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PERFECT CODES IN UNIT GRAPH OF SOME COMMUTATIVE RINGS

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Abstract

A unit graph of a ring R is a graph with vertex set R and two distinct vertices x and y are adjacent if and only if $x+y$ is a unit of R. A subset C of the vertex set in a graph Γ is called a perfect code if $S_1(c)$ form a partition of the vertex set, when c runs through C. In this paper, we characterize the commutative rings with identity in which their associated unit graphs accept the order 1 and order 2 perfect codes. In addition, we prove an order 2 perfect code in complement unit graph of a ring R having complete bipartite form. Moreover, we prove an order $(p+1)/2$ perfect code in complement unit graph of a division ring of $O(R) = p$, p is prime and an infinite order perfect code in complement unit graph of a division ring with $Char(R) = 0$. We also characterize some of the commutative rings R in which their associated unit graph as well as complement unit graphs do not accept the perfect codes.

1. Introduction

The study of ring $\mathbb{Z}_n$ by the notion of a unit graph has been first proposed in [1]. Then, this graph was generalized for an arbitrary ring R in [2]. Later on, Su and Zhou [3] proved that the girth of unit graph of a ring R is equal to 3, 4, 6, or ∞. However, Su and Wei [4] conducted a research on the unit graph associated with rings; in particular to determine the diameter of $\Gamma(R)$.

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In a graph Γ, a subset $C \subseteq V(\Gamma)$ is called a code. The code C is a perfect code in Γ if $S_1(c)$ form a partition of $V(\Gamma)$, when c runs through C, where $S_1(c)$ is the closed neighbourhood of c with radius 1. In other words, C is a perfect code in Γ if for all $c \in C$, satisfying $\bigcap S_1(c) = \phi$ and $\bigcup S_1(c) = V(\Gamma)$ [5]. The number of elements in C is called the order of the perfect code C; if C contains exactly k elements, where $k = 1, 2, 3, \ldots, n$, then C is called an order k perfect code. The investigation of perfect codes on graph was initiated by Biggs [6] to determine the non-trivial perfect codes in distance-transitive graph. Afterwards, Kratochvil [7] proved the perfect codes existence in products and second powers of graphs. Recently, the researches on perfect codes in graphs were extended to the perfect codes in algebraic graphs. Ma [8] characterized the finite groups associated with the power graphs which accept or do not accept the perfect codes. In another paper, Ma [9] investigated the perfect codes in proper reduced power graphs associated with finite groups and proved some mathematical results on determination of perfect codes in the mentioned graphs. Also there are some number of researches focused on determination of perfect codes over Cayley graphs, see [10-12].

In this paper, the research on perfect codes in graphs associated with groups is extended to graphs associated with rings by mainly focusing on determining the perfect codes in unit graphs and complement unit graphs associated with some commutative rings. In Section 2, we determine the perfect codes in unit graphs of some commutative rings. In Section 3, we determine the perfect codes in complement unit graphs of some commutative rings.

2. Perfect Code in Unit Graphs

In this section, by establishing some theorems and corollaries the perfect codes in unit graphs associated with the commutative rings are investigated.

Theorem 2.1. *Let* $\Gamma(R)$ *be the unit graph associated with* R*. Then,* $\Gamma(R)$ *accepts the order* 1 *perfect code if and only if* R *is a division ring of*

(i) $O(R) = p$.

(ii) $Char(R) = 0$.

Proof. (1) Assume that R is a division ring with $O(R) = p$, where p is prime. This means that R has p distinct elements, which can be represented as follows:

$$R = \{r_i : i = 1, 2, 3, \ldots, p, \text{ where } p \text{ is prime}\}.$$

Further assume that R^* is the set of all $0 \neq r_i \in R$, thus every $0 \neq r_i \in R$ has a multiplicative inverse as $r_j \in R^*$ such that $r_i \cdot r_j = e$, because R is a division ring with $O(R) = p$. Suppose that $U(R)$ is the set of units of R, then it can be represented as

$$U(R) = \{r_i : r_i \cdot r_j = e, 1 \leq i \leq j \leq p\}.$$

By the definition of $\Gamma(R)$, the vertex set $V(\Gamma(R)) = R$ and $\forall r_i, r_j \in V(\Gamma(R)), r_i \neq r_j, r_i$ is connected to r_j if and only if $r_i + r_j \in U(R)$. Thus, with respect to $V(\Gamma(R))$, there exists a special vertex 0, such that for all $0 \neq r_i \in V(\Gamma(R)), 0 + r_i \in U(R)$. Also there exist some $0 \neq r_i \in V(\Gamma(R))$ such that $r_i + r_j \in U(R)$. Therefore, according to p, three cases are considered:

Case 1. If $p = 2$, then $|V(\Gamma(R))| = 2$. Since $0 + r_i \in U(R)$, thus $\Gamma(R)$ has a K_2 form.

Case 2. If $p = 3$, then $|V(\Gamma(R))| = 3$. Since $0 + r_i \in U(R)$, thus $\Gamma(R)$ has a $K_{1, 2}$ form.

Case 3. If $p \geq 5$, then $|V(\Gamma(R))| \geq 5$. Since, $0 + r_i \in U(R)$ and also $r_j \neq 0$ in $V(\Gamma(R))$ is not adjacent to $r_i \neq 0$ if $r_i + r_j = 0$, otherwise these vertices are adjacent. This gives that $\Gamma(R)$ has a complete $(p+1)/2$-partite form. To show $\Gamma(R)$ accepts only the order 1 perfect code, the closed neighbourhood of each r_i of $\Gamma(R)$ with radius 1 is obtained as follows:

$$\begin{aligned} S_1(r_i) &= \{r_j \in V(\Gamma(R)) : d(r_i, r_j) \leq 1\} \\ &= V(\Gamma(R)) \text{ if } r_i = 0 \qquad (1) \end{aligned}$$

$$S_1(r_i) = \{r_j \in V(\Gamma(R)) : d(r_i, r_j) \leq 1\}$$
$$\neq V(\Gamma(R)) \text{ if } r_i \neq 0 \tag{2}$$

According to (1) and (2), we obtain that $C = \{r_i\}$ for $r_i = 0$ is the order 1 perfect code accepted by $\Gamma(R)$, since $S_1(0) = V(\Gamma(R))$. Let $|C| > 1$, then $\bigcap S_1(r_i) \neq \phi$ for all $r_i \in C$, which is a contradiction.

(ii) Assume that R is a division ring with $Char(R) = 0$, that is R has infinite distinct elements as $r_1, r_2, r_3, \ldots, r_i, \ldots,$ which can be represented $R = \{r_i\}_{i=1}^{\infty}$. Let R^* represents the set of $0 \neq r_i \in R$, i.e. $R^* = \{r_i \neq 0\}_{i=1}^{\infty}$. Since R is a division ring, thus every $r_i \in R^*$ has a multiplicative inverse as $r_j \in R^*$ such that $r_i \cdot r_j = e$. Suppose that $U(R)$ is the set of units of R, thus $U(R) \in R^*$. According to (i), there is a special vertex $0 \in V(\Gamma(R))$ which is adjacent to every $r_i \neq 0$, i.e. $0 + r_i \in U(R)$. Besides, the non-zero vertices r_i and r_j in $\Gamma(R)$ are not connected if $r_i + r_j = 0$, otherwise they are connected. These adjacency of the vertices show that $\Gamma(R)$ is a complete infinite-partite graph as $K_{1,2,2,2,\ldots,2,2,2,\ldots}$. Next, to show $\Gamma(R)$ accepts only the order 1 perfect code, it follows from Part (i).

Conversely, if the perfect code accepted by the $\Gamma(R)$ is of order 1, this means that the perfect code C contains an element r_i such that $S_1(r_i) = V(\Gamma(R))$. It shows that a special vertex r_i exists in $\Gamma(R)$ such that $r_i + r_j \in U(R)$. This condition is true only when $r_i = 0$. Hence, the ring R contains only the non-unit element $r_i = 0$ which is indeed either a division ring of $O(R) = p$ or a division ring of $Char(R) = 0$.

The following corollaries are the immediate consequences of the proof of Theorem 2.1.

Corollary 2.2. *Let* $\Gamma(R)$ *be the unit graph associated with* R. *Then* $\Gamma(R)$ *accepts the order* 1 *perfect code if and only if* R *is a simple ring*.

Corollary 2.3. *Let* $\Gamma(R/I)$ *be the unit graph associated with the quotient* *ring* R/I. *Then* $\Gamma(R/I)$ *accepts the order* 1 *perfect code if and only if I is a maximal ideal of R.*

Theorem 2.4. *Let* $\Gamma(R)$ *be the unit graph associated with R. Then,* $\Gamma(R)$ *accepts an order* 2 *perfect code if and only if R is a commutative ring with*

(i) $Char(R) = O(R) = 2p$, $p \geq 3$ *is prime.*

(ii) $O(R) = 4$ *and* $Char(R) = 2$.

Proof. (i) Assume R is a commutative ring with $Char(R) = O(R) = 2p$, $p \geq 3$ is prime. Thus, R has $2p$ distinct elements as $r_1, r_2, r_3, \ldots, r_{2p}$. Let $U(R)$ indicates the set of all units of R, then $|U(R)| = p - 1$. Hence, $|NU(R)| = p + 1$. Further assume that $\Gamma(R)$ is the unit graph associated with R, thus by the definition of a unit graph, $V(\Gamma(R)) = R$ and $E(\Gamma(R)) = \{\{r_i, r_j\} : r_i + r_j \in U(R)$ for all $r_i \neq r_j\}$. By the vertices adjacency of $\Gamma(R)$, for any distinct vertices $r_i + r_j \in U(R)$, then $r_i + r_j \notin U(R)$. Also there exists a vertex $m \in NU(R)$ such that for any distinct vertices $m \neq r_k, m \neq r_l \in NU(R)$, $r_k + r_l \notin U(R)$. Similarly, for $r_i \in U(R)$ and for $0 \neq r_k \in NU(R)$, then $m + r_j \notin U(R)$ and $m + r_j \in U(R)$. This gives that $V(\Gamma(R))$ can be partitioned into two partite sets $V_1(\Gamma(R)) = U(R) \cup m$ and $V_2(\Gamma(R)) = NU(R) \setminus \{m\}$ of order p, where each element of $V_1(\Gamma(R))$ is exactly adjacent with $p - 1$ elements of $V_2(\Gamma(R))$ and vice versa. Hence, $\Gamma(R)$ is a bipartite graph, i.e. $\Gamma(R) = K_{p,\, p}$.

Next, we prove that $\Gamma(R)$ accepts an order 2 perfect code. Suppose $V_1(\Gamma(R)) = \{x_i : i = 1, 2, 3, \ldots, p\}$ and $V_2(\Gamma(R)) = \{y_i : i = 1, 2, 3, \ldots, p\}$. Then the closed neighbourhood of each x_i and y_i can be obtained as

$$S_1(x_i) = \{r_i \in V(\Gamma(R)) : d(x_i, r_i) \leq 1\}$$
$$= V_2(\Gamma(R)) \setminus \{y_i\} \cup \{x_i\}, \tag{3}$$
$$S_1(y_i) = \{r_i \in V(\Gamma(R)) : d(y_i, r_i) \leq 1\}$$
$$= V_1(\Gamma(R)) \setminus \{x_i\} \cup \{y_i\}, \tag{4}$$

if $x_i + y_i \notin U(R)$. According to (3) and (4), $S_1(x_i) \neq V(\Gamma(R))$ and $S_1(y_i) \neq V(\Gamma(R))$, which does not satisfy the conditions of perfect codes.

However, $S_1(x_i) \cap S_1(y_i) = \phi$ and $S_1(x_i) \cup S_1(y_i) = V(\Gamma(R))$ for all $x_i \in V_1(\Gamma(R))$ and $y_i \in V_2(\Gamma(R))$, which satisfy the conditions of perfect codes. Hence, $C = \{x_i, y_i\}$, where $x_i + y_i \notin U(R)$, is an order 2 perfect code accepted by $\Gamma(R)$, Let $|C| > 2$, then $\bigcap S_1(r_i) \neq \phi$ for all $r_i \in C$, which is a contradiction.

(ii) Suppose that $Char(R) = 2$ and $O(R) = 4$, that is $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where $|U(R)| = 1$. Therefore, the vertices adjacencies of $\Gamma(R)$ yield that $\Gamma(R)$ is a $2K_2$ graph. By Theorem 2.1 (i), a K_2 graph accepts an order 1 perfect code, thus a $2K_2$ graph accepts an order 2 perfect code.

Conversely, if $\Gamma(R)$ accept an order 2 perfect code, that is $C = \{x_i, y_i\}$. This implies that the closed neighbourhood of the vertices x_i and y_i partition the set of vertices of $\Gamma(R)$ into two disjoint sets namely, $S_1(x_i)$ and $S_1(y_i)$, where $x_i + y_i \notin U(R)$. Therefore, it gives that $\Gamma(R)$ has either a bipartite form such that every vertex of one partite is adjacent to the $p - 1$ vertex in another partite or it has a $2K_2$ form. If $\Gamma(R)$ is a bipartite graph, this means that each partite set has order p, hence $Char(R) = O(R) = 2p$, $p \geq 3$ is prime. If $\Gamma(R)$ is a $2K_2$ graph, then it yields that $Char(R) = 2$ and $O(R) = 4$.

Theorem 2.5. *Let $\Gamma(R)$ be the unit graph associated with R. Then $\Gamma(R)$ does not accept the perfect code if R is a commutative ring with $Char(R)$ $= O(R) = 2^k$, $k \geq 2$.*

Proof. Assume R is a commutative ring with $Char(R) = O(R) = 2^k$, $k \geq 2$. Thus R has 2^k distinct elements which can be displayed as follows:

$$R = \{r_i : i = 1, 2, 3, \ldots, 2^k\}.$$

Suppose $U(R)$ is the set of units of R. i.e. $U(R)$ contains those r_i and r_j

of R that satisfy the condition $r_i \cdot r_j = e$. According to the order of R, there exist 2^{k-1} distinct elements in R such that $r_i \cdot r_j = e$. Hence, $|U(R)| = 2^{k-1}$. Suppose that $NU(R)$ denote the set of non-units of R, thus $|NU(R)| = 2^{k-1}$. Since, $\Gamma(R)$ is the unit graph of ring R, thus $V(\Gamma(R)) = R$ and $E(\Gamma(R)) = \{\{r_i, r_j\} : r_i + r_j \in U(R)$ for all $r_i \neq r_j\}$. It follows that for any two distinct vertices $r_i, r_j \in U(R), r_i + r_j \notin U(R)$ and for any two distinct vertices $r_i, r_m \in NU(R), r_k + r_m \notin U(R)$. However, for any $r_i \in U(R)$ and $r_k \in NU(R)$, then $r_i + r_k \in U(R)$. Hence, $\Gamma(R)$ is a complete bipartite graph with partite sets $U(R)$ and $NU(R)$. Now suppose that $C \subseteq V(\Gamma(R))$ be a code, thus C is a perfect code in $\Gamma(R)$ if the following two conditions holds:

(i) $\bigcup S_1(r_i) = V(\Gamma(R))$,

(ii) $\bigcap S_1(r_i) = \phi$ for all $r_i \in C$.

Let $|C| > 1$, thus the neighbourhood of code word $r_i \in C$ with radius 1 gives that $S_1(r_i) \neq V(\Gamma(R))$. If $|C| > 1$, thus the neighbourhood of code word $r_i \in C$ with radius 1 gives that $\bigcap S_1(r_i) = \phi$, which shows that C does not satisfy the conditions of perfect codes. Hence, C is not a perfect code in $\Gamma(R)$.

3. Perfect Codes in Complement Unit Graphs

In this section, we prove the perfect codes in complements unit graphs of some commutative rings with identity.

Theorem 3.1. *Let $\Gamma^c(R)$ be the complement unit graph associated with R. Then, $\Gamma^c(R)$ accepts an order $(p+1)/2$ perfect code if and only if R is a division ring of $O(R) = p$, where p is an odd prime.*

Proof. Suppose that R is a division ring of $O(R) = p$, where p is an odd prime. Then $|U(R)| = p - 1$. By the complement unit graph, $V(\Gamma^c(R)) = R$ and $E(\Gamma^c(R)) = \{\{r_i, r_j\} : r_i + r_j \notin U(R)$ for all $r_i \neq r_j\}$. If $r_i \in U(R)$, then $0 + r_i \in U(R)$, that is 0 is an isolated vertex in $\Gamma^c(R)$. Similarly, for each

$r_i \in U(R)$, there exists a unique r_j such that $r_i + r_j \notin U(R)$. This shows that $\Gamma^c(R)$ contains $(p+1)/2$ components of K_2, since $|U(R)| = p - 1$. Hence, $\Gamma^c(R) = K_1 + (p-1)/2K_2$. Let $C \subseteq V(\Gamma^c(R))$ be a code, then we prove that C is of order $(p+1)/2$ perfect code. Assume that $C = \{r_k : r_k$ is an independent vertex $\}$. Then the closed neighbourhoods of code words r_k with radius 1 give that $\cap S_1(r_k) = \phi$ and $\cup S_1(r_k) = V(\Gamma^c(R))$ for all $r_k \in C$. Since $\Gamma^c(R)$ contains $(p+1)/2$ independent vertex, hence C is an order $(p+1)/2$ perfect code accepted by $\Gamma^c(R)$.

Conversely, suppose that $\Gamma^c(R)$ accepts an order $(p+1)/2$ perfect code, this means that the elements of C partition the vertex set of $\Gamma^c(R)$ into $(p+1)/2$ disjoint set as $\bigcup_{k=1}^{(p+1)/2} S_1(r_k)$. It follows that $\Gamma^c(R) = K_1$ $+(p-1)/2K_2$. Let $\Gamma^c(R) = (p-1)/2K_2$, then no ring can be associated with $\Gamma^c(R)$, which is a contradiction. Since 0 is an isolated vertex and every non-zero vertex r_i is adjacent to a unique non-zero vertex r_i, this implies that R is a division ring of $O(R) = p$, where p is an odd prime.

Theorem 3.2. *Let R be a division ring with $Char(R) = 0$ and $\Gamma^c(R)$ be the complement unit graph associated with R. Then, $\Gamma^c(R)$ accepts an infinite order perfect code.*

Proof. Assume R is a division ring with $Char(R) = 0$. Then $|U(R)| = \infty$. By the complement unit graph, $V(\Gamma^c(R)) = R$ and $E(\Gamma^c(R)) = \{\{r_i, r_j\} : r_i + r_j \notin U(R)$ for all $r_i \neq r_j\}$. If $r_i \in U(R)$, then $0 + r_i \in U(R)$, that is 0 is an isolated vertex in $\Gamma^c(R)$. Similarly, for each $r_i \in U(R)$, there exists a unique r_j such that $r_i + r_j \notin U(R)$. Since $|U(R)| = \infty$, thus $\Gamma^c(R)$ contains infinite pieces of 2^k. Hence, $\Gamma^c(R) = K_1 + \cup_{k=1}^{\infty} K_2$. Let $C \subseteq V(\Gamma^c(R))$ be a code, then we prove that any code C of infinite order is perfect. Assume that $C = \{r_k : r_k$ is an independent vertex$\}$. Then the closed neighbourhoods of code words r_k with radius 1 give that $\bigcup_{k=1}^{\infty} S_1(r_k) = \phi$ and $\bigcup_{k=1}^{\infty} S_1(r_k) = V(\Gamma^c(R))$ for all $r_k \in C$. Hence, C is an infinite order perfect code in $\Gamma^c(R)$.

Theorem 3.3. *Let $\Gamma^c(R)$ be the complement unit graph associated with R. Then $\Gamma^c(R)$ accepts an order 2 perfect code if R is a ring with $Char(R) = O(R) = 2^k$, $k \geq 2$.*

Proof. Assume R is a commutative ring with $Char(R) = O(R) = 2^k$, $k \geq 2$. By Theorem 2.5, $\Gamma(R)$ is a complete bipartite graph with partite set $U(R)$ and $NU(R)$, respectively. Suppose that $\Gamma(R)$ is the complement unit graph associated with ring R, then $V(\Gamma^c(R)) = R$ and $E(\Gamma(R)) = \{\{r_i, r_j\} : r_i + r_j \in U(R)$ for all $r_i \neq r_j\}$. Since for any distinct elements $r_i, r_j \in U(R)$, then $r_i + r_j \notin U(R)$ and for any distinct elements $r_k, r_m \in NU(R)$, then $r_k + r_m \in NU(R)$, while for any $r_i \in U(R)$ and $r_k \in NU(R)$, then $r_i + r_j \in U(R)$. This shows that $\Gamma^c(R)$ consists of two pieces of complete graphs, i.e., $\Gamma^c(R) = K_{m_1} + K_{m_2}$, where $m_1 = |U(R)|$ and $m_2 = |U(R)|$. Now, we show that $\Gamma^c(R)$ accepts an order 2 perfect code. Suppose that $C_1 \subseteq V(K_{m_1})$ and $C_2 \subseteq V(K_{m_2})$ be the set of code words of order 1, then $C = C_1 \cup C_2$ is the set of code words of order 2. Since C satisfies the following two conditions of perfect codes:

(i) $\bigcup S_1(r_i) = V(\Gamma^c(R))$,

(ii) $\bigcap S_1(r_i) = \phi$ for all $r_i \in C$.

Hence, C is an order 2 perfect code accepted by $\Gamma^c(R)$.

Suppose that C is a perfect code of order $|C| \neq 2$ in $\Gamma^c(R)$. If $|C| = 1$, then $S_1(r_i) \neq V(\Gamma^c(R))$ for $r_i \in C$, and if $|C| > 2$, then $\bigcap S_1(r_i) = \phi$ for all $r_i \in C$, which is a contradiction.

Theorem 3.4. *Let R be a commutative ring with $Char(R) = O(R) = 2p$,*

$p \geq 3$ is prime and $\Gamma^c(R)$ be the complement unit graph associated with R. Then, $\Gamma^c(R)$ does not accept the perfect codes.

Proof. Assume R is a commutative ring with $Char(R) = O(R) = 2p$, $p \geq 3$ is prime. Thus, by Theorem 2.4 (i), $\Gamma(R) = K_{p,\, p}$, where every vertex of one partite set is adjacent to exactly $p - 1$ vertices of another partite set and vice versa. Suppose that $\Gamma^c(R)$ is the complement unit graph associated with R.

By the definition of the complement unit graph, $V(\Gamma^c(R)) = R$ and $E(\Gamma(R)) = \{\{r_i, r_j\} : r_i + r_j \in U(R) \text{ for all } r_i \neq r_j\}$. The vertices adjacency of $\Gamma^c(R)$ shows that there are p edges incident with each vertex of $\Gamma^c(R)$. Therefore, $\Gamma^c(R)$ is a p-regular graph. Suppose that $C \subseteq V(\Gamma^c(R))$ be a code, we show that no code C is perfect. If $|C| = 1$, then $S_1(r_i) \neq V(\Gamma^c(R))$ for $r_i \in C$. If $|C| > 1$, then $\bigcap S_1(r_i) \neq \phi$ for all $r_i \in C$. Hence, $\Gamma^c(R)$ does not accept the perfect codes.

4. Conclusion

This research presents the perfect codes in unit graphs associated with some commutative rings R with identity. Some results have been established which show the unit graphs associated with R are either a K_2 or a $K_{1,\, 2}$ or a complete $(p+1)/2$-partite graph or a complete infinite-partite graph as $K_{1,\, 2,\, 2, \ldots,\, 2,\, 2,\, 2, \ldots}$ or a bipartite graph or a complete bipartite graph. In addition, the results show that the complement unit graph of R are either a $K_1 + \bigcup_{i=1}^{(p-1)/2} K_2$ graph, or a $K_1 + \bigcup_{i=1}^{\infty} K_2$ graph or a $K_{m_1} + K_{m_2}$ graph or a p-regular graph. As a consequence, K_2, $K_{1,\, 2}$, a complete $(p+1)/2$-partite graph and a complete infinite-partite graph as $K_{1,\, 2,\, 2, \ldots,\, 2,\, 2,\, 2, \ldots}$ graphs accept the order 1 perfect code, while the bipartite graph and the complement of the complete bipartite graph accept an order 2 perfect code. However, a complete bipartite graph and the complement of bipartite graph do not accept the perfect code of any order. Moreover, the findings show that the complement unit graph of a division ring of $O(R) = p,\ p \geq 3$ accepts an order $(p+1)/2$ perfect code and the complement unit graph associated with a division ring with $Char(R) = 0$ accepts an infinite order perfect code.

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