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A COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPS OF TYPE (E) IN *L*-FUZZY METRIC SPACE

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Abstract

In this paper, we introduce the notion of compatible of type (E) in L-fuzzy metric space and prove a common fixed point theorem of self maps with the property of (C) in the complete L-fuzzy metric space.

1. Introduction

In 1986, the concept of fuzzy set was introduced by Zadeh [20]. Then fuzzy metric space was initiated by Kramosil and Michalek [9]. George and Veeramani [7] modified the notion of fuzzy metric space with the help of continuous t-norm. Using to idea of L-fuzzy set [8] Saadatietal, introduced the notion of L fuzzy metric spaces with the help of continuous t-norms as a generalization of fuzzy metric space due to George and Veeramani. In 2007,

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M. R. Singh and Y. M. Singh introduced the concept of compatible mappings of type (E) in metric space. The aim of this paper, we introduce the notion of compatible of type (E) in L-fuzzy metric space and prove a common fixed point theorem of self mapping with the property of (C) in the complete L-fuzzy metric space.

2. Preliminaries

Definition 2.1. Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and U be a non empty set is called universe. An *$\mathcal{L}$-fuzzy set* $\mathcal{A}$ on U is defined as a mapping $\mathcal{A} : U \to L.$ For each u in U, $\mathcal{A}(u)$ represents the degree to which u satisfies $\mathcal{A}$.

Definition 2.2. A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $T : L^2 \to L$ satisfying the following conditions.

(i) $T(x, 1_{\mathcal{L}}) = x$ for all $x \in L$ (boundary condition)

(ii) $T(x, y) = T(y, x)$ for all $x, y \in L^2$ (commutativity)

(iii) $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in L^3$ (associativity)

(iv) $x \leq_L x'$ and $y \leq_L y' \Rightarrow T(x, y) \leq_L T(x', y')$ (monotonicity)

Definition 2.3. A t-norm T on $\mathcal{L}$ is said to be *continuous* if for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converges to x and y. We have $\lim_{n\to\infty} T(x_n, y_n) = T(x, y)$

For example, $T(x, y) = \min(x, y)$ and $T(x, y) = xy$ are two continuous t-norms on [0, 1].

A t-norm can also be defined recursively as an $(n+1)$-ary operation $(n \in \mathbb{N})$ by $T' = T$ and $T^n(x_1, \ldots, x_{n+1}) = T(T^{n-1}(x_1, \ldots, x_n), x_{n+1})$ for $n \geq 2$ and $x_i \in L$.

Definition 2.4. The 3-tuple $(X, \mathcal{M}, T)$ is said to be an *$\mathcal{L}$-fuzzy metric* *space* if X is an arbitrary (nonempty) set, T is a continuous t-norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set $X^2 \times (0, +\infty)$ on satisfying the following conditions for every x, y, z in X and t, s in $(0, +\infty)$

(a) $\mathcal{M}(x, y, t) >_{\mathcal{L}} 0_{\mathcal{L}}$

(b) $\mathcal{M}(x, y, t) = 1_{\mathcal{L}}$ for all $t > 0$ if and only if $x = y$

(c) $\mathcal{M}(x, y, t) = \mathcal{M}(x, y, t)$

(d) $T(\mathcal{M}(x, y, t), \mathcal{M}(x, y, \mathrm{s}) \leq_{\mathcal{L}} \mathcal{M}(x, z, t + s)$

(e) $\mathcal{M}(x, y, \cdot) : (0, +\infty) \to L$ is continuous. In this case $\mathcal{M}$ is called an $\mathcal{L}$-*fuzzy metric***.** If $\mathcal{M} = \mathcal{M}_{M, N}$ is an intuitionistic fuzzy set, then the 3-tuple $(X, \mathcal{M}_{M, N}, T)$ is said to be an *intuitionistic fuzzy metric space*.

Example 2.5. Let (X, d) be a metric space. Denote $T(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* and let M and N be fuzzy sets on $X^2 \times (0, +\infty)$ defined as follows.

$$\mathcal{M}_{M, N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{ht^n}{ht^n + md(x, y)}, \frac{md(x, y)}{ht^n + md(x, y)}\right)$$

for all $t, h, m, n \in R^+$. Then $(X, \mathcal{M}_{M, N}, T)$, is an intuitionistic fuzzy metric space.

Definition 2.6. A sequence $\{x_n\}_{n \in N}$ is an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, T)$ is called a *Cauchy Sequence*, if for each $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in N$ such that for all $m \geq n \geq n_0 \, (n \geq m \geq n_0)$, $\mathcal{M}(x_m, x_n, t) >_L \mathcal{N}(\varepsilon)$.

The sequence $\{x_n\}_{n \in N}$ is said to be *convergent* to $x \in X$ in the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, T)$ (denoted by $x_n \xrightarrow{\mathcal{M}} x$) if $\mathcal{M}(x_n, x, t) = \mathcal{M}(x, x_n, t) \to 1_{\mathcal{L}}$ as $n \to +\infty$ for every $t > 0$.

An $\mathcal{L}$-fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

Henceforth, we assume that T is a continuous t-norm on the lattice $\mathcal{L}$ such that for every $\mu \in L/\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ there is a $\lambda \in L/\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $T^{n-1}(\mathcal{N}(\lambda), \ldots, \mathcal{N}(\lambda)) >_L \mathcal{N}(\mu)$.

Definition 2.7. An $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ has the *property* (C) if it satisfies the following condition $\mathcal{M}(x, y, t) = C$ for all $t > 0$ implies $C = 1_{\mathcal{L}}$.

Lemma 2.8. *Let* $(X, \mathcal{M}, \mathcal{T})$ *be an* $\mathcal{L}$*-fuzzy metric space. If we define* $E_{\lambda, \mathcal{M}} : X^2 \to R^+ \cup \{0\}$ *by* $E_{\lambda, \mathcal{M}}(x, y) = \inf\{t > 0 : \mathcal{M}(x, y, t) >_L \mathcal{N}(\lambda)\}$ *for each* $\lambda \in L/\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ *and* $x, y \in X$. *Then*

(i) *For any* $\mu \in L/\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ *there exists* $\lambda \in L/\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ *such that*

$E_{\mu, \mathcal{M}}(x_1, x_n) \leq E_{\lambda, \mathcal{M}}(x_1, x_2) + E_{\lambda, \mathcal{M}}(x_2, x_3) + \ldots + E_{\lambda, \mathcal{M}}(x_{n-1}, x_n)$ *for any* $x_1, \ldots, x_n \in X$.

(ii) *The sequence* $\{x_n\}_{n \in N}$ *is convergent w. r. t.* $\mathcal{L}$*-fuzzy metric space* $\mathcal{M}$ *if and only if* $E_{\lambda, \mathcal{M}}(x_n, x) \to 0$. *Also the sequence* $\{x_n\}_{n \in N}$ *is Cauchy w. r. t.* $\mathcal{L}$*-fuzzy metric space* $\mathcal{M}$ *if and only if it is Cauchy with* $E_{\lambda, \mathcal{M}}$.

Lemma 2.9. *Let* $(X, \mathcal{M}, \mathcal{T})$ *be an* $\mathcal{L}$*-fuzzy metric space which has the property* (C). *If for all* $x, y \in X, t > 0$ *and for a number* $k \in (0, 1)$, $\mathcal{M}(x, y, kt) \geq_L \mathcal{M}(x, y, t)$. *Then* $x = y$.

3. Compatible Maps of Type (E)

Definition 3.1. Let S and T be two mapping from an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ into itself and $\{x_n\}$ be a sequence in X such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = u$ for some $u \in X$. Then the mapping S and T are said to be *Compatible of type* (E)

iff $\lim_{n\to\infty} \mathcal{M}(SSx_n, ST_n, t) = 1_{\mathcal{L}}$,

$\lim_{n\to\infty} \mathcal{M}(SSx_n, Tx, t) = 1_{\mathcal{L}}$,

$\lim_{n\to\infty} \mathcal{M}(STx_n, Tx, t) = 1_{\mathcal{L}}$,

$\lim_{n\to\infty} \mathcal{M}(TTx_n, TSx_n, t) = 1_{\mathcal{L}}$,

$\lim_{n\to\infty} \mathcal{M}(TTx_n, Sx, t) = 1_{\mathcal{L}}$,

$\lim_{n\to\infty} \mathcal{M}(TSx_n, Sx, t) = 1_{\mathcal{L}}$ for all $t > 0$.

Proposition 3.2. *If S and T are compatible mappings of type (E) on a $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ into itself. If one of S and T is continuous, $\{x_n\}$ is a sequence in X such that* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ *for some* $x \in X$.

Then (i) $S(x) = T(x)$ *and*

$$\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} TTx_n = \lim_{n\to\infty} STx_n = \lim_{n\to\infty} TSx_n$$

(ii) *If there exists* $u \in X$ *such that* $Su = Tu = x$, *then* $STu = TSu$.

Proof. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some $x \in X$.

Then by definition of compatible of type (E), we have $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} STx_n = T(x)$

If S is a continuous mapping. Then we get $\lim_{n\to\infty} SSx_n = S(\lim_{n\to\infty} Sx_n)$ $= S(x) \Rightarrow T(x) = S(x)$

Also $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} TTx_n = \lim_{n\to\infty} STx_n = \lim_{n\to\infty} TSx_n$.

Similarly, if T is continuous, we get the same result.

Again, suppose $Su = Tu = x$ for some $u \in X$.

Then $STu = S(Tu) = S(x)$ and $TSu = T(su) = T(x)$

From (i) we have $S(x) = T(x)$. Hence $STu = TSu$ ■

Theorem 3.3. *If* $(X, \mathcal{M}, \mathcal{T})$ *is a complete L fuzzy metric space with the property of* (C). *If one of the self mappings* (A, S) *and* (B, T) *of* X *is continuous such that*

(i) $AX \subset TX$, $BX \subset SX$

(ii) $\mathcal{M}(Ax, By, kt) \geq_L \mathcal{M}(Sx, Ty, t)$ *for all* $x, y \in X$ *and* $k \in [0, 1]$

(iii) *If* (A, S) *and* (B, T) *compatible of type* (E). *Then* A, B, S *and* T *have a unique common fixed point.*

Proof. Let $x_0 \in X$ from condition (i) there exists a point $x_1, x_2 \in X$ such that $Ax_0 = Tx_1 = y_0$ and $Bx_1 = Sx_2 = y_1$

Therefore by induction, we construct a sequence $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ for $n = 0, 1, 2, \ldots$

We first prove $\{y_n\}$ is a Cauchy sequence in $(X, \mathcal{M}, \mathcal{T})$

$$\mathcal{M}(y_{2n}, y_{2n+1}, t) = \mathcal{M}(Ax_{2n}, Bx_{2n+1}, t)$$

$$\geq_L \mathcal{M}(Sx_{2n}, Tx_{2n+1}, t/k)$$

$$= \mathcal{M}(y_{2n-1}, y_{2n}, t/k)$$

$$\mathcal{M}(y_n, y_{n+1}, t) \geq_L \mathcal{M}(y_{n-1}, y_n, t/k)$$

$$\geq_L \mathcal{M}(y_{n-2}, y_{n-1}, t/k^2) \ldots \geq_L \mathcal{M}(y_0, y_1, t/k^n)$$

This implies $E_{\lambda, \mathcal{M}}(y_n, y_{n+1}) \leq k^n E_{\lambda, \mathcal{M}}(y_0, y_1)$

Therefore, for every $\mu \in L/\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ there exists $\gamma \in L/\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $E_{\mu, \mathcal{M}}(y_n, y_m) \leq E_{\gamma, \mathcal{M}}(y_n, y_{n+1}) + E_{\gamma, \mathcal{M}}(y_{n+1}, y_{n+2}) + \ldots + E_{\gamma, \mathcal{M}}(y_{m-1}, y_m)$

$$\leq k^n + E_{\gamma, \mathcal{M}}(y_0, y_1) + k^{n+1} + E_{\gamma, \mathcal{M}}(y_0, y_1) + \ldots + k^{m-1} + E_{\gamma, \mathcal{M}}(y_0, y_1)$$

$$\leq E_{\lambda, \mathcal{M}}(y_0, y_1) \sum_{j=n}^{m-1} k^j \to 0 \text{ as } m, n \to \infty.$$

Then by lemma 2.8, $\{y_n\}$ is a Cauchy sequence.

Since $(X, \mathcal{M}, \mathcal{T})$ is complete, $\{y_n\}$ converges to some point $z \in X$ and so that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z,\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z$$

If A and S are compatible of type (E), one of the mapping of the pair (A, S) continuous.

Then by proposition 3.2, we have $Az = Sz$.

Since $AX \subset TX$, there exists a point ω in X such that $Az = T\omega$

To prove $Az = T\omega$

Using condition (ii) put $x = z,\ y = \omega$;

$\mathcal{M}(Az, B\omega, kt) \geq_L \mathcal{M}(Sz, T\omega, t) = \mathcal{M}(Az, Az, t) = 1_L$

$\mathcal{M}(Az, B\omega, kt) = 1_L$ and we get $Az = B\omega$

Now to prove $Sz = Az = z$

Put $x = z,\ y = x_{2n+1}$ in (ii) $\mathcal{M}(Az, Bx_{2n+1}, kt) \geq_L \mathcal{M}(Sz, Tx_{2n+1}, t)$ as $n \to \infty$

$$\mathcal{M}(Az, z, kt) \geq_L \mathcal{M}(Az, z, t)$$

Hence we get $Sz = Az = z$

z is a common fixed point of A and S.

Again if B and T are compatible of type (E) and one of the mappings of (B, T) discontinuous, $B\omega = T\omega = Az = z$. By proposition 3.2, $BB\omega = BT\omega = TB\omega = TT\omega$. Thus $Bz = Tz$

Put $x = x_{2n},\ y = z$ in (ii) $\mathcal{M}(Ax_{2n}, Bz, kt) \geq_L \mathcal{M}(Sx_{2n}, Tz, t)$ as $n \to \infty$

$$\mathcal{M}(z, Bz, kt) \geq_L \mathcal{M}(z, Bz, t)$$

We have $Bz = Tz = z$

z is common fixed point of B and T.

For uniqueness, suppose that $A\omega$ is another common fixed point of A, B, S and T.

Then using (ii), we put $x = Az,\ y = A\omega$;

$$\mathcal{M}(AAz, BA\omega, kt) = \mathcal{M}(Az, A\omega, kt) \geq_L \mathcal{M}(SAz, TA\omega, t) \geq_L \mathcal{M}(Az, A\omega, t)$$

Therefore $Az = A\omega = z$

Thus z is a unique common fixed point of A, B, S and T.

Corollary 3.4. *If* $(X, \mathcal{M}, \mathcal{T})$ *is a complete L fuzzy metric space with the property of (C). If one of the self mappings* (A, B) *of X is continuous such that* $\mathcal{M}(Ax, By, kt) \geq_L \mathcal{M}(x, y, t)$ *for all* $x, y \in X$ *and* $k \in [0, 1]$ *and if* (A, B) *is compatible of type (E). Then A and B have a unique common fixed point.*

Proof. If we take $S = T = I_x$ an identity mapping of X in the theorem 3.2, we get the result.

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