2}(G(ABx, STy, STy) + G(Qy, Px, STy)), \\ \frac{1}{2}(G(Px, STy, STy) + G(Qy, STy, STy)) \end{array} \right\}$$

for all $x, y \in X$, *where* $h \in (0, 1)$;

(4) *One of* $P(X)$, $Q(X)$, $AB(X)$ *and* $ST(X)$ *is a complete subset of* X;

Then the pairs (P, AB) *and* (Q, ST) *have a coincidence point. Further if* (P, AB) *and* (Q, ST) *are weakly compatible, then* P, Q, S, T, A *and* B *have a unique common fixed point in X.*

Proof. Suppose that the pair (Q, ST) satisfies the property E.A, then there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} STx_n = t$ for some $t \in X$. Since $Q(X) \subseteq AB(X)$, there exist a sequence $\{y_n\}$ in X such that $Qx_n = ABy_n$. Hence $\lim_{n\to\infty} ABy_n = t$. We claim that $\lim_{n\to\infty} Py_n = t$ on contrary suppose that this is not true. Then from 3^{rd} condition, we have

$$G(Py_n, Qx_n, Qx_n)$$

$$\leq h \max \left\{ \begin{array}{c} G(ABy_n, STx_n, STx_n), G(Qx_n, ABy_n, STx_n), G(Py_n, STx_n, STx_n), \\ G(Qx_n, STx_n, STx_n), G(Qx_n, Py_n, STx_n), G(Py_n, ABy_n, ABy_n), \\ G(STx_n, ABy_n, ABy_n), G(Qx_n, ABy_n, ABy_n), G(Py_n, Qx_n, ABy_n), \\ G(Py_n, ABy_n, STx_n), G(ABy_n, Qx_n, Qx_n), G(Qy_n, Qx_n, STx_n), \\ \frac{1}{2}(G(Py_n, STx_n, STx_n) + G(Qx_n, ABy_n, ABy_n)), \\ \frac{1}{2}(G(ABy_n, STx_n, STx_n) + G(Qx_n, Py_n, STx_n)), \\ \frac{1}{2}(G(Py_n, STx_n, STx_n) + G(Qx_n, STx_n, STx_n)) \end{array} \right\}$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} G(Py_n, t, t) \leq h \max \left\{ \begin{array}{c} G(t, t, t), G(t, t, t), \lim_{n\to\infty} G(Py_n, t, t), \\ G(t, t, t), \lim_{n\to\infty} G(t, Py_n, t), \lim_{n\to\infty} G(Py_n, t, t), \\ G(t, t, t), G(t, t, t), \lim_{n\to\infty} G(Py_n, t, t), \\ \lim_{n\to\infty} G(Py_n, t, t), G(t, t, t), G(t, t, t), \\ \frac{1}{2}\left(\lim_{n\to\infty} G(Py_n, t, t) + G(t, t, t)\right), \\ \frac{1}{2}\left(G(t, t, t) + \lim_{n\to\infty} G(t, Py_n, t)\right), \\ \frac{1}{2}\left(\lim_{n\to\infty} G(Py_n, t, t) + G(t, t, t)\right) \end{array} \right\}$$

It implies that

$\lim_{n\to\infty} G(Py_n, t, t) \leq h \lim_{n\to\infty} G(Py_n, t, t) < G(Py_n, t, t)$ as $h \in (0, 1)$ is a contradiction.

Hence $\lim_{n\to\infty} Py_n = t.$

Now suppose that $AB(X)$ is a complete subset of X, then $t = ABu$ for some $u \in X$.

Now we will show that $Pu = ABu = t.$

Again from 3^{rd} condition, we have

$$G(Pu, Qx_n, Qx_n)$$

$$\leq h \max \left\{ \begin{array}{c} G(ABu, STx_n, STx_n), G(Qx_n, ABu, STx_n), G(Pu, STx_n, STx_n), \\ G(Qx_n, STx_n, STx_n), G(Qx_n, Pu, STx_n), G(Pu, ABu, ABu), \\ G(STx_n, ABu, ABu), G(Qx_n, ABu, ABu), G(Pu, Qx_n, ABu), \\ G(Pu, ABu, STx_n), G(ABu, Qx_n, Qx_n), G(Qu, Qx_n, STx_n), \\ \frac{1}{2}(G(Pu, STx_n, STx_n) + G(Qx_n, ABu, ABu)), \\ \frac{1}{2}(G(ABu, STx_n, STx_n) + G(Qx_n, Pu, STx_n)), \\ \frac{1}{2}(G(Pu, STx_n, STx_n) + G(Qx_n, STx_n, STx_n)) \end{array} \right\}$$

Taking the limit as $n \to \infty$, we get

$$G(Pu, t, t) \leq h \max \left\{ \begin{array}{c} G(t, t, t), G(t, t, t), G(Pu, t, t), \\ G(t, t, t), G(t, Pu, t), G(Pu, t, t), \\ G(t, t, t), G(t, t, t), G(Pu, t, t), \\ G(Pu, t, t), G(t, t, t), G(t, t, t), \\ \frac{1}{2}(G(Pu, t, t) + G(t, t, t)), \\ \frac{1}{2}(G(t, t, t) + G(t, Pu, t)), \\ \frac{1}{2}(G(Pu, t, t) + G(t, t, t)) \end{array} \right\}.$$

It implies that

$$\lim_{n\to\infty} G(Pu, t, t) \le h \lim_{n\to\infty} G(Pu, t, t) < G(Pu, t, t)$$

Hence

$$Pu = ABu \tag{1}$$

Therefore u is a coincidence point of the pair (P, AB), the weak compatibility of P and AB implies that $PABu = ABPu$ and hence $PPu = PABu = ABPu = ABABu$

Since $P(X) \subseteq ST(X)$, there exists $v \in X$ such that

$$Pu = STv \tag{2}$$

Again from 3^{rd} condition, we have

$G(Pu, Qv, Qv)$

$$\le h \max \begin{Bmatrix} G(ABu, STv, STv), G(Qv, ABu, STv), G(Pu, STv, STv), \\ G(Qv, STv, STv), G(Qv, Pu, STv), G(Pu, ABu, ABu), \\ G(STv, ABu, ABu), G(Qv, ABu, ABu), G(Pu, Qv, ABu), \\ G(Pu, ABu, STv), G(ABu, Qv, Qv), G(Qv, Qv, STv), \\ \frac{1}{2}(G(Pu, STv, STv) + G(Qv, ABu, ABu)), \\ \frac{1}{2}(G(ABu, STv, STv) + G(Qv, Pu, STv)), \\ \frac{1}{2}(G(Pu, STv, STv) + G(Qv, STv, STv)) \end{Bmatrix}$$

From (1) and (2) we have

$G(STv, Qv, Qv)$

$$\le h \max \begin{Bmatrix} G(Pu, Pu, Pu), G(Qv, Stv, Stv), G(STv, STv, STv), \\ G(Qv, STv, STv), G(Qv, STv, STv), G(Pu, Pu, Pu), \\ G(STv, STv, STv), G(Qv, STv, STv), G(STv, Qv, STv), \\ G(STv, STv, STv), G(STv, Qv, Qv), G(Qv, Qv, STv), \\ \frac{1}{2}(G(STv, STv, STv) + G(Qv, STv, STv)), \\ \frac{1}{2}(G(Pu, Pu, Pu) + G(Qv, STv, STv)), \\ \frac{1}{2}(G(STv, STv, STv) + G(Qv, STv, STv)) \end{Bmatrix}.$$

It implies that $STv = Qv$. Thus

$$Pu = ABu = STv = Qv = t. \tag{3}$$

Now we take $x = t$ and $y = v$ then from 3^{rd} condition, we have

$$G(Pt, Qv, Qv) \leq h \max \left\{ \begin{array}{c} G(ABt, STv, STv), G(Qv, ABt, STv), G(Pt, STv, STv), \\ G(Qv, STv, STv), G(Qv, Pt, STv), G(Pt, ABt, ABt), \\ G(STv, ABt, ABt), G(Qv, ABt, ABt), G(Pt, Qv, ABt), \\ G(Pt, ABt, STv), G(ABt, Qv, Qv), G(Qv, Qv, STv), \\ \frac{1}{2}(G(Pt, STv, STv) + G(Qv, ABt, ABt)), \\ \frac{1}{2}(G(ABt, STv, STv) + G(Qv, Pt, STv)), \\ \frac{1}{2}(G(Pt, STv, STv) + G(Qv, STv, STv)) \end{array} \right\}$$

On using (3), we obtain $Pt = t$.

Since the pair (P, AB) is weakly compatible, therefore

$$PABu = ABPu = Pt = ABt.$$

Hence we have $Pt = ABt = t$.

The same result can be obtained for the pair (Q, ST).

For Uniqueness: Suppose that there exist another fixed point z of P, Q, S, T, A and B such that $x = t$ and $y = z$.

Then by condition 3^{rd}, we have

$$G(Pt, Qz, Qz) \leq h \max \left\{ \begin{array}{c} G(ABt, STz, STz), G(Qz, ABt, STz), G(Pt, STz, STz), \\ G(Qz, STz, STz), G(Qz, Pt, STz), G(Pt, ABt, ABt), \\ G(STz, ABt, ABt), G(Qz, ABt, ABt), G(Pt, Qz, ABt), \\ G(Pt, ABt, STz), G(ABt, Qz, Qz), G(Qz, Qz, STz), \\ \frac{1}{2}(G(Pt, STz, STz) + G(Qz, ABt, ABt)), \\ \frac{1}{2}(G(ABt, STz, STz) + G(Qz, Pt, STz)), \\ \frac{1}{2}(G(Pt, STz, STz) + G(Qz, STz, STz)) \end{array} \right\}$$

$$G(t, z, z) \leq h \max \begin{Bmatrix} G(t, z, z), G(z, t, z), G(t, z, z), \\ G(z, z, z), G(z, t, z), G(t, t, t), \\ G(z, t, t), G(z, t, t), G(t, z, t), \\ G(t, t, z), G(t, z, z), G(z, z, z), \\ \frac{1}{2}(G(t, z, z) + G(z, t, t)), \\ \frac{1}{2}(G(t, z, z) + G(z, t, z)), \\ \frac{1}{2}(G(t, z, z) + G(z, z, z)) \end{Bmatrix}$$

It implies that $t = z$.

Therefore t is a unique common fixed point of P, Q, S, T, A and B.

Theorem 3.2. *Let* (X, G) *be a G-metric space and* $P, Q, S, T, A, B : X \to X$ *be six self-mappings such that*:

(1) $P(X) \subseteq ST(X)$ *and* $Q(X) \subseteq AB(X)$;

(2) *One of the pairs* (P, AB) *and* (Q, ST) *satisfies the property E.A*;

(3) $G(Px, Qy, Qy)$

$$\leq h \max \begin{Bmatrix} G(ABx, STy, STy), G(Qy, ABx, STy), G(Px, STy, STy), \\ G(Qy, STy, STy), G(Qy, Px, STy), G(Px, ABx, ABx), \\ G(Sty, ABx, ABx), G(Qy, ABx, ABx), G(Px, Qy, ABx), \\ G(Px, ABx, STy), G(ABx, Qy, Qy), G(Qy, Qy, STy) \end{Bmatrix}$$

for all $x, y \in X$, *where* $h \in (0, 1)$;

(4) *One of* $P(X), Q(X), AB(X)$ *and* $ST(X)$ *is a complete subset of X.*

Then the pairs (P, AB) *and* (Q, ST) *have a coincidence point. Further if* (P, AB) *and* (Q, ST) *are weakly compatible, then* P, Q, S, T, A *and B have a unique common fixed point in X.*

Proof. Suppose that the pair (Q, ST) satisfies the property E.A, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n \to \infty} Qx_n = \lim_{n \to \infty} STx_n = t$ for some $t \in X$. Since $Q(X) \subseteq AB(X)$, there exists a sequence $\{y_n\}$ in X such that $Qx_n = ABy_n$. Hence $\lim_{n\to\infty} ABy_n = t$. We claim that $\lim_{n\to\infty} Py_n = t$ on contrary suppose that this is not true. Then from 3^{rd} condition, we have

$$G(Py_n, Qx_n, Qx_n)$$

$$\leq h \max \left\{ \begin{array}{c} G(ABy_n, STx_n, STx_n), G(Qx_n, ABy_n, STx_n), G(Py_n, STx_n, STx_n), \\ G(Qx_n, STx_n, STx_n), G(Qx_n, Py_n, STx_n), G(Py_n, ABy_n, ABy_n), \\ G(STx_n, ABy_n, ABy_n), G(Qx_n, ABy_n, ABy_n), G(Py_n, Qx_n, ABy_n), \\ G(Py_n, ABy_n, STx_n), G(ABy_n, Qx_n, Qx_n), G(Qx_n, Qx_n, STx_n) \end{array} \right\}$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} G(Py_n, t, t) \leq h \max \left\{ \begin{array}{c} G(t, t, t), G(t, t, t), \lim_{n\to\infty} G(Py_n, t, t), \\ G(t, t, t), \lim_{n\to\infty} G(t, Py_n, t), \lim_{n\to\infty} G(Py_n, t, t), \\ G(t, t, t), G(t, t, t), \lim_{n\to\infty} G(Py_n, t, t), \\ \lim_{n\to\infty} G(Py_n, t, t), G(t, t, t), G(t, t, t) \end{array} \right\}$$

It implies that

$\lim_{n\to\infty} G(Py_n, t, t) \leq h \lim_{n\to\infty} G(Py_n, t, t) < G(Py_n, t, t)$ as $h \in (0, 1)$ is a contradiction.

Hence $\lim_{n\to\infty} Py_n = t$.

Now suppose that $AB(X)$ is a complete subset of X, then $t = ABu$ for some $u \in X$.

Now we will show that $Pu = ABu = t$.

Again from 3^{rd} condition, we have

$$G(Pu, Qx_n, Qx_n)$$

$$\leq h \max \left\{ \begin{array}{c} G(ABu, STx_n, STx_n), G(Qx_n, ABu, STx_n), G(Pu, STx_n, STx_n), \\ G(Qx_n, STx_n, STx_n), G(Qx_n, Pu, STx_n), G(Pu, ABu, ABu), \\ G(STx_n, ABu, ABu), G(Qx_n, ABu, ABu), G(Pu, Qx_n, ABu), \\ G(Pu, ABu, STx_n), G(ABu, Qx_n, Qx_n), G(Qx_n, Qx_n, STx_n) \end{array} \right\}$$

Taking the limit as $n \to \infty$, we get

$$G(Pu, t, t) \leq h \max \left\{ \begin{array}{c} G(t, t, t), G(t, t, t), G(Pu, t, t), \\ G(t, t, t), G(t, Pu, t), G(Pu, t, t), \\ G(t, t, t), G(t, t, t), G(Pu, t, t), \\ G(Pu, t, t), G(t, t, t), G(t, t, t) \end{array} \right\}$$

It implies that

$$\lim_{n\to\infty} G(Pu, t, t) \leq h \lim_{n\to\infty} G(Pu, t, t) < G(Pu, t, t)$$

Hence

$$Pu = ABu \tag{1}$$

Therefore u is a coincidence point of the pair (P, AB), the weak compatibility of P and AB implies that

$PABu = ABPu$ and hence $PPu = PABu = ABPu = ABABu$.

Since $P(X) \subseteq ST(X)$, there exists $v \in X$ such that

$$Pu = STv \tag{2}$$

Again from 3^{rd} condition, we have

$$G(Pu, Qv, Qv)$$

$$\leq h \max \left\{ \begin{array}{c} G(ABu, STv, STv), G(Qv, ABu, STv), G(Pu, STv, STv), \\ G(Qv, STv, STv), G(Qv, Pu, STv), G(Pu, ABu, ABu), \\ G(STv, ABu, ABu), G(Qv, ABu, ABu), G(Pu, Qv, ABu), \\ G(Pu, ABu, STv), G(ABu, Qv, Qv), G(Qv, Qv, STv) \end{array} \right\}$$

From (1) and (2) we have

$$G(STv, Qv, Qv)$$

$$\leq h \max \left\{ \begin{array}{c} G(Pu, Pu, Pu), G(Qv, STv, STv), G(STv, STv, STv), \\ G(Qv, STv, STv), G(Qv, STv, STv), G(Pu, Pu, Pu), \\ G(STv, STv, STv), G(Qv, STv, STv), G(STv, Qv, STv), \\ G(STv, STv, STv), G(STv, Qv, Qv), G(Qv, Qv, STv) \end{array} \right\}$$

It implies that $STv = Qv$.

Thus

$$Pu = ABu = STv = Qv = t. \tag{3}$$

Now we take $x = t$ and $y = v$ then from 3rd condition, we have

$$G(Pt, Qv, Qv)$$

$$\leq h \max \begin{Bmatrix} G(ABt, STv, STv), G(Qv, ABt, STv), G(Pt, STv, STv), \\ G(Qv, STv, STv), G(Qv, Pt, STv), G(Pt, ABt, ABt), \\ G(STv, ABt, ABt), G(Qv, ABt, ABt), G(Pt, Qv, ABt), \\ G(Pt, ABt, STv), G(ABt, Qv, Qv), G(Qv, Qv, STv) \end{Bmatrix}$$

On using (3), we obtain $Pt = t$.

Since the pair (P, AB) is weakly compatible, therefore

$$PABu = ABPu = Pt = ABt.$$

Hence we have $Pt = ABt = t$.

The same result can be obtained for the pair (Q, ST).

For Uniqueness: Suppose that there exists another fixed point z of P, Q, S, T, A and B such that $x = t$ and $y = z$. Then by condition 3^{rd} we have

$$G(Pt, Qz, Qz)$$

$$\leq h \max \begin{Bmatrix} G(ABt, STz, STz), G(Qz, ABt, STz), G(Pt, STz, STz), \\ G(Qz, STz, STz), G(Qz, Pt, STz), G(Pt, ABt, ABt), \\ G(STz, ABt, ABt), G(Qz, ABt, ABt), G(Pt, Qz, ABt), \\ G(Pt, ABt, STz), G(ABt, Qz, Qz), G(Qz, Qz, STz) \end{Bmatrix}$$

$$G(t, z, z) \leq h \max \begin{Bmatrix} G(t, z, z), G(z, t, z), G(t, z, z), \\ G(z, z, z), G(z, t, z), G(t, t, t), \\ G(z, t, t), G(z, t, t), G(t, z, t), \\ G(t, t, z), G(t, z, z), G(z, z, z) \end{Bmatrix}$$

It implies that $t = z$. Therefore t is a unique common fixed point of P, Q, S, T, A and B.

4. Conclusion

In this paper, we use the notion of the property E.A, to prove common fixed point theorems for weakly compatible maps in G-metric space. Our result generalizes, extends and unifies several existing fixed point results in metric space and G-metric space. This can be further extended for more number of self-mappings satisfying a more complex class of inequality.

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