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PRIME DISTANCE ANTI-MAGIC LABELING AND ITS SPECTRAL VALUES ON SOME SPECIAL GRAPHS

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Abstract

A connected graph $G(V, E)$ is said to admit a prime distance anti-magic labeling if there exists an one-one function $f: V(G) \to Z$ such that $|f(u) - f(v)|$ is a prime number for every pair of adjacent vertices u and v in G and also all the edge labeling must be distinct. In this paper we have discussed about prime distance anti-magic labeling of some special kind of graphs such as caterpillar, spider, Bi-Star, $C_n @ P_m$ and binary tree graphs. Also we have calculated determinant, characteristic polynomial and characteristic roots of caterpillar and spider graphs.

1. Introduction

In this research we mean a graph G by finite connected undirected graph with p vertices and q edges. Let $|V(G)|$ and $|E(G)|$ be the number of vertices and number edges of the graph respectively.

For detailed survey of graph labeling we refer Gallian's [1] work.

In 1994, N. Hartsfield and Ringel [2] introduced the concept of anti-magic graph. Each vertex labeling f of a graph $G = (V, E)$ from $\{0, 1, 2, \ldots, |E(G)|\}$ induces an edge labeling f^* where $f^*(e)$ is sum the labels of end vertices of an edge e. Labeling f is called anti-magic if and only if all the edge labeling are pairwisely distinct.

In 2013, Laison et al. [3] have defined a graph G to be a prime distance

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graph if there exists a one-one labeling of its vertices given by $f: V(G) \to Z$ such that for any two adjacent vertices u and v, the integer $|f(u) - f(v)|$ is a prime and f is called a prime distance labeling of G. They also defined $f(uv)|f(u) - f(v)|$ and called f a prime distance labeling of G. Therefore, G is a prime distance graph if and only if there exists a prime distance labeling of G. Prime distance properties were studied in [4]. For a various graph theoretic notation and terminology, we follow [5].

2. Preliminaries

We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 2.1. Caterpillar is a tree with all vertices either on a single central path or distance one away from it. The central path may be considered to be the largest path in the caterpillar, so that both end vertices have valency one.

Definition 2.2. A spider $SP(P_{n,2})$ is a caterpillar $S(X_1, X_2, \ldots, X_n)$ where $X_n = 2$ and $X_i = 0, i = 1, 2, \ldots, n-1$.

Definition 2.3. The graph $G = C_n @ P_m$ is called dragon consists of a cycle C_n together with a path P_m one end vertex a_1 of P_m is joined with a node u_n of C_n. That is $|E(G)||V(G)|\{V_1 \cup V_2\}$ where $V_1 = \{u_1, u_2, \ldots, u_n\}$ of vertices of the cycle C_n and $V_2 = \{a_1, a_2, \ldots, a_m\}$ of vertices of the path P_m. Therefore $V(G) = \{v_1, v_2, \ldots, v_{n+m}\}$ and $E(G) = E(C_n) \cup E(P_m) \cup \{u_n a_1\}$.

Hence $C_n @ P_m$ contains $n + m$ vertices and equal number of edges.

Definition 2.4. Bi-star is the graph obtained by joining apex vertices of two copies of $K_{1,n}$.

3. Prime Distance Graphs

Here we discussed about prime distance labeling of some special kind of graphs.

Theorem 3.1. *Every caterpillar admits prime distance anti-magic labeling.*

Proof. Let u_i be the vertices of the central path of the caterpillar and $1 \leq i \leq n$.

Let a_i be the end vertices which is attached to the corresponding vertices of the path u_i.

Define $f : V(G) \to Z$ by

$f(a_1) = 1;$

$f(u_1) = f(a_1) + p_1;$

$f(u_2) = f(a_2) + p_2; \ldots$

In general

$f(u_n) = f(u_{n-1}) + p_n;$

$f(v_n) = f(u_n) + p_{n+1}$

Also $| f(v_n) - f(u_n) | = p_{n+1} = (n+1)^{th}$ prime.

Now to get the edge labeling, for any two adjacent vertices u and v the integer $| f(u) - f(v) |$ is prime and distinct.

Hence the caterpillar admits prime distance anti-magic labeling.

Example 3.2. The following figure 3.1 is prime distance anti-magic labeling of caterpillar $T = S(X_1, X_2, X_3)$

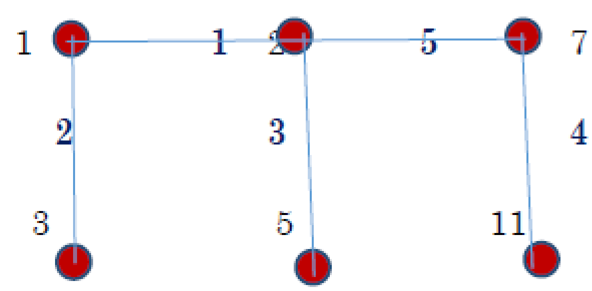

Figure 3.1.

Theorem 3.3. *Every spider* $SP(P_{n,2})$ *admits prime distance anti-magic labeling.*

Proof. Consider the graph $G = SP(P_{n,2})$. It has $n+1$ vertices and $n+1$ edges. Here the path vertices of the spider are denoted as $u_1, u_2, \ldots, u_n$ and a_1, a_2 are the end vertices which is attached to the first vertex of the path p_n. Therefore the cardinality of the vertex set of G is $n+2$.

Here $E(G) = \{v_i v_{i+1} / 1 \le i \le n-1\} \cup \{u_1 a_i / I = 1, 2\}$.

Hence $|E(G)| = n+1$.

Define $f : V(G) \to Z$ by $f(a_1) = 1$, $f(a_2) = 6$.

$f(u_1) = 3 = p_2$

$f(u_2) = f(u_1)p_3$

In general

$f(u_n) = f(u_{n-1}) + p_{n-1}$.

The assignment of edge labeling satisfies the condition that for any two adjacent vertices u and $|f(u) - f(v)|$ is prime and all are distinct from each other.

Hence the spider admits prime distance anti-magic labeling.

Example 3.4. The below figure 3.2 shows the prime distance anti-magic labeling of the spider $SP(2, 2)$.

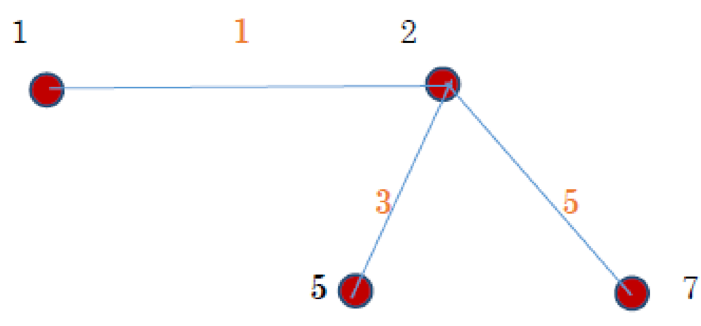

Figure 3.2.

Theorem 3.5. *The graph* $C_n @ P_m$ *admits prime distance anti-magic labeling.*

Proof. Consider $G = C_n @ P_m$ and it has $n + m$ vertices and $n + m$ edges.

Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle C_n.

Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_m.

Here $V(G) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_m\}$

Therefore $|V(G)| = n + m.$

Here $E(G) = \{u_i u_{i+1} / 1 \le i \le n-1\} \cup \{u_n u_1\} \cup \{v_i v_{i+1} / 1 \le i \le m-1\} \cup \{u_n v_1\}$

Hence $|E(G)| = n + m.$

Let $f : V(G) \to Z$ be a $1-1$ labeling defined by

$f(u_1) = 1;$

$f(u_2) = f(u_1) + p_1;$

$f(u_n) = f(u_{n-1}) + p_{n-1};$

$f(u_n) = f(v_{n-1}) + p_{2n-2}$

where p_n is the n^{th} prime, such that for any two adjacent vertices u and v satisfies $|f(u) - f(v)|$ is prime and distinct. Hence the graph $C_n @ P_m$ admits prime distance anti-magic labeling.

Example 3.6. The figure 3.3 is the prime distance anti-magic labeling of $C_n @ P_2$.

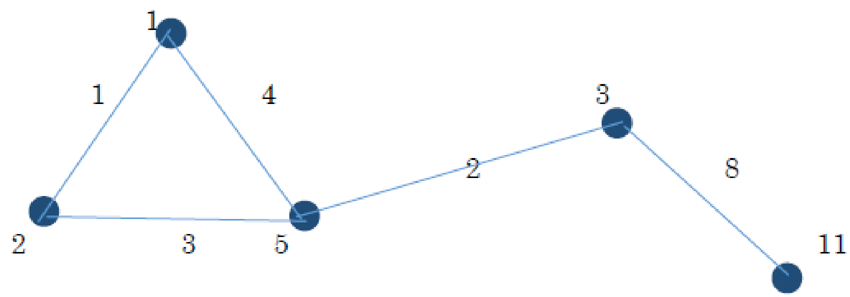

Figure 3.3.

Theorem 3.7. *Bi-star* $B_{n,n}$ *admits prime distance anti-magic labeling*.

Proof. Let u_1, u_2 be the vertices of the apex vertex.

Let $a_1, a_2, \ldots, a_n$ be the vertices which is attached to the apex vertex u_1 and $b_1, b_2, \ldots, b_n$ be vertices which is attached to apex vertex of u_2. The vertex assignments are $f(u_1) = 1; f(u_2) = 2; f(a_n) = f(u_1) + (n+1)^{th}$ prime and $f(b_n) = f(u_n) + (n+2)^{th}$ prime. To get the edge labeling using the condition of $|f(u) - f(v)|$ and it must be distinct. Hence the proof.

Theorem 3.8. *Binary tree admits prime distance labeling*.

Proof. The vertex labeling is $f : V(G) \to Z$ such that for any two adjacent vertices u and v, the integer $|f(u) - f(v)|$ is a prime and the edge labeling is also defined as $f(uv) = |f(u) - f(v)|$ and all are distinct. Therefore, binary tree is a prime distance anti-magic graph.

Theorem 3.9. *If the graph G has a prime distance anti-magic labeling, then G contains a unique cycle*.

Proof. Consider the graph G.

Suppose it has more than cycle then the edge labeling of G fails to the condition of prime.

Hence it is proved.

4. Spectral Values of Caterpillar and Spider Graphs

We present the eigenvalues and characteristic polynomial of caterpillar and spider graphs.

Theorem 4.1. *The characteristic polynomial of the caterpillar is always admitting even powers of the terms.*

Proof. Since the cardinality of the caterpillar is always even. Hence its characteristic polynomial has even powers of term.

Thus proved.

Theorem 4.2. *The determinant value of adjacency matrix of the caterpillar is always* ± 2.

Proof. For example, consider the caterpillar $T = S(X_1, X_2, X_n)$

$$\text{Its } m(T) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} = A$$

$\det(A) = -1.$

$$\text{Eigenvalues } = \begin{pmatrix} -1.9319 \\ -1 \\ -0.5176 \\ 0.5176 \\ 1 \\ 1.9319 \end{pmatrix}$$

Theorem 4.3. *The characteristic roots of caterpillar have always real and distinct.*

Proof. By using characteristic polynomial, we get the result.

Theorem 4.4. *The characteristic roots of caterpillar has always in the form of* $\pm a$.

Proof. By using theorem 8, we can directly get the result.

Observations 4.5

1. The determinant value of the spider is always zero.

2. If the spider has even degree then its characteristic polynomial admits even power of terms only.

3. If $|V(G)|$ of the spider is even then its characteristic roots of the form is $\pm a$.

References

[1] J. A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combination (2013), #DS6.

[2] N. Hartsfield and G. Ringel, Pearls in Graph Theory, Academic Press, Boston-San Diego, New York, London, 1990.

[3] J. D. Laison, C. Starr and A. Walker, Finite prime distance graphs and 2-odd graphs, Discrete Mathematics 313 (2013), 2281-2291.

[4] A. Parthiban and N. Gnanamalar David, Prime distance labeling of some path related graphs, International Journal of Pure and Applied Mathematics 120 (2018), 59-67.

[5] D. B. West, Introduction to Graph Theory, Prentice Hall of India, New Delhi, 2005.