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GENERAL POSITION PROBLEM OF HEXAGONAL DERIVED NETWORKS

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Abstract

For a graph G, the general position problem aims to obtain a general position set S of maximum number of vertices in G, in which no three vertices lie on a same geodesic in G. Such a general position set is referred to as a gp-set of G. The gp-number of G, $gp(G)$ denotes the cardinality of a gp-set S in G. In this paper, we solve the general position problem of hexagonal derived networks such as hexagonal, honeycomb, silicate and oxide networks and compute their general position numbers i.e., $gp(G)$.

1. Introduction

Given a graph G, the general position problem is to find a general position set S of maximum number of vertices in which no three vertices lie on a same geodesic in G. The set S is referred to as a gp-set of G and the gp-number, $gp(G)$ denotes the cardinality of S. General Position Subset

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Selection Problem [5, 15] and the no-three-in-line problem [4], motivated the authors to introduce the concept of general position problem in [10]. Also, in [10] they have proved the NP-completeness of the above problem in general. Recently general position problem has been studied in [1, 6, 8, 11].

Honeycomb, hexagonal networks are known as natural architectures as they bear resemblance to atomic or molecular lattice structures. Hexagon tessellations are used to build honeycomb networks recursively [16]. Honeycomb networks are extensively applied in image processing [2], cellular phone base station [14], computer graphics [9] and as the representation of Carbon Hexagons of Carbon Nanotubes [7] and benzenoid hydrocarbons [16] in chemistry. The largest and the hardest class of minerals are silicates so far. Fusing metal carbonates or metal oxides with sand, silicates were obtained.

Throughout this article, $G = (V, E)$ represents a simple and connected graph. We refer [3] for the basic definitions. In the following sections, we solve the general position problem of hexagonal, honeycomb, silicate and oxide networks and compute their *gp*-numbers.

2. Hexagonal Networks

$HX(n)$, $n > 1$ denotes a hexagonal network of dimension n. $HX(2)$ consists of six triangles. $HX(n)$ is constructed by adding triangles over the boundary edges of $HX(n-1)$. $HX(n)$ consists of six three degree vertices called as corner vertices and from each of the corner vertices there is a center vertex at a distance $n-1$. In $HX(n)$, there are $3(n^2 - n) + 1$ vertices and $3(3n^2 - 5n + 2)$ edges. There are exactly 6 three degree vertices, $6n - 12$ four degree vertices and $3n^2 - 9n + 7$ six degree vertices [12].

For our convenience we call the vertices on the boundary of $HX(n)$ as the boundary vertices and the remaining vertices as the interior vertices. The set of interior vertices in $HX(n)$ is called its interior and is denoted as $\text{int}(HX(n))$.

Lemma 2.1. *Let H be a subgraph of* $HX(n)$ *induced by the vertices of a* *path of hexagons parallel to a major axis. Then* $gp(H) = 4$.

Proof. Let H be a subgraph of $HX(n)$ induced by the vertices of a path of hexagons along the Z-axis in $HX(n)$ and S be a general position set of H. Consider the 3 parallel paths P_1, P_2, P_3 of H (Refer Figure 2). Since $gp(P_n) = 2, |S \cap V(P_1)| \leq 2$. Without loss of generality, assume $|S \cap V(P_1)| = 2$. Denote the two vertices as a and c respectively. One can easily verify that, $|S \cap V(P_2)| \leq 1$. Further, if $|S \cap V(P_2)| = 1$, then $S \cap V(P_3) = \varnothing$ and if $S \cap V(P_2) = \varnothing$, then $|S \cap V(P_3)| \leq 2$. Hence $gp(H) \leq 4$. Choose two vertices, say b and d on P_3 such that b, d is the mirror image about the Z-axis of a, c respectively (Refer Figure 2). Now $\{a, b, c, d\}$ is the required gp-set of H. Hence the proof.

Lemma 2.2. *Let* S *be a general position set of* $HX(n)$ *and suppose* $p, r \in S$ *such that* p, r *do not lie on a same path parallel to the major axes. Let* H *be the triangular grid subgraph of* $HX(n)$ *bounded by the parallelogram pqrs. Then any vertex of H lies on a* p, r*-geodesic.*

Proof. The proof can be easily verified by completing the parallelogram *pqrs* (Refer Figure 1).

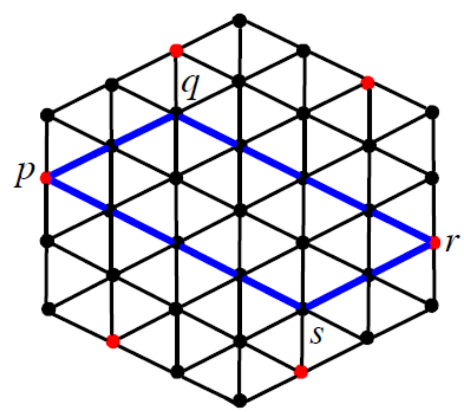

Figure 1. Vertices in red form a gp-set of $HX(4)$.

Lemma 2.3. *For* $n \geq 3$, $gp(HX(n)) \leq 6$.

Proof. Let S be a general position set of $HX(n)$. Consider the subgraph H of $HX(n)$ induced by the vertices of the central path of hexagons along the Z-axis. By Lemma 2.1, $|S \cap V(H)| \leq 4$. Without loss of generality, assume $|S \cap V(H)| = 4$ and denote the four vertices by a, b, c, d. Denote the right hemisphere of $HX(n)$- that is, the subgraph of $HX(n)$ on the plane $Z \geq 0$ by R. Suppose any other vertex, say e in R belongs to S. Then consider the two intersecting lines, say l_1 and l_2 passing through e in R (Refer Figure 2). Denote the two boundary vertices in R lying on l_1, l_2 by m, n respectively. Consider the triangular grid subgraph of $HX(n)$ bounded by the triangle emn, denoted by T.

Claim: $S \cap (V(R) \setminus V(T)) = \varnothing$.

On the contrary, assume that there exists a vertex $x \in V(R) \setminus V(T)$ above l_1. Then e will lie inside the parallelogram formed with c, x as diagonally opposite vertices (Refer Figure 2). Similarly if there exists a vertex $y \in V(R) \setminus V(T)$ below l_2, then e will lie inside the parallelogram with a, y as diagonally opposite vertices. Lemma 2.2 completes the proof of our claim.

Consider the subgraph T of R. We claim that $|\operatorname{int}(T) \cap S| \leq 1$. Suppose on the contrary it contains two vertices say f and h, then f will lie on a h, e- geodesic by Lemma 2.2. Now assume $f \in \operatorname{int}(T)$.

Next consider the subgraph on the other hemisphere of $HX(n)$ denoted by R' on the plane $Z \leq 0$.

Case 1. If $f \in S$, then $(V(R') \setminus \{a, c\}) \cap S = \varnothing$. On the contrary, suppose there exists a vertex, say g in $(V(R') \setminus \{a, c\}) \cap S$ then e lies on a g, f- geodesic by Lemma 2.2.

Case 2. If $f \notin S$, then $|(V(R') \setminus \{a, c\}) \cap S| \leq 1$. Suppose $|(V(R') \setminus \{a, c\}) \cap S| = 2$, then one vertex will lie on a geodesic between the other vertex and e, by Lemma 2.2. Hence $|S| \leq 6$ and this completes the proof of the lemma.

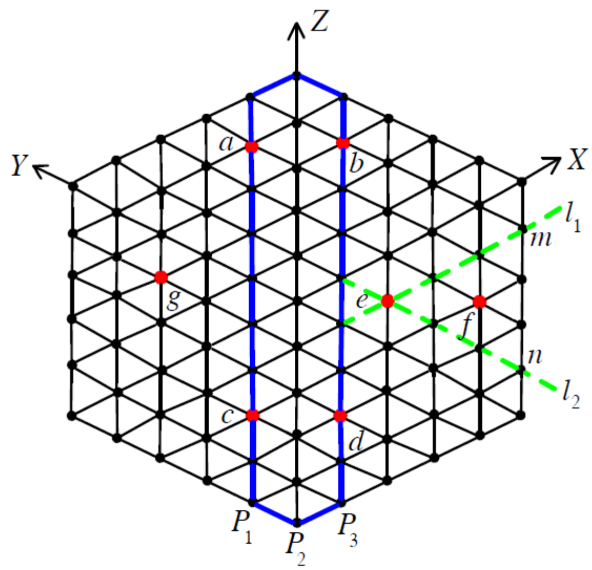

Figure 2. $HX(6)$.

Theorem 2.1. *If* $n \geq 3$, *then* $gp(HX(n)) = 6$.

Proof. By Lemma 2.3, $gp(HX(n)) \leq 6$. Let S be a set of 4-degree boundary vertices, one from each face of $HX(n)$ such that the distance between any two consecutive vertices is $n-1$. Then S is the required gp-set of $HX(n)$ (Refer Figure 1).

3. Honeycomb Networks

Honeycomb network is based on hexagonal networks. $HC(1)$ consist of a hexagon. We obtain $HC(2)$, by placing hexagons on the boundary edges of $HC(1)$. Inductively, by placing hexagons on the boundary edges of $HC(n-1)$ we obtain $HC(n)$ (Refer Figure 3). There are $6n^2$ vertices and $9n^2 - 3n$ edges in $HC(n)$. There are $6n$ two degree vertices and the remaining are three degree vertices [12].

Theorem 3.1. *If* $n \geq 2$, *then* $gp(HC(n)) = 6$.

Proof. Honeycomb network is based on hexagonal tessellations. Hence the proof is similar to the proof of Theorem 2.1, because the paths induced by the central path of hexagons of $HC(n)$ parallel to a major axis behave exactly similar to the paths P_1 and P_3 of $HX(n)$ (Refer Figure 2 and Figure 3).

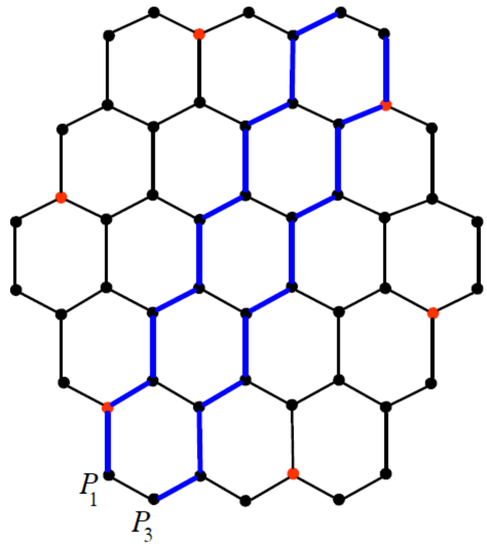

Figure 3. A *gp*-set of $HC(3)$.

4. Silicate Networks

A honeycomb network is used to construct a silicate network, $SL(n)$. Consider $HC(n)$. On every vertex of $HC(n)$ place silicon ions. Divide all the edges of $HC(n)$ once and on the new vertices place oxygen ions. At each two degree silicon ions, $6n$ new pendant edges are introduced and at these pendant vertices place oxygen ions. The three adjacent oxygen ions are associated with every silicon ion to form a tetrahedron.

The resultant network is $SL(n)$. $SL(n)$ consists of $15n^2 + 3n$ vertices and $36n^2$ edges. There are $6n^2 + 6n$ three degree vertices and $9n^2 - 3n$ six degree vertices. The three degree oxygen nodes are called as boundary nodes of $SL(n)$ [13].

Lemma 4.1 [10]**.** *The set of simplicial vertices of a graph form a general position set.*

Lemma 4.2. *If S is a gp-set of* $SL(n)$, *then S does not contain any of the* 6-*degree vertices of* $SL(n)$.

Proof. Suppose on the contrary S is a *gp*-set of $SL(n)$ containing a 6-degree vertex, say v. Observe that v is incident on two axis l_1 and l_2. Consider the axis l_1 with terminal vertices a and b (Refer Figure 4). Since $v \in S$, both a and b cannot belong to S. Without loss of generality, assume $a \in S$. Similarly since $v \in S$, either one of the terminal vertices say c or d of l_2 belongs to S. Since $a \in S$, $c \in S$. Also v is adjacent to two silicon vertices e and f in the neighboring two tetrahedrons. By a similar argument, $e \in S$ and $f \notin S$. Now $S' = (S \setminus \{v\}) \cup \{b, d, f\}$ is a general position set of $SL(n)$ such that $|S'| > |S|$. Hence the proof.

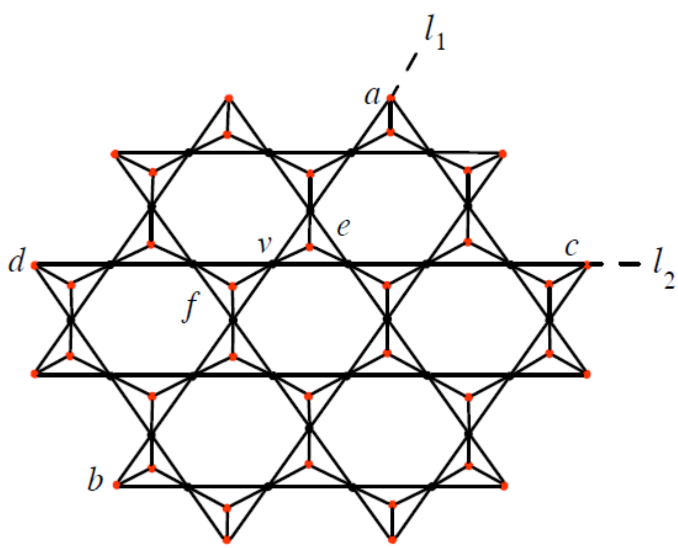

Figure 4. A gp-set of $SL(2)$.

Theorem 4.1. If $n \geq 1$, then $gp(SL(n)) = 6n^2 + 6n$.

Proof. Let S be the set of all simplicial vertices of $SL(n)$. Observe that S contains only the 3-degree vertices of $SL(n)$ (Refer Figure 4). From Lemma 4.1, S is a general position set of $SL(n)$. By Lemma 4.2, we observe that S is the required *gp*-set of $SL(n)$.

Corollary 4.1. *The gp-set of* $SL(n)$ *is unique*.

5. Oxide Networks

The oxide network $OX(n)$ is obtained by deleting all the silicon ions from the silicate network. $OX(n)$ consists of $9n^2 + 3n$ vertices and $18n^2$ edges [13].

Theorem 5.1. *If* $n \geq 1$, *then* $gp(OX(n)) = 6n$.

Proof. We first claim that any *gp*-set of $OX(n)$ cannot contain any of the 4-degree vertices of $OX(n)$. On the contrary, suppose S is a *gp*-set of $OX(n)$ containing a 4-degree vertex, say v. Note that v is incident on two axis l_1 and l_2. Let a and b be the terminal vertices of l_1. Since $v \in S$, both a and b cannot belong to S. Without loss of generality, assume $a \in S$. If c and d are the terminal vertices of l_2, since $a, v \in S$, $c \in S$. Now $S' = (S \setminus \{v\}) \cup \{b, d\}$ is a general position set of $OX(n)$ such that $|S'| > |S|$. This proves our claim. Therefore $gp(OX(n)) \leq 6n$.

Let T be the set of all simplicial vertices of $OX(n)$. Observe that T contains only the 2-degree vertices of $OX(n)$ and $|T| = 6n$. By Lemma 4.1, T forms a general position set of $OX(n)$ and T is the required *gp*-set of $OX(n)$ Refer Figure 5).

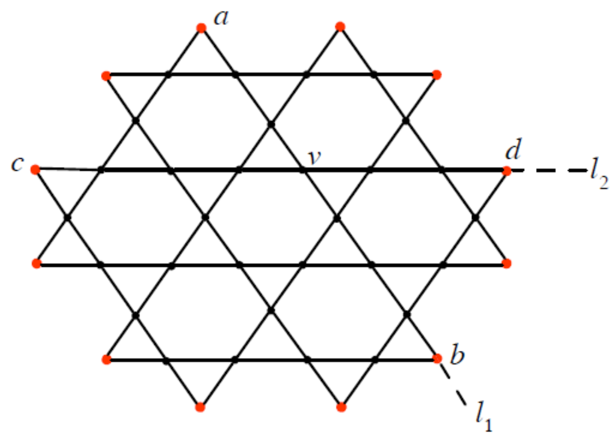

Figure 5. A *gp*-set of $OX(2)$.

Corollary 5.1. *The gp-set of* $OX(n)$ *is unique.*

6. Conclusion

In this article, the general position problem of hexagonal derived networks such as hexagonal, honeycomb, silicate and oxide networks are solved and their general position numbers are determined.

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