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BIPOLAR FUZZY GRADATION OF OPENNESS

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Abstract

In this paper Bipolar fuzzy gradation of openness and bipolar fuzzy gradation of closedness are introduced and analysed.

1. Introduction

In Zhang [6] introduced the notion of a bipolar fuzzy set. In, Kim, et al. [5] defined bipolar fuzzy point and obtain some of its properties and introduced the concept of bipolar fuzzy topology in the sense of chang [1]. Definition of bipolar fuzzy set- A bipolar fuzzy set in a non-empty set is a pair $A_{bp} = (A_{bp}^{+}, A_{bp}^{-})$ where $A_{bp}^{+} : X \to [0, 1]$ and $A_{bp}^{-} : X \to [-1, 0]$". In Hazra, et al. [4] introduced the concept of gradation of openness. Using this concept they gave the new definition of bipolar fuzzy topology. In this articles [2, 3, 4], the authors developed a theory on gradation of openness. They associated a Chang fuzzy topology with every gradation of openness and vice versa. Also they introduced the definition of gradation preserving maps. In this paper, the concept of gradation of openness is extended to bipolar fuzzy topological spaces.

2. Preliminary Definitions

Definition: 2.1 [5]**.** Let X be a non-empty set. Then a pair

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$A_{bp} = (A_{bp}^{+}, A_{bp}^{-})$ is called *bipolar-valued fuzzy set or bipolar fuzzy set* in, X where $A_{bp}^{+} : X \to [0, 1]$ and $A_{bp}^{-} : X \to [-1, 0]$. The set of all bipolar fuzzy set in X is denoted as $BPF(X)$.

Definition: 2.2 [5]. The *bipolar fuzzy null set* denoted by $0_{bp} = (0_{bp}^{+}, 0_{bp}^{-})$ is a bipolar fuzzy set in X defined as $0_{bp}^{+}(x) = 0$ and $0_{bp}^{-}(x) = 0$, for each $x \in X$.

Definition: 2.3 [5]. The *bipolar fuzzy whole set* denoted by $1_{bp} = (1_{bp}^{+}, 1_{bp}^{-})$ is a bipolar fuzzy set in X defined as $1_{bp}^{+}(x) = 1$ and $1_{bp}^{-}(x) = -1$, for each $x \in X$.

Definition: 2.4 [5]. Let X be a non-empty set and let $A_{bp} = (A_{bp}^{+}, A_{bp}^{-})$, $B_{bp} = (B_{bp}^{+}, B_{bp}^{-})$ be bipolar fuzzy sets in X. Then

(i) A_{bp} is a *subset* of B_{bp}, denoted by $A_{bp} \subset B_{bp}$ is defined as

$A_{bp}^{+}(x) \leq B_{bp}^{+}(x)$ and $A_{bp}^{-}(x) \geq B_{bp}^{-}(x)$, for each $x \in X$.

(ii) The *complement* of A_{bp} is denoted by $A_{bp}^{c} = ((A_{bp}^{c})^{+}, (A_{bp}^{c})^{-})$ is a bipolar fuzzy set in X defined as

$((A_{bp}^{c})^{+}(x) = 1 - A_{bp}^{+}(x)$ and $(A_{bp}^{c})^{-}(x) = -1 - A_{bp}^{-}(x)$, for each $x \in X$.

(iii) The *intersection* of A_{bp} and, B_{bp} denoted by $A_{bp} \cap B_{bp}$ is a bipolar fuzzy set in X defined as

$(A_{bp} \cap B_{bp})(x) = (A_{bp}^{+}(x) \wedge B_{bp}^{+}(x), A_{bp}^{-}(x) \vee B_{bp}^{-}(x))$, for each $x \in X$.

(iv) The *union* of A_{bp} and B_{bp}, denoted by $A_{bp} \cup B_{bp}$ is a bipolar fuzzy set in X defined as

$(A_{bp} \cup B_{bp})(x) = (A_{bp}^{+}(x) \vee B_{bp}^{+}(x), A_{bp}^{-}(x) \wedge B_{bp}^{-}(x))$, for each $x \in X$.

(v) The *intersection* of $((A_{bp})_{\lambda})_{\lambda \epsilon \Lambda}$, a collection of bipolar fuzzy subsets in

X denoted by $\bigcap_{\lambda \epsilon \Lambda}(A_{bp})_{\lambda}$ is a bipolar fuzzy set in X defined as

$$(\bigcap_{\lambda \epsilon \Lambda} (A_{bp})_{\lambda})(x) = (\wedge_{\lambda \epsilon \Lambda} ((A_{bp})^{+}_{\lambda})(x), \vee_{\lambda \epsilon \Lambda} ((A_{bp})^{-}_{\lambda})(x)), \text{ for each } x \in X.$$

(vi) The *union* of $((A_{bp})_{\lambda})_{\lambda \epsilon \Lambda}$, a collection of bipolar fuzzy subsets in X denoted by $\bigcup_{\lambda \epsilon \Lambda}(A_{bp})_{\lambda}$, is a bipolar fuzzy set in defined as

$$(\bigcup_{\lambda \epsilon \Lambda} (A_{bp})_{\lambda})(x) = (\vee_{\lambda \epsilon \Lambda} ((A_{bp})^{+}_{\lambda})(x), \wedge_{\lambda \epsilon \Lambda} ((A_{bp})^{-}_{\lambda})(x)), \text{ for each } x \in X.$$

Definition: 2.5 [5]**.** Let X be a non-empty set and let $\mathfrak{B} \subset BPF(X)$. Then $\mathfrak{B}$ is called a *bipolar fuzzy topology on X*, if it satisfies the following axioms:

(i) $0_{bp}, 1_{bp} \epsilon \mathfrak{B}$.

(ii) $A_{bp} \cap B_{bp} \epsilon \mathfrak{B}$, for any $A_{bp}, B_{bp} \subset \mathfrak{B}$.

(iii) $\bigcup_{\lambda \epsilon \Lambda}(A_{bp})_{\lambda} \epsilon \mathfrak{B}$, for any $((A_{bp})_{\lambda})_{\lambda \epsilon \Lambda} \subset \mathfrak{B}$.

In this case the pair $(X, \mathfrak{B})$ is called a bipolar fuzzy topological space and each member of $\mathfrak{B}$ is called a bipolar fuzzy open set $(BPFOS)$ in X. $A_{bp} \in BPF(X)$ is said to be closed in X, if $A^{c}_{bp} \in \mathfrak{B}$. The set of all bipolar fuzzy topologies on X is denoted as $BPFT(X)$.

Definition: 2.6. Let $(X, \mathfrak{B})$ be a bipolar fuzzy topological space. Let $Y \subseteq X$. Let $A_{bp} = (A^{+}_{bp}, A^{-}_{bp}) \in \mathfrak{B}$. Define $A_{bp}/Y = (A^{+}_{bp}/Y, A^{-}_{bp}/Y)$ such that $(A^{+}_{bp}/Y)(z) = A^{+}_{bp}(z)$ and $(A^{-}_{bp}/Y)(z) = A^{-}_{bp}(z)$, for all $z \in Y$. Define $(\mathfrak{B}/Y) = \{(A_{bp}/Y) \mid A_{bp} \in \mathfrak{B}\}$ Then $(\mathfrak{B}/Y)$ is called the *bipolar fuzzy subspace topology on* Y and $(Y, \mathfrak{B}/Y)$ is called a *bipolar fuzzy subspace of* $(X, \mathfrak{B})$ or simply Y is called a *bipolar fuzzy subspace* of X.

Definition: 2.7 [5]**.** Let X and Y be a non-empty set, let $A_{bp} \in BPF(X)$ and $B_{bp} \in BPF(Y)$ let $\theta : X \to Y$ be a mapping. Then

(i) The image of A_{bp} under θ, denoted by $\theta(A_{bp}) = (\theta(A^{+}_{bp}), (A^{-}_{bp}))$, is a bipolar fuzzy set in Y defined as follows: for each $y \in Y$.

$$[\theta(A_{bp}^{+})(y) = \begin{cases} \vee_{x \in \theta^{-1}(y)} A_{bp}^{+}(x), \text{ if } \theta^{-1}(y) \neq \phi \\ 0, \text{ otherwise} \end{cases}$$

and

$$[\theta(A_{bp}^{-})(y) = \begin{cases} \wedge_{x \in \theta^{-1}(y)} A_{bp}^{-}(x), \text{ if } \theta^{-1}(y) \neq \phi \\ 0, \text{ otherwise} \end{cases}$$

(ii) The pre-image of B_{bp} under θ, denoted by $\theta^{-1}(B_{bp}) = (\theta^{-1}(B_{bp}^{+}), \theta^{-1}(B_{bp}^{-}))$, is a bipolar fuzzy set in defined as follows: for each $x \in X$.

$$[\theta^{-1}(B_{bp}^{+})](x) = B_{bp}^{+}(\theta(x)) \text{ and } [\theta^{-1}(B_{bp}^{-})](x) = B_{bp}^{-}(\theta(x))$$

Definition: 2.8. Let $(X, \mathfrak{B}_1)$ and $(Y, \mathfrak{B}_2)$ be two bipolar fuzzy topological spaces. A mapping $\theta : X \to Y$ is a bipolar fuzzy continuous if for all $A_{bp} \in \mathfrak{B}_2$, $\theta^{-1}(A_{bp}) \in \mathfrak{B}_1$.

3. Bipolar Fuzzy Gradation of Openness

Definition: 3.1. Let be a non-empty set. A mapping $\mathcal{G} : BPF(X) \to I$ is said to be bipolar fuzzy gradation of openness over X, iff the following axioms are satisfied:

$(BPFGO1)\, \mathcal{G}(0_{bp}) = \mathcal{G}(1_{bp}) = 1$

$(BPFGO2)\, \mathcal{G}((A_{bp})_i) > 0$, for $i = 1$ to m

$$\Rightarrow \mathcal{G}(\cap_{i=1}^{m} (A_{bp})_i) = 0.$$

$(BPFGO3)\, \mathcal{G}((A_{bp})_\lambda) > 0$, for all $\lambda \in \Lambda$

$$\Rightarrow \mathcal{G}(\cup_{\lambda \in \Lambda} (A_{bp})_\lambda) > 0$$

Therefore $(X, \mathcal{G})$ is called bipolar fuzzy gradation space.

Definition: 3.2. Let be a non-empty set. A mapping $\mathfrak{F} : BPF(X) \to I$ is said to be bipolar fuzzy gradation of closedness over X iff the following conditions are satisfied:

$(BPFGC1) \mathfrak{F}(0_{bp}) = \mathfrak{F}(1_{bp}) = 1.$

$(BPFGC2) \mathfrak{F}((A_{bp})_\lambda) > 0,$ for $\lambda \in \Lambda$

$$\Rightarrow \mathfrak{F}(\cap_{\lambda \in \Lambda} (A_{bp})_\lambda) > 0$$

$(BPFGC3) \mathfrak{F}((A_{bp})_i) > 0,$ for $i = 1$ to m

$$\Rightarrow \mathfrak{F}(\cup_{i=1}^{m} (A_{bp})_i) > 0$$

Definition: 3.3. Let $(X, \mathcal{G})$ be a bipolar fuzzy gradation space. Then the bipolar fuzzy topology on X induced by is given by $\mathfrak{B}(\mathcal{G}) = \{A_{bp} \in BPF(X) \mid \mathcal{G}(A_{bp}) > 0\}.$

Definition: 3.4. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two bipolar fuzzy gradation of openness on X. Then $(\mathcal{G}_1) \geq (\mathcal{G}_2)$ if $\mathcal{G}_1(A_{bp}) \geq \mathcal{G}_2(A_{bp}),$ for all $A_{bp} \in BPF(X).$

Definition: 3.5. Let $(X, \mathcal{G})$ be a bipolar fuzzy gradation space and $A_{bp} \in BPF(X).$ Then $\mathcal{G}$-closure of $A_{bp},$ denoted by $\mathcal{G}cl(A_{bp}) = \cap\{B_{bp} \in BPF(X); \mathfrak{F}_{\mathcal{G}}(B_{bp}) > 0, B_{bp} \supseteq A_{bp}\}$

Note:

(i) $\mathfrak{F}_{\mathcal{G}}(\mathcal{G}cl(A_{bp})) > 0$ and

(ii) For every $A_{bp}, B_{bp} \in BPF(X), A_{bp} \supseteq B_{bp}$ implies that $\mathcal{G}cl(A_{bp}) \supseteq \mathcal{G}cl(B_{bp})$

Definition: 3.6. Let $(X, \mathcal{G}_1), (Y, \mathcal{G}_2)$ be two bipolar fuzzy gradation spaces. Then a map $\theta : X \to Y$ is called

(i) a bipolar fuzzy gradation preserving map, if $(\mathcal{G}_2)(A_{bp}) \leq (\mathcal{G}_1)(\theta^{-1}(A_{bp})),$ for each $A_{bp} \in BPF(Y).$

(ii) a bipolar fuzzy strongly gradation preserving map, if $(\mathcal{G}_2)(A_{bp})$

$= (\mathcal{G}_1)(\theta^{-1}(A_{bp}))$, for each $A_{bp} \in BPF(Y)$.

(iii) a bipolar fuzzy weakly gradation preserving map, if

$(\mathcal{G}_2)(A_{bp}) > 0 \Rightarrow (\mathcal{G}_1)(\theta^{-1}(A_{bp})) > 0$, for each $A_{bp} \in BPF(Y)$.

Definition: 3.7. Let be a non-empty set. A mapping $\mathcal{G}_* : 2^X \to I$ is said to be crisp gradation of openness on X, iff the following conditions are satisfied:

(i) $\mathcal{G}_*(\phi) = \mathcal{G}_*(X) = 1$

(ii) $\mathcal{G}_*(A_i) > 0$, for $i = 1$ to m

$\Rightarrow \mathcal{G}_*(\bigcap_{i=1}^{m} A_i) > 0$

(iii) $\mathcal{G}_*(A_\lambda) > 0$, for $\lambda \in \Lambda$

$\Rightarrow \mathcal{G}_*(\bigcup_{\lambda \in \Lambda} A_\lambda) > 0$

Therefore $(X, \mathcal{G}_*)$ is a crisp gradation space. Then the topology on X induced by $\mathcal{G}_*$ is $\mathfrak{B}(\mathcal{G}_*) = \{A \in 2^X / \mathcal{G}_*(A) > 0\}$

Definition: 3.8. Given $A = 2^X$ define $A_{bp} = (A_{bp}^+, A_{bp}^-)$ as $A_{bp}^+ : X \to [0, 1]$ such that $A_{bp}^+(x) > 0$, if $x \in A$ and $A_{bp}^+(x) = 0$, if $x \notin A$, $A_{bp}^- X \to [-1, 0]$ such that $A_{bp}^-(x) < 0$, if $x \in A$ and $A_{bp}^-(x) = 0$, if $x \notin A$ Therefore $A_{bp} \in BPF(X)$.

Definition : 3.9. Let $\mathcal{G} : BPF(X) \to I$ be a bipolar fuzzy gradation of openness on X. Define $\mathcal{G}_* : 2^X \to I$ such that $\mathcal{G}_*(A) = \mathcal{G}(A_{bp})$. Then $\mathcal{G}_*$ is a crisp gradation of openness on X.

Theorem: 3.10. *Let $\mathcal{G}$ be a bipolar fuzzy gradation of openness on X and $\mathfrak{F}_{\mathcal{G}} : BPF(X) \to I$ be a mapping defined by $\mathfrak{F}_{\mathcal{G}}(A_{bp}) = \mathcal{G}(A_{bp}^c)$. Then $\mathfrak{F}_{\mathcal{G}}$ is a bipolar fuzzy gradation of closedness* on X.

Proof: Let $\mathcal{G}$ be a bipolar fuzzy gradation of openness on X.

To prove: $\mathfrak{F}_{\mathcal{G}}$ is a bipolar fuzzy gradation of closedness on X

(i) $\mathfrak{F}_{\mathcal{G}}(0_{bp}) = \mathcal{G}(0_{bp}') = \mathcal{G}(1_{bp}) = 1 \quad (by(BPFGO1))$

$\mathfrak{F}_{\mathcal{G}}(1_{bp}) = \mathcal{G}(1_{bp}') = \mathcal{G}(0_{bp}) = 1 \quad (by(BPFGO1))$

(ii) $\mathfrak{F}_{\mathcal{G}}(\bigcup_{i=1}^{m} (A_{bp})_i) = \mathcal{G}(\bigcup_{i=1}^{m} (A_{bp})_i)^c$

$$= \mathcal{G}(\bigcap_{i=1}^{m} (A_{bp})_i^c) > 0 \quad (by(BPFG02))$$

(iii) $\mathfrak{F}_{\mathcal{G}}(\bigcap_{\lambda\in\Lambda} (A_{bp})_{\lambda}) = \mathcal{G}(\bigcap_{\lambda\in\Lambda} (A_{bp})_{\lambda})^c$

$$= \mathcal{G}(\bigcup_{\lambda\in\Lambda} (A_{bp})_{\lambda}^c) \quad (by(BPFGO3))$$

$$= \mathfrak{F}_{\mathcal{G}}(\bigcap_{\lambda\in\Lambda} (A_{bp})_{\lambda}) > 0$$

Therefore $\mathfrak{F}_{\mathcal{G}}$ is a bipolar fuzzy gradation of closedness on X.

Theorem: 3.11. *Let $\mathfrak{F}$ be a bipolar fuzzy gradation of closedness on X and $\mathcal{G}_{\mathfrak{F}} : BPF(X) \to I$ be a mapping defined by $\mathcal{G}_{\mathfrak{F}}(A_{bp}) = \mathfrak{F}(A_{bp}^c)$. Then $\mathcal{G}_{\mathfrak{F}}$ is a bipolar fuzzy gradation of openness on X.*

Proof: Let $\mathfrak{F}$ be a bipolar fuzzy gradation of closedness on X.

To prove: $\mathcal{G}_{\mathfrak{F}}$ is a bipolar fuzzy gradation of openness on X

(i) $\mathcal{G}_{\mathfrak{F}}(0_{bp}) = \mathfrak{F}(0_{bp}^c) = \mathfrak{F}(1_{bp}) = 1 \quad (by(BPFGC1))$

$\mathcal{G}_{\mathfrak{F}}(1_{bp}) = \mathfrak{F}(1_{bp}^c) = \mathfrak{F}(0_{bp}) = 1 \quad (by(BPFGC1))$

(ii) $\mathcal{G}_{\mathfrak{F}}(\bigcap_{i=1}^{m} (A_{bp})_i) = \mathfrak{F}(\bigcap_{i=1}^{m} (A_{bp})_i)^c$

$$= \mathfrak{F}(\bigcup_{i=1}^{m} (A_{bp})_i^c) > 0 \quad (by(BPFGC2))$$

$\Rightarrow \mathcal{G}_{\mathfrak{F}}(\bigcap_{i=1}^{m} (A_{bp})_i) > 0$

(iii) $\mathcal{G}_{\mathfrak{F}}((\bigcup_{\lambda\in\Lambda} (A_{bp})_{\lambda}) = \mathfrak{F}(\bigcup_{\lambda\in\Lambda} (A_{bp})_{\lambda})^c$

$$= \mathfrak{F}(\cap_{\lambda\in\Lambda} (A_{bp})^c_\lambda) \quad (by(BPFGC3))$$

$$\Rightarrow \mathcal{G}_{\mathfrak{F}}(\cup_{\lambda\in\Lambda} (A_{bp})_\lambda) > 0$$

Therefore $\mathcal{G}_{\mathfrak{F}}$ is a bipolar fuzzy gradation of openness on X.

Remark: 3.12. Let $\mathcal{G}$ be a bipolar fuzzy gradation of openness on X and $\mathfrak{F}$ be a bipolar fuzzy gradation of closedness on X. Then

(i) $\mathcal{G}_{\mathfrak{F}_{\mathcal{G}}} = \mathcal{G}$

(ii) $\mathfrak{F}_{\mathcal{G}_{\mathfrak{F}}} = \mathfrak{F}$

Theorem: 3.13. *Let* $(X, \mathcal{G})$ *be a bipolar fuzzy gradation space. Then*

(i) $\mathcal{G}cl(0_{bp}) = 0_{bp}$

(ii) $\mathcal{G}cl(A_{bp}) \supseteq A_{bp}$

(iii) $\mathcal{G}cl((A_{bp})_1 \cup (A_{bp})_2) = \mathcal{G}cl(A_{bp})_1 \cup \mathcal{G}cl(A_{bp})_2$

(iv) $\mathcal{G}cl(\mathcal{G}cl(A_{bp}) = \mathcal{G}cl(A_{bp})$

Proof: Proof of (i) and (ii) are obvious

To prove (iii)

Let $(A_{bp})_1$ and $(A_{bp})_2 \in BPF(X)$, from (ii),

$\mathcal{G}cl(A_{bp})_2 \supseteq (A_{bp})_2$ and $\mathcal{G}cl((A_{bp})_2 \supseteq (A_{bp})_2$

$\Rightarrow \mathcal{G}cl(A_{bp})_1 \cup \mathcal{G}cl(A_{bp})_2 \supseteq (A_{bp})_1 \cup (A_{bp})_2$

From the definition of bipolar fuzzy gradation of closedness,

$\mathfrak{F}_{\mathcal{G}} = ((A_{bp})_i) > 0,$ for $i = 1$ to 2

$\Rightarrow \mathfrak{F}_{\mathcal{G}}(\mathcal{G}cl(A_{bp})_1 \cup \mathcal{G}cl(A_{bp})_2) > 0$

Since $\mathfrak{F}_{\mathcal{G}}(\mathcal{G}cl(A_{bp})_1 \cup \mathcal{G}cl(A_{bp})_2) > 0$ and $\mathcal{G}cl(A_{bp})_1 \cup \mathcal{G}cl(A_{bp})_2 \supseteq (A_{bp})_1 \cup (A_{bp})_2$

Then $\mathcal{G}cl(A_{bp})_1 \cup \mathcal{G}cl(A_{bp})_2 \supseteq \mathcal{G}cl((A_{bp})_1 \cup (A_{bp})_2) \ldots (1)$

Now to prove $\mathcal{G}cl((A_{bp})_1 \cup (A_{bp})_2) \supseteq \mathcal{G}cl(A_{bp})_1 \cup \mathcal{G}cl(A_{bp})_2$

We know that $(A_{bp})_1 \subseteq (A_{bp})_1 \cup (A_{bp})_2$ and $(A_{bp})_2 \subseteq (A_{bp})_1 \cup (A_{bp})_2$

$\Rightarrow \mathcal{G}cl(A_{bp})_1 \subseteq \mathcal{G}cl(A_{bp})_1 \cup (A_{bp})_2)$ and $\mathcal{G}cl(A_{bp})_2 \subseteq \mathcal{G}cl((A_{bp})_1 \cup (A_{bp})_2)$

$\Rightarrow \mathcal{G}cl(A_{bp})_1 \cup \mathcal{G}cl(A_{bp})_2 \subseteq \mathcal{G}cl((A_{bp})_1 \cup (A_{bp})_2)$

$\Rightarrow \mathcal{G}cl((A_{bp})_1 \cup (A_{bp})_2) \supseteq \mathcal{G}cl(A_{bp})_1 \cup (A_{bp})_2 \ldots (2)$

from (1) and (2) $\mathcal{G}cl((A_{bp})_1 \cup (A_{bp})_2) = \mathcal{G}cl(A_{bp})_1 \cup \mathcal{G}cl(A_{bp})_2$

Proof of (iv) is obvious.

Theorem : 3.14. *Let* $(X, \mathcal{G})$ *be a bipolar fuzzy gradation space. Then for each* $A_{bp} \in BPF(X)$, $\mathfrak{F}_{\mathcal{G}}(A_{bp}) > 0$ *iff* $A_{bp} = \mathcal{G}cl(A_{bp})$.

Proof: Let $(X, \mathcal{G})$ be a bipolar fuzzy gradation space. Assume, $\mathfrak{F}_{\mathcal{G}}(A_{bp}) > 0$, for each $A_{bp} \in BPF(X)$. To prove: $A_{bp} = \mathcal{G}cl(A_{bp})$ $\mathcal{G}cl(A_{bp}) = \bigcap\{B_{bp} \in BPF(X) \mid \mathfrak{F}_{\mathcal{G}}(B_{bp}) > 0, B_{bp} \supseteq A_{bp}\}$

$= A_{bp}$

(since $\mathfrak{F}_{\mathcal{G}}(A_{bp}) > 0$, then A_{bp} is a member of above collections and also every member contains A_{bp}) therefore $A_{bp} = \mathcal{G}cl(A_{bp})$. Conversely, Assume $A_{bp} = \mathcal{G}cl(A_{bp})$ To prove: $\mathfrak{F}_{\mathcal{G}}(A_{bp}) > 0$ By the definition of $\mathcal{G}$-closure of A_{bp}

$\mathcal{G}cl(A_{bp}) = \bigcap\{B_{bp} \in BPF(X) \mid \mathfrak{F}_{\mathcal{G}}(B_{bp}) > 0, B_{bp} \supseteq A_{bp}\}$ it is clear that $\mathfrak{F}_{\mathcal{G}}(\mathcal{G}cl(A_{bp})) > 0$

$\Rightarrow \mathfrak{F}_{\mathcal{G}}(A_{bp}) > 0.$

Theorem: 3.15. *Let* $\{\mathcal{G}_k, k = 1, 2, \ldots n\}$ *be a finite family of bipolar fuzzy gradation of openness on X Then* $\mathcal{G} = \bigcap_{k=1}^{n} \mathcal{G}_k$ *is a bipolar fuzzy gradation of openness on X.*

Proof: Let $\{\mathcal{G}_k, k = 1, 2, \ldots n\}$ be a finite family of bipolar fuzzy gradation of openness on X. Let $\mathcal{G} = \bigcap_{k=1}^{n} \mathcal{G}_k$ To prove: $\mathcal{G} = \bigcap_{k=1}^{n} \mathcal{G}_k$ is a bipolar fuzzy gradation of openness on X.

(i) $\mathcal{G}(0_{bp}) = \bigcap_{k=1}^{n} \mathcal{G}_k(0_{bp})$

$$= \mathcal{G}_1(0_{bp}) \wedge \mathcal{G}_2(0_{bp}) \wedge \mathcal{G}_3(0_{bp}) \ldots \wedge \mathcal{G}_n(0_{bp})$$

$$= 1$$

(since for each $\mathcal{G}_k$, $k = 1, 2, \ldots n$ is a $BPFG01$)

$\mathcal{G}(1_{bp}) = \bigcap_{k=1}^{n} \mathcal{G}_k(1_{bp})$

$$= \mathcal{G}_1(1_{bp}) \wedge \mathcal{G}_2(1_{bp}) \wedge \mathcal{G}_3(1_{bp}) \ldots \wedge \mathcal{G}_n(1_{bp})$$

$$= 1$$

(since for each $\mathcal{G}_k$, $k = 1, 2, \ldots n$ satisfies the condition $BPFG01$)

(ii) $\mathcal{G}_k(A_{bp})_i = \bigcap_{k=1}^{n} \mathcal{G}_k(A_{bp})_i$, for $i = 1$ to m

$$= \mathcal{G}_1(A_{bp})_i \wedge \mathcal{G}_2(A_{bp})_i \wedge \mathcal{G}_3(A_{bp})_i \ldots \wedge \mathcal{G}_n(A_{bp})_i > 0,$$

for $i = 1$ to m

$\Rightarrow \mathcal{G}(A_{bp})_i > 0$

(since for each $\mathcal{G}_k(A_{bp})_i > 0$, for $i = 1$ to m)

$\Rightarrow \mathcal{G}(\bigcap_{1=1}^{m} (A_{bp})_i) = \bigcap_{k=1}^{n} \mathcal{G}_k(\bigcap_{1=1}^{m} (A_{bp})_i)$

$= \mathcal{G}_1(\bigcap_{1=1}^{m} (A_{bp})_i) \wedge \mathcal{G}_2(\bigcap_{1=1}^{m} (A_{bp})_i) \wedge \ldots \wedge \mathcal{G}_n(\bigcap_{1=1}^{m} (A_{bp})_i) > 0$

(since for each $\mathcal{G}_k(\bigcap_{1=1}^{m} (A_{bp})_i) > 0$, $k = 1, 2, \ldots n$)

$\Rightarrow \mathcal{G}(\bigcap_{1=1}^{m} (A_{bp})_i) > 0$

(iii) $\mathcal{G}(A_{bp})_\lambda = \bigcap_{k=1}^{n} \mathcal{G}_k(A_{bp})_\lambda$, for all $\lambda \in \Lambda$

$$= \mathcal{G}_1(A_{bp})_\lambda \wedge \mathcal{G}_2(A_{bp})_\lambda \wedge \mathcal{G}_3(A_{bp})_\lambda \ldots \wedge \mathcal{G}_n(A_{bp})_\lambda > 0,$$

for all $\lambda = \Lambda$

$\Rightarrow \mathcal{G}(A_{bp})_{\lambda} > 0$

(since for each $\mathcal{G}_k(A_{bp})_{\lambda} > 0$, for all $\lambda \in \Lambda$)

$\Rightarrow \mathcal{G}(\bigcup_{\lambda \in \Lambda} (A_{bp})_{\lambda}) = \bigcap_{k=1}^{n} \mathcal{G}_k(\bigcup_{\lambda \in \Lambda} (A_{bp})_{\lambda})$

$= \mathcal{G}_1(\bigcup_{\lambda \in \Lambda} (A_{bp})_{\lambda}) \wedge \mathcal{G}_2(\bigcup_{\lambda \in \Lambda} (A_{bp})_{\lambda}) \wedge \ldots \wedge \mathcal{G}_n(\bigcup_{\lambda \in \Lambda} (A_{bp})_{\lambda})$, $k = 1, 2, \ldots, n$

$\Rightarrow \mathcal{G}(\bigcup_{\lambda \in \Lambda} (A_{bp})_{\lambda}) > 0$

(since for each $\Rightarrow \mathcal{G}_k(\bigcup_{\lambda \in \Lambda} (A_{bp})_{\lambda}) > 0$, $k = 1, 2 \ldots 2, n$)

Therefore $\mathcal{G} = \bigcap_{k=1}^{n} \mathcal{G}_k$ is a bipolar fuzzy gradation of openness on X.

Theorem : 3.16. *Let* $(X, \mathcal{G}_1), (Y, \mathcal{G}_2)$ *be two bipolar fuzzy topological spaces and be a* $\theta : X \to Y$ *function. Then the following conditions are equivalent.*

(i) θ *is a bipolar fuzzy weakly gradation preserving map.*

(ii) $\theta(\mathcal{G}cl(A_{bp})) \leq \mathcal{G}cl(\theta(A_{bp}))$, *for all* $A_{bp} \in BPF(X)$.

Proof: Let $(X, \mathcal{G}_1), (Y, \mathcal{G}_2)$ be two bipolar fuzzy topological spaces and $\theta : X \to Y$ be a Function

To prove:- (i) $\Rightarrow$ (ii)

Let us assume that (i) holds, that is $\mathcal{G}_2(A_{bp}) > 0 \Rightarrow \mathcal{G}_1(\theta^{-1}(A_{bp})) > 0$, for each $A_{bp} \in BPF(X)$. Then for each $A_{bp} \in BPF(X)$

$\theta^{-1}(\mathcal{G}cl(\theta(A_{bp}))) = \theta^{-1}[\bigcap \{B_{bp} \in BPF(Y) : \mathfrak{F}_{\mathcal{G}_2}(B_{bp}) > 0$ and $B_{bp} \supseteq \theta(A_{bp})\}]$

$= \theta^{-1}[\bigcap \{B_{bp} \in BPF(Y) : \mathfrak{F}_{\mathcal{G}_1}(\theta^{-1}(B_{bp})) > 0$ and $B_{bp} \supseteq \theta(A_{bp})\}]$

$\geq \bigcap\{\theta^{-1}(B_{bp}) \in BPF(X) : \mathfrak{F}_{\mathcal{G}_1}(\theta^{1}(B_{bp})) > 0, \theta^{-1}(B_{bp}) \geq \theta^{-1}(\theta(A_{bp})) \geq A_{bp}\}$

$\geq \bigcap\{D_{bp} \in BPF(X) : \mathfrak{F}_{\mathcal{G}_1}(D_{bp}) > 0, D_{bp} \supseteq A_{bp}\}$

$\geq \mathcal{G}cl(A_{bp})$

$\Rightarrow \theta^{-1}(\mathcal{G}cl(\theta(A_{bp}))) \geq \mathcal{G}cl(A_{bp})$

$\Rightarrow \mathcal{G}cl(\theta(A_{bp})) \geq (\mathcal{G}cl(A_{bp}))$

$\Rightarrow \theta(\mathcal{G}cl(A_{bp})) \leq \mathcal{G}cl(\theta(A_{bp}))$, for all $A_{bp} \in BPF(X)$

Therefore (i) $\Rightarrow$ (ii)

To prove:- (ii) $\Rightarrow$ (i)

Let us assume (ii) holds, that is $\theta(\mathcal{G}cl(A_{bp})) \leq \mathcal{G}cl(\theta(A_{bp}))$, for all $A_{bp} \in BPF(X)$. From the theorem – ( ), we have, for each $B_{bp} \in BPF(Y)$.

$\mathcal{G}_2(B_{bp}) > 0 \Rightarrow \mathfrak{F}_{\mathcal{G}}((B_{bp})^c) > 0$ iff $\mathcal{G}cl(Bbp))^c = (B_{bp}^c)$

Since $\theta(\mathcal{G}cl(\theta^{-1}(B_{bp}^c))) \subseteq \mathcal{G}cl(\theta^{-1}(B_{bp}^c))) \subseteq (B_{bp}^c)$, we have $\mathcal{G}cl(\theta^{-1}(B_{bp}^c)) \subseteq \theta^{-1}(B_{bp}^c)$

Hence,

$$\begin{aligned}\mathfrak{F}_{\mathcal{G}_1}(\theta^{-1}(B_{bp}^c)) > 0 &\Rightarrow \mathfrak{F}_{\mathcal{G}_1}(\theta^{-1}(B_{bp}))^c > 0\\ &\Rightarrow \mathcal{G}_1((\theta^{-1}(B_{bp})^c)^c > 0\\ &\Rightarrow \mathcal{G}_1(\theta^{-1}(B_{bp})) > 0\end{aligned}$$

Therefore θ is a bipolar fuzzy weakly gradation preserving map.

Theorem: 3.17. *let* $\theta : (X, \mathcal{G}_1) \to (Y, \mathcal{G}_2)$ *is a bipolar fuzzy weakly gradation preserving map iff* $\theta : (X, \mathfrak{B}(\mathcal{G}_1)) \to (Y, \mathfrak{B}(\mathcal{G}_2))$ *is a bipolar fuzzy continuous.*

Proof: Assume is a bipolar fuzzy gradation preserving map.

To prove: $\theta : (X, \mathfrak{B}(\mathcal{G}_1)) \to (Y, \mathfrak{B}(\mathcal{G}_2))$ is a bipolar fuzzy continuous.

Let $A_{bp} \in \mathfrak{B}(\mathcal{G}_2)$

$\Rightarrow \mathcal{G}_2(A_{bp}) > 0$ (since $\mathfrak{B}(\mathcal{G}) = \{A_{bp} \in BPF(X)/\mathcal{G}(A_{bp}) > 0\}$)

$\Rightarrow \mathcal{G}_2(\theta^{-1}(A_{bp})) > 0$ (since θ is a bipolar fuzzy weakly gradation preserving map)

$\Rightarrow \theta^{-1}(A_{bp}) \in \mathfrak{B}(\mathcal{G}_1)$

Therefore θ is bipolar fuzzy continuous

Conversely,

Assume $\theta : (X, \mathfrak{B}(\mathcal{G}_1)) \to (Y, \mathfrak{B}(\mathcal{G}_2))$ is a bipolar fuzzy continuous.

To prove: $\theta : (X, \mathfrak{B}(\mathcal{G}_1)) \to (Y, \mathfrak{B}(\mathcal{G}_2))$ is a bipolar fuzzy weakly gradation preserving map

Let $\mathcal{G}_2(A_{bp}) > 0$

$\Rightarrow A_{bp} \in \mathfrak{B}(\mathcal{G}_2)$

$\Rightarrow \theta^{-1}(A_{bp}) \in \mathfrak{B}(\mathcal{G}_1)$

$\Rightarrow \mathcal{G}_2(\theta^{-1}(A_{bp})) > 0$

Therefore θ is a bipolar fuzzy weakly gradation preserving map.

Theorem: 3.18. *For* $A \in 2^I$, $A \subseteq [0, 1] = I$, *define* $(A_I)_{bp} = ((A_I)^+_{bp}, (A_I)^-_{bp})$ *as* $(A_I)^+_{bp} : I \to [0, 1]$ *such that* $(A_I)^+_{bp}(x) = A$, *for all* $x \in \mathrm{X}$ *and* $(A_I)^-_{bp} : I \to [-1, 0]$ *such that* $(A_I)^-_{bp}(x) = 0$, *for all* $x \in X$. *Therefore* $(A_I)_{bp} \in BPF(I)$.

Let $\mathcal{G}$ *be a bipolar fuzzy gradation of openness on X. Define* $(\mathcal{G}_I)_* : 2^I \to I$ *such that* $(\mathcal{G}_I)_*(A) = \mathcal{G}(A_I)_{bp})$. *Then* $(\mathcal{G}_I)_*$ *is a crisp gradation of openness on X.*

Theorem: 3.19. *Given* $A_{bp} \in BPF(I)$, *where* $A_{bp}^{+} : I \to [0, 1]$ *and* $A_{bp}^{-} : I \to [-1, 0]$. *Fix* $\alpha \in I$ *Define* $(A_{bp})_{\alpha} \subseteq I$ *such that* $(A_{bp})_{\alpha} = \{A_{bp}^{+}(\alpha)\}$, *singleton set. Let* $(\mathcal{G}_{*} : 2^{I} \to I$ *be a crisp gradation of openness on I. Define* $\mathcal{G}_{\alpha} : BPF(I) \to I$ *such that* $\mathcal{G}_{\alpha}(A_{bp}) = \mathcal{G}_{*}((A_{bp})_{\alpha})$. *Then* $\mathcal{G}_{\alpha}$ *is a bipolar fuzzy gradation of openness on X.*

Theorem: 3.20. *Let* $\mathcal{G}_{*} : 2^{I} \to I$ *be a crisp gradation of openness on I. Let X be any non-empty set. Fix* $x \in X, \mathcal{G}_{x} : BPF(X) \to I$ *such that* $\mathcal{G}_{x}(A_{bp}) = \mathcal{G}_{*}\{A_{bp}^{+}(x)\}$, *singleton* $\{A_{bp}^{+}(x)\}$. *Then* $\mathcal{G}_{x}$ *is a bipolar fuzzy gradation of openness on X.*

Theorem: 3.21. *Let* $\mathcal{G} : BPF(X) \to I$ *be a bipolar fuzzy gradation of openness on X. Let* $\mathcal{G}_{c} : BPF(X) \to I$ *such that* $\mathcal{G}_{c}(A_{bp}) = \mathcal{G}(A_{bp}^{c})$. *Then* $\mathcal{G}_{c}$ *is a bipolar fuzzy gradation of openness on*.

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