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JUST CHROMATIC EXCELLENCE IN INTUITIONISTIC FUZZY GRAPHS

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Abstract

Let G be an intuitionistic fuzzy graph. A family $C = \{c_1, \ldots, c_k\}$ of intuitionistic fuzzy sets on a set V is called a k-vertex coloring of $G = (V, E)$ if (i) $\upsilon\, c_1(x) = V$, for all $x \in V$, (ii) $c_i \wedge c_j = 0$ and (iii) For every strong edge xy of G, $\min \{c_i(\mu_1(x)), c_i(\mu_1(y))\} = 0$ and $\max \{c_i(\gamma_1(x)), c_i(\gamma_1(y))\} = 1\ (1 \leq i \leq k)$.

The chromatic number $(\chi(G))$ of the intuitionistic fuzzy graph G is the least value of k for which the G has a k-vertex coloring. The chromatic partition C is the partitioning the vertex set into independent sets of vertices where each set has the same color. If every vertex of G emerges as a singleton in exactly one χ-partitions of G, then it is just χ-excellent. If just χ-excellent G of order n has exactly n number of χ-partition then it is tight just χ-excellent graph. This article is focusing on the concepts called just chromatic excellence and tight just chromatic excellence in intuitionistic fuzzy graphs.

1. Introduction

Krassimir T. Atanassov introduced Intuitionistic fuzzy sets [7] and Intuitionistic fuzzy graph [8] in 1986 and 1999 respectively. R. Parvathi et al. expanded the intuitionistic fuzzy graph and its properties [11, 12]. S. Ismail Mohideen et al. conferred coloring of intuitionistic fuzzy graph using α, β-cuts

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[5] in 2015. M. A. Rifayathali et al. examined strong intuitionistic fuzzy graph coloring [13] in 2017 and intuitionistic fuzzy graph coloring [17] in 2018. Intuitionistic Fuzzy Graph coloring has been applied to many real world problems like Job scheduling, allocation, telecommunications and bioinformatics, etc.

2. Preliminaries

Definition 2.1. Intuitionistic Fuzzy Graph (IFG) is of the form $G = (V, E)$ where

(i) $V = \{v_1, v_2, \ldots, v_n\}$ such that $\mu_1 : V \to [0, 1]$ and $\gamma_1 : V \to [0, 1]$ denote the degrees of membership and non-membership of the element $v_i \in V$ respectively and $0 \leq \mu_1(v_i) + \gamma_1(v_i) \leq 1,$ for every $v_i \in V, (i = 1, 2, \ldots, n).$

(ii) $E \subset V \times V$ where $\mu_2 : V \times V \to [0, 1]$ and $\gamma_2 : V \times V \to [0, 1]$ are such that $\mu_2(v_i, v_j) \leq \min [\mu_1(v_i), \mu_1(v_j)], \gamma_2(v_i, v_j) \leq \max [\gamma_1(v_i), \gamma_1(v_j)].$ And $0 \leq \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \leq 1$ for every $(v_i, v_j) \in E.$

Definition 2.2. The arc (u, v) in IFG G is said to be a strong arc if $\frac{1}{2} \min \{\mu_1(u), \mu_1(v)\} \leq \mu_2(u, v)$ and $\frac{1}{2} \max \{\gamma_1(u), \gamma_1(v)\} \leq \gamma_2(u, v).$

Definition 2.3. Let G be an IFG. A family $C = \{c_1, \ldots, c_k\}$ of intuitionistic fuzzy sets on a set V is called a k-vertex coloring of $G = (V, E)$ if (i) $\vee c_i(x) = V,$ for all $x \in V,$ (ii) $c_i \wedge c_j = 0$ and (iii) For every strong edge xy of $G, \min \{c_i(\mu_1(x)), c_i(\mu_1(y))\} = 0$ and $\max \{c_i(\gamma_1(x)), c_i(\gamma_1(y))\} = 1 \, (1 \leq i \leq k).$

The chromatic number $(\chi(G))$ of the IFG G is the least value of k for which the G has a k-vertex coloring.

Definition 2.4. The chromatic partition C is the partitioning the vertex set into independent sets of vertices where each set has the same color.

Definition 2.5. A graph G is an intuitionistic fuzzy chromatic excellent if for every vertex of $v \in V(G)$ there exists a intuitionistic fuzzy chromatic partition C such that $\{v\} \in C.$

3. Just Chromatic Excellence in Intuitionistic Fuzzy Graphs

Definition 3.1. If every vertex of *G* emerges as a singleton in exactly one χ-partitions of *G*, then it is just χ-excellent.

Theorem 3.2. *If G is a just χ-excellent IFG and $G \neq K_n$, then any χ-partition of can contain exactly one singleton.*

Proof. Let us assume that there exists a χ-partition *C* of *G* containing more than one singleton. Let $C_1 = \{\{u_1\}, \{u_2\}, V_2, V_\chi\}$ be a partition of *G*. Since *G* is just χ-excellent and $G \neq K_n$, no vertex of $V(G)$ is a full degree vertex. Therefore there exists $v_1 \in V(G)$ such that u_1 and v_1 are not adjacent such that $\mu_2(u_1, v_1) > \min[\mu_1(u_1), \mu_1(v_1)]$ and $\gamma_2(u, v) > \max[\gamma_1(u_1), \gamma_1(v_1)]$. Let $v_1 \in V_i$, $3 \leq i \leq \chi$. Clearly, $|V_i| \geq 2$, for if $V_i = \{v_1\}$, then u_1 and v_1 are adjacent. Let $C_2 = \{\{u_1, v_1\}, \{u_2\}, V_2, \ldots, V_i - \{v_1\}, \ldots, V_\chi\}$. Then C_2 is a χ-partition containing $\{u_2\}$, which is a contradiction to *G* is just χ-excellent.

Theorem 3.3. *If G and H are just χ-excellent IFG and one of them is not complete if other is K_1 then $G + H$ is not just χ-excellent.*

Proof. Let $G = K_1$. Then *H* is not complete IFG. Then $G + H$ is not complete but it has a full degree vertex. Therefore $G + H$ is not just χ-excellent graph. Let $G \neq K_1$ and $h \neq K_1$. Since *G* and *H* are just χ-excellent, $G, H \neq \overline{K_n}$ for $n \geq 2$. Then any χ-partition of *G* and *H* contains atleast two elements. Then for any χ-partition of *G* with a singleton element, we can associate several χ-partitions of *H*, giving a χ-partition of $G + H$. Therefore $G + H$ is not just χ-excellent.

Definition 3.4. Let $C = \{V_1, V_2, \ldots, V_\chi\}$ be a χ-partition of *G*. Let $u \in V_i$ is said to be intuitionistic-fuzzy colorful vertex if *u* is adjacent to every color class in *C*-partition but not adjacent to V_i such that $\mu_2(u, v_i) \leq \min[\mu_1(u), \mu_1(v_i)]$ and $\gamma_2(u, v_i) \leq \max[\gamma_1(u), \gamma_1(v_i)]$ for some vertex $v_i \in V_1, \ldots, V_{i-1}, \ldots, V_x$ and $\mu_2(u, v_j) > \min[\mu_1(u), \mu_1(v_j)]$ and $\gamma_2(u, v_j) > \max[\gamma_1(u), \gamma_1(v_j)]$ for every $v_j \in V_i$.

Theorem 3.5. *Let G be a just* χ*-excellent IFG which is not complete. Let* $u \in V(G)$ *and let* $C = \{\{u\}, V_2, \ldots, V_\chi\}$ *be a* χ*-partition of G. If* $|V_i| \geq 3$ *for some* $2 \leq i \leq \chi$ *then there exists at least some* V_j *with* $|V_j| \geq 3$ *containing a vertex not adjacent to u.*

Proof. Suppose let u is adjacent to every vertex in V_i with $|V_i| \geq 3 (2 \leq i \leq \chi)$.

Case (1): Let $|V_i| \geq 3$ for all $i, 2 \leq i \leq \chi$. Then is a full degree vertex and it appears singleton in every χ-partition of G, which is a contradiction to G is just χ-excellent and $G \neq K_n$.

Case (2): Let $|V_i| \geq 3$ for all $i, 2 \leq i \leq t$ and $|V_{t+1}| = 2$. Let $|V_{t+1} = \{v_1, v_2\}|$. Suppose there exists $V_{t+1}, V_{t+2}, \ldots, V_\chi$ such that $|V_{i+j}| = 2, 2 \leq j \leq x - t$ (Note that no $V_i, (2 \leq i \leq \chi)$ is a singleton since G is just χ-excellent). Since C is a χ-partition, u is adjacent with atleast one vertex in each of $V_{t+1}, V_{t+2}, \ldots, V_\chi$. Suppose u is adjacent with v_1 and not adjacent with v_2 in V_{t+1} such that $\mu_2(u, v_1) \leq \min [\mu_1(u), \mu_1(v_1)]$ and $\gamma_2(u, v_1) \leq \max [\gamma_1(u), \gamma_1(v_1)]$ and $\mu_2(u, v_2) > \min [\mu_1(u), \mu_1(v_2)]$ and $\gamma_2(u, v_2) > \min [\gamma_1(u), \gamma_1(v_2)]$ for $v_1, v_2 \in V_{t+1}$. Then u is adjacent with every vertex $V_{t+j}, 2 \leq j \leq \chi - 1$ such that $\mu_2(u, v_i) \leq \min [\mu_1(u), \mu_1(v_i)]$ and $\gamma_2(u, v_i) \leq \max [\gamma_1(u), \gamma_1(v_i)]$ for every $v_i \in V_{t+j}, 2 \leq j \leq \chi - 1$. For, otherwise there exists some vertex $w \in V_{t+j}$ not adjacent with u. Therefore $C_1 = \{\{u, v_2, w\}, V_2, \ldots, V_t, \{v_1\}, \ldots, V_{t+j} - \{w\}, \ldots, V_\chi\}$ which is a contradiction to G is just χ-excellent. Hence u is adjacent with every vertex in $V - \{v_1\}$. (Note that if $V_{t+1} = V_\chi$ then also is adjacent with every vertex in $V - \{v_2\}$). Since G is just χ-excellent there exists a χ-excellent $C_2 = \{\{v_2\}, V_2', \ldots, V_\chi'\}$. Therefore $u \in V_i'$, a contradiction since u is adjacent with every vertex in $V - \{v_2\}$ such that $\mu_2(u, v_i) \leq \min [\mu_1(u), \mu_1(v_i)]$ and $\gamma_2(u, v_i) \leq \max [\gamma_1(u), \gamma_1(v_i)]$ for every vertex $v_i \in V - \{v_2\}$. Hence the theorem.

4. Tight Just Chromatic Excellence in Intuitionistic-Fuzzy Graphs

Definition 4.1. If just χ-excellent IFG G of order n has exactly n number of χ-partition then it is tight just χ-excellent graph.

Example 4.2.

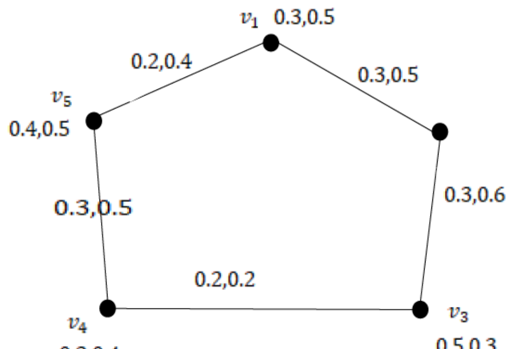

Figure 4.1

The intuitionistic-fuzzy coloring $C = \{c_1, c_2, c_3\}$

$$c_1(v_i) = \begin{cases} (0.4, 0.6) & i = 2 \\ (0, 1) & \text{otherwise} \end{cases}, \quad c_2(v_i) = \begin{cases} (0.3, 0.5) & i = 2 \\ (0.3, 0.4) & i = 4 \\ (0, 1) & \text{otherwise} \end{cases},$$

$$c_3(v_i) = \begin{cases} (0.5, 0.3) & i = 3 \\ (0.4, 0.5) & i = 5 \\ (0, 1) & \text{otherwise} \end{cases}.$$

For the above IFG, $\chi(G) = 3$. Similarly, the χ-partitions are

$$C_1 = \{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}, C_2 = \{\{v_2\}, \{v_1, v_4\}, \{v_3, v_5\}\},$$

$$C_2 = \{\{v_3\}, \{v_1, v_4\}, \{v_2, v_5\}\}, C_4 = \{\{v_4\}, \{v_1, v_3\}, \{v_2, v_5\}\},$$

$$C_5 = \{\{v_5\}, \{v_1, v_3\}, \{v_2, v_4\}\}.$$

The graph is just χ-excellent and it has exactly 5, χ-partitions. Hence the graph is tight just χ-excellent.

Theorem 4.3. *A just χ-excellent graph G is tight just χ-excellent graph if and only if* $n = 2\chi - 1$.

Proof. Suppose that G be a just χ-excellent graph with $n = 2, \chi - 1$. Since G is just χ-excellent, then there exists a χ-partition contains for given any vertex u. The remaining $\chi - 1$ partitions must have at least two elements each. Since in a just χ-excellent graph no χ-partitions can contain two singletons. Therefore the minimum number of elements in any χ-partition are $2(\chi - 1) + 1 = 2\chi - 1 = n =$ total number of elements. Therefore every χ-partition contains singleton and other sets are two elements set. If a χ-partition does not contain singleton, then the total number of elements in the partition are at least $2\chi > n$ which is a contradiction. Hence G is tight just χ-excellent. If G is tight just χ-excellent of order n, then G contains n, χ-partitions and every χ-partition must contain singleton and other $\chi - 1$ partitions are two element sets. Then the number in the partition are $2(\chi - 1) + 1 = 2\chi - 1 - n =$ total number of elements.

Theorem 4.4. *C_{2n+1} is just χ-excellent but not tight just χ-excellent if $n \geq 1$. Further there exists a chromatic partition in which every vertex of the cycle is colorful if and only if $2n + 1 \equiv 0 (\text{mod } 3)$.*

Proof. Let us take C_{3n} where n is odd. Then the intuitionistic-fuzzy chromatic number is 3. Then the χ-partition $C = \{\{u_1, u_4, \ldots, u_{3n-2}\}, \{u_2, u_5, \ldots, u_{3n-1}\}, \{u_3, u_6, \ldots, u_{3n}\}\}$ in which every vertex is intuitionistic-fuzzy colorful. Consider C_{2n+1} where n is even. A χ-partition giving $3n - 1$ intuitionistic-fuzzy colorful vertices is $\{\{u_1, u_4, \ldots, u_{3n-2}\}, \{u_2, u_5, \ldots, u_{2n-1}\}, \{u_3, u_5, \ldots, u_{3n}\}\}$. In above χ-partition except u_1 and u_{3n+1} are colorful. Let $C = \{V_1, V_2, V_3\}$ be a χ-partition of C_{3n+1}. (n-even). For any $v_i, u_i \in V_i$ then u_{i-2} and $u_{i+2} \notin V_i$. Hence $V_1 = \{u_1, u_4, \ldots\}, V_2 = \{u_3, u_6, \ldots\}, V_3 = \{u_2, u_5, \ldots\}$. Since the total number of vertices is $3n + 1$, there exists at least one V_i such that $|V_i| \geq n + 1$. Suppose that $|V_1| \geq n + 1$. If $|V_1| = n + 1$, then the $(n + 1)^{\text{th}}$ term in V_1 is u_{2n+1} which is adjacent to $u_1 \in V_1$, which is a contradiction. Similarly, contradiction arises if $|V_1| > n + 1$. Therefore $|V_1| \leq n$. Similarly $|V_2| \leq n$ and $|V_3| \leq n$ which is a contradiction to $|V| = 3n + 1$. If

$V_1 = \{u_1, u_4, \dots u_{3n-2}\}$, $V_2 = \{u_2, u_5, \dots, u_{3n-1}\}$ and $V_3 = \{u_3, u_6, \dots u_{3m}\}$, then u_{2n+1} cannot be accommodated in V_1 and V_2, since they contain the adjacent vertices u_1 and u_{3n} respectively. Therefore u_{3n+1} has to be included in V_2. Here u_{3n} and u_1 will not be intuitionistic-fuzzy colorful. Hence the number of intuitionistic-fuzzy colorful vertices is at most $3n - 1$. Since we have already shown that there exists a χ-partition containing $3n - 1$ colorful vertices. Hence the maximum number of intuitionistic-fuzzy colorful vertices in any χ-partition of C_{3n+2} (*n* even) is $3n - 1$. Similarly, we can prove that for C_{3n+2} where *n* is odd, the maximum number of intuitionistic-fuzzy colorful vertices in any χ-partition is $3n$.

Theorem 4.5. *There is no χ-partition containing exactly* $(n - 1)$ *intuitionistic-fuzzy colorful vertices in* C_{3n}.

Proof. Let $\{u_1, u_2, \dots u_{3n}\}$ be the vertices in C_{3n}. Assume that there exists a χ-partition $C = \{V_1, V_2, V_3\}$ containing exactly $(n - 1)$ intuitionistic-fuzzy colorful vertices. Since exactly one vertex u_i is not intuitionistic-fuzzy colorful, u_{i-1}, u_{i+1} belong to the same color class of *C* say V_1 such that $\mu_2(u_{i-1}, u_i) > \min [\mu_1(\mu_{i-1}), \mu_1(\mu_i)]$ and $\gamma_2(u_{i-1}, u_i) > \max [\gamma_1(\mu_{i-1}), \gamma_1(\mu_i)]$ and $\mu_2(u_i, u_{i+1}) > \min [\mu_1(\mu_i), \mu_1(\mu_{i+1})]$ and $\gamma_2(u_i, u_{i+1}) > \max [\gamma_1(u_i), \gamma_1(u_{i+1})]$. Then every element of V_1 and V_2 is colorful. Let us take $V_2 = \{u_{i1}, u_{i2}, u_{ir}\}$ such that $\mu_2(u_{it}, u_{it+1}) > \min [\mu_1(\mu_{it}), \mu_1(\mu_{it+1})]$ and $\gamma_2(u_{it}, u_{it+1}) > \max [\gamma_1(u_{it}), \gamma_1(u_{it+1})]$, $t = 1, 2, \dots r$, where $(i1 < i2 < \dots < ir)$ and $V_2 = \{u_{j1}, u_{j2}, \dots, i_{js}\}$ such that $\mu_2(u_{jt}, u_{jt+1}) > \min [\mu_1(u_{jt}), \mu_1(u_{jt+1})]$ and $\gamma_2(u_{jt}, u_{jt+1}) > \max [\gamma_1(u_{jt}), \gamma_1(u_{jt+1})]$, $t = 1, 2, \dots, s$, where $(j1 < j2 < \dots < js)$. In the color classes V_2 and V_3, ik and $ik + 1$ must have difference 3 and also in jk and $jk + 1$. Therefore in V_2 and V_3 the maximum cardinality of vertices satisfying above property is *n*. Then no V_i can have cardinality more than *n* since $\beta_0(C_{3n}) = n$. If $|V_1| < n$ or $|V_2| < n$ or $|V_3| < n$, then one or two of the remaining elements of the partition will have more than elements a contradiction. Therefore $|V_1| = n = |V_2| = |V_3|$. Since V_1 and V_2 satisfy the property that the difference between any to suffixes is $3, V_1$ also satisfies the same condition, which is a contradiction. Therefore exactly $n - 1$ intuitionistic-fuzzy colorful vertices in a χ-partition is not possible.

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